INTRODUCTORY ELECTRODYNAMICS FOR ENGINEERS

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FOR

ENGINEERS

BY

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PREFACE

Text-books are written in the attempt to meet a need which is felt from some pedagogical point of view. The view which has led to the preparation of this text, has been presented in a paper entitled "Engineering Courses for the Functional Rather than the Industrial Divisions of Engineering."[1] Stated briefly, this view is that the prevailing educational practice of placing all students of engineering, regardless of interest and aptitude, in the same classes in the important introductory work in the mathematical and physical science of the first two years of college is a very grave educational blunder. It is maintained that separate provisions should be made in this introductory work for the needs of the men of superior aptitude and for those of moderate aptitude by providing the first two of the following types of treatment of these subjects.

TYPES OF TREATMENT OF SUBJECTS OF STUDY

Profoundly technical treatment

Adapted to the needs of men with superior aptitude and keen interest in the subject. Object: to facilitate the acquirement of a *profound understanding* of the subject.

Moderately technical treatment

Adapted to the needs of men with a moderate aptitude and moderate interest in the subject. Object: to facilitate the acquirement of skill or *proficiency* in the use or application of the principles of the subject to the *moderate* extent necessary to effective work in the man's natural field.

Non-technical treatment

Adapted to the needs of men who have either little aptitude for, or a non-professional interest in, the subject. Object: to faciliate the acquirement of a *non-technical appreciation* of the general content, the methods, and the possibilities of the science.

The text represents the authors' notion of the type of *introduction* which engineering students belonging to the first group should receive to the electrical theory and the electrical principles which are fundamental to design, development, research, and technical supervision in the electrical field. The introduction

[1] Jour. A.I.E.E. (Nov. 1923).

is in the form of a connected development in which the observations, the definitions, the units, and the laws are taken up in the sequence in which the units of the electrostatic system are defined, each in terms of those preceding it.

The text has been used in mimeographed form during the past seven years at the University of Wisconsin in the introductory course in Electrical Engineering. A brief statement of the position of this course in the curriculum may contribute to an understanding of some features of the text. The course starts at the middle of the second year, at a time when the student has completed the first semester of calculus and is entering upon the second. One of the aims in starting this course before the completion of the work in calculus is to strengthen the work in calculus by giving the student object lessons in the engineering applications of the calculus. To this end, a number of the laboratory experiments, which are such an essential part of the course, are of the type in which the student predicts (from calculation) what the value of an effect should be, and then proceeds to check his prediction by actual measurement. As an illustration, in studying the engineering applications of Coulomb's inverse-square law of force in the fifth week of the course, the student takes the dimensions of a telephone line and a power line which parallel each other in a long passage-way. By calculation he predicts the value of the potential difference between the telephone wires in terms of the potential difference between the power wires. He then proceeds to check this prediction by measurements with electrostatic voltmeters, and to make other observations on electrostatic induction.

In this introductory course as given by the authors, the following sections are omitted. The portions omitted all come at the ends of chapters, and are not a prerequisite to the study of the sections which follow.

Sections which may be omitted in a first course.

Chap.	V	Sec. 93 to end
	VII	Sec. 151 to 156
	VIII	Sec. 185 to end
	IX	Sec. 210 to end
	XI	Sec. 261 to end
	XIII	Sec. 308 to end
	XV	All

The emphasis placed on the following features makes them to some extent the distinctive features of the text.

1. All interpretative discussions have been carried on in terms of the forces on electric charge. This makes it possible to correlate seemingly diverse phenomena, by showing that the customary formulas relating to these phenomena are but alternative and restricted ways of writing the more fundamental equations expressing the force on electrons. To emphasize this feature, the book has been given the title "Introductory Electrodynamics." The text begins with the forces between stationary charges, then takes up the various forces which produce or oppose the motion of charges in electric circuits, and then treats of the forces which constitute the magnetic effects in steady fields. The accelerating forces in non-steady fields are finally considered.

2. By this treatment in terms of the forces on electric charge, the magnetic pole concept has been eliminated from the theory of magnetism, except as a convenient alternative means of making calculations in a few special cases. The real forces in a magnetic field are shown to be the forces on electric charge. These forces are always normal to the lines of magnetic flux density. The fictitious character of the set of forces which is assumed to act *along* these lines on magnetic poles is emphasized.

3. The treatment is algebraic in character. That is to say, the electrical quantities, charge, current, potential increase, electromotive force, flux, etc., are defined in an algebraic sense, the algebraic sign usually relating to a direction along a line or across a surface. Those algebraic conventions which are necessary in addition to the algebraic definitions are also clearly stated and attention is repeatedly drawn to them. Non-algebraic treatments in which the definitions deal with numerical values only, and in which signs are not used clearly and consistently, can be successfully used only in the simplest cases.

4. The entire treatment of electric and magnetic theory is given in terms of a single system of units instead of in a mixture of three systems. (See Sec. 10 and 11.) This system, the rationalized Practical System, has been freed of the irrational 4π factors and of the multiplicity of troublesome conversion factors by three expedients:

First: By using the ampere-turn and the ampere-turn per cm. as the units of magnetomotive force and of magnetic intensity.

Second: By using the weber and the weber per sq. cm. as the units of magnetic flux and of flux density.

Third: By assigning to the permittivity, p_0 of free space such a value that the electrostatic force between charges is expressed by the formula

$$f(\text{dyne-sevens}) = \frac{q_1 q_2\,(\text{coulombs})}{4\pi p l^2(\text{cm.})}$$

The student must in the end familiarize himself with the three systems of units, because he will find it necessary to read articles in which the three systems appear, perhaps in a single paragraph. But he will experience much less difficulty in dealing with these articles if his own thinking is founded on a complete statement of the relations in a single factor-free system.

5. Much mental confusion is to be traced to the failure to recognize the relative parts played by observation and by definition in any system of knowledge, and the consequent failure to realize that many relations are largely matters of definition and convention. With the object of stimulating the student to distinguish between *definitions*, *experimentally determined relations*, *deduction*, and *generalizations*, one of these designations has been appended to the name of each important relation which appears in the text. The more important definitions, laws, and principles have been printed in bold face type.

We wish to acknowledge our indebtedness to L. J. Peters of the University of Wisconsin for his constructive criticisms.

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MADISON, WIS.
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INTRODUCTORY ELECTRODYNAMICS FOR ENGINEERS

CHAPTER I

QUALITATIVE VERSUS QUANTITATIVE KNOWLEDGE—UNITS

1. Purpose.—This text is an inquiry dealing with those fundamental conceptions pertaining to electrical phenomena, and with those definitions and quantitative relations which have made possible the electrical engineering achievements in the broad fields of electric power generation, transmission, conversion, and utilization, electrochemistry, electrothermics, electric lighting, telegraphy, telephony, and radio communication. It deals largely with the development and application of those **precise** definitions of quantity of electricity, electromotive force, current, electric intensity, magnetic intensity, magnetic flux, resistance, etc. which give to electric theory its quantitative character, and which have served to make applied electricity so exact an art within the short space of a century.

The aim is to carry on the inquiry in the spirit of those seeking to obtain, not simply a **qualitative** grasp, but rather a **quantitative** grasp, and if possible a **profound** understanding of the relations.

2. Types of Knowledge and Kinds of Interest.—It may be well at the outset to contrast the three types of knowledge of the relations in a given field—the qualitative, the quantitative, and the profound—and to consider the **kind** of interest attaching to the pursuit of each type.

An extreme example of a statement of the purely qualitative type is the first law of electrostatics, namely: "Bodies charged with like electricity repel each other; bodies charged with unlike

electricity attract each other." This law contains no information which is of service in the computation of the magnitude of the forces between the charged bodies. Few statements are as purely qualitative as this. More generally, the statements express relations which enable one to furnish a rough answer to the questions, How much? or How many? For certain purposes, such statements may be classed as quantitative. All too frequently, however, there is a failure to realize that for many other purposes these statements are essentially qualitative.

As an illustration of the contrast between the qualitative and the quantitative, let us consider the knowledge relating to the principles involved in the common pole-top transformer to which the residential lighting circuits are connected. The usual qualitative knowledge of these principles includes little more than that the transformer is a device which receives a small current at the high electrical pressure (electromotive force or voltage) suitable for transmitting power, and delivers a large current at the low pressure suitable for lamps and motors. The principle involved in the transformation of the voltage is that if two coils are wound upon a common iron core, and if a varying magnetic field is set up in the core by a varying current in one of the coils, the relative magnitude of the electromotive forces induced in the two coils by the varying magnetic field is approximately proportional to the ratio of the number of turns in the two coils. To transform, therefore, from the 2200 volts of the power-distributing lines to the 110 volts for the lighting circuit, the transformer should contain two coils wound upon the same iron core—one coil containing approximately twenty times as many turns as the other.

3. Quantitative Knowledge.—Contrast this with the quantitative knowledge required in designing the transformer, or in passing judgment upon the merits of two transformers. In the course of the design, after having determined the maximum power which the transformer is to be rated to deliver, and the way in which the demand upon the transformer for power will vary (upon the average) through the day and from season to season during the year, the designer must raise and answer such questions as the following:

a. Of the many possible geometrical arrangements of the copper windings in relation to the iron core, which is the best?

b. To what extent should the two coils be subdivided and interleaved?

c. How many turns of wire should be used in each winding, and what diameters should the wires have?

d. What should be the cross-sectional proportions and length of the iron core?

e. What should be the exact (not approximate, as given in the qualitative explanation) ratio of the turns in the two coils to give a specified ratio of transformation under specified conditions?

f. To what extent will this ratio of transformation vary as the demand upon the transformer for power is varied from no load to full load?

g. What will be the power loss in the transformer at different loads?

h. How hot will the different parts of the transformer get?

i. Cooling questions—for example, what oil ducts should be provided through the windings and the iron core to permit the cooling medium to circulate and carry away the heat? What cooling medium should be used?

j. Of two grades of steel, one costing 6.5 cents a pound and having a loss per cubic centimeter as shown by curve A, and the other costing 2.5 cents a pound and having a loss as shown by curve B, which should be used for the steel core?

k. Questions as to methods of insulating the windings.

These questions, or problems, are not independent questions which may be taken up one at a time and independently answered. The solution of question x hinges upon the solution of question y, and so some of the questions must be worked through a number of times, starting with tentative assumptions and using the data from the solutions so obtained to correct the assumptions for a second approximation, and so on. The answers to these questions hinge not alone upon the physical properties of the materials entering into the transformer, but upon such matters as the cost of generating the power which is to be transformed, the cost of copper, the cost of steel, the cost of the labor in the manufacture of the different designs, the estimated life of the transformer, and the interest rate upon money. A marked

change in any one of these items may mean a change in many of the answers—and a marked change in the design.

4. Qualitative versus Quantitative.—The qualitative statement of what the transformer tank contains and how it functions is, in its apparent simplicity, highly gratifying. In the absence of such a statement, the transformer tank is a veritable sealed Pandora's box—a thing which it is not in human nature to tolerate. The statement opens the box and releases its content, to bless or to vex, depending upon the spirit in which it is received and followed up.

The recital of the above quantitative questions (and they may be duplicated for almost any piece of apparatus) is an indication of the infinite complexity of nature. **The attainment of quantitative knowledge in such a situation is through a long, exacting discipline in which the acquiring of precise ideas through the painstaking consideration of many details plays a large part.** It is the necessity of submission to this discipline which makes the pursuit of quantitative knowledge irksome and distasteful to so many natures.

The general interest in qualitative knowledge lies in the fact that it is essentially **explanatory**—explanatory in the sense of accounting for the striking features of the particular complex phenomenon under immediate consideration in terms of more remote relations having a broader application. The qualitative explanation invariably relates, moreover, not to the properties of the actual device under observation, but to a simplified make-believe device. The explanation brings into play the make-believe practice of childhood and of the childhood of the race, a game which is always a delightful recreation. In the design (and design is another word for engineering) of **actual** transformers, motors, power stations, radio stations, and transmission lines, however, qualitative explanations do not carry the designer very far.

Even in those cases in which the explanation (so-called) is no more fundamental than the plain recital of the phenomenon itself, things appear to become more intelligible, or at least less puzzling, since, by the explanation, the ultimate unresolvable fact is frequently pushed farther and farther into the background.

All too often, however, the qualitative explanation is of the type occurring in one of the Hindu myths, in which the earth is said to rest upon the back of an elephant, and the elephant to stand upon the back of a tortoise, and for the footing of the tortoise—no necessity is felt!

5. Profound Knowledge.—This leads to a consideration of what is involved in a **profound knowledge** of a subject. A profound knowledge implies at least a realization that the present-day accounts[1] of nature (everyday affairs) undoubtedly contain elephants and tortoises which are as unnecessary and as stultifying to growth in clear thinking and in clear seeing as those of any ancient cult. While it may in all truth be said that no forest primeval ever presented a more formidable aspect to timid natures than does the array of questions relating to the quantitative properties of a transformer, or of any other device, it may with equal truth be said that no virgin region ever presented to our hunting and fishing ancestry a more alluring prospect of high adventure and rich reward than is presented to the pioneering spirit who, engaged in the pursuit of quantitative knowledge, has caught a glimpse of the game with which the region abounds.

6. Explanatory versus Descriptive Accounts.—Dropping the allegory, the kind of interest which is necessary to a profound knowledge is an interest not alone in the recital of physical relations, but in the way in which the mind operates upon this food for thought. The interest necessary to a profound knowledge is an interest in the sorting and classifying of ideas. As an example, one ought to recognize that all accounts of phenomena resolve into two broad classes, or into a mixture of the two, namely:

Explanatory accounts.
Descriptive accounts.

[1] In this connection it is pertinent to say that, if the proposed inquiry is to be pursued in the spirit of adventure portrayed in this paragraph, the disciple should know that the explanations and doctrines advanced in the text (or by the instructor) are frequently erroneous, and that even some of the so-called facts are not real but are figments of the imagination. Such an understanding is absolutely essential to the creation of the sense of adventure in a living, growing world.

There must be an interest in resolving any involved account into these elements. There must be a recognition that if any inquiry as to the "how" or "why" is pushed deep enough, it always ends with a purely descriptive statement—an unresolvable statement of fact for which there is (at that time) no explanation in terms of anything more elemental.[2] With this should come a realization that the intense dissatisfaction with the descriptive account and the craving for the explanatory account is a human trait to be alternately nourished and suppressed. For each—the description and the explanation—there is a proper time and place. In some situations, the craving for a tortoise, or at least an elephant, is the criterion of the pioneer; in other situations the craving is utterly unwarranted and the surrender to it is fraught with the gravest possibilities of stagnation.

7. Interest versus Citizenship.—It is not within the powers of any man to carry into very many of his fields of interest the exacting discipline necessary to quantitative knowledge, or the critical attitude necessary to profound knowledge. For the sake of recreation and of sanity, his interests in many fields must necessarily be qualitative. But having in mind the growing complexity of the social organization, with the consequent change in the constraints which is resulting from the diversion of human interests into new and unnatural lines, may we not be warranted in saying that the three kinds of interest under consideration play the following parts in good citizenship?

[2] In this connection it may be well to direct attention to the grossly misleading character of the constantly reiterated statement that, "We do not know what electricity is." The statement, "We do not know what iron is," is rarely made. Yet, what is the difference between the character of our knowledge of iron and of electricity? It is this: While our knowledge of electrical relations and phenomena is far more complete and fundamental in character than is our knowledge of iron, we do not see, or handle, or smell electricity in the sense that we do iron. Since, however, ordinary seeing, touching, and smelling constitute such a very, very small part of our knowledge of a thing, the real state of affairs would be more correctly expressed by the statements:

"Although we do not identify electricity by the senses of sight, smell, or touch, our knowledge of it is very satisfactory. On the other hand, while we can see and feel and smell iron, our knowledge of it is less complete."

Qualitative Interests.—An element in good citizenship from the viewpoint of the necessity for stabilizing recreations.

Quantitative Interest.—A necessary element in good citizenship from the economic point of view. A good workman must know "how much."[3]

Profound Interest.—A necessary element in good citizenship from the spiritual or moral side. The thing men have in common is the same mind stuff; loyalty to the critical interest in intellectual operations and achievements may well constitute one of the common ties.[4]

This text has been written from the point of view of an electrical engineer. It is, consequently, an attempt in the main to develop the quantitative relations of the subject.

8. Classification of Relations.—In the interest of clear thinking, an attempt has been made in this text to distinguish between the four types of relations dealt with, namely:

1. Relations which are matters of **definition.**
2. Relations which have been **determined** purely by observation and **experiment.**
3. Relations which have been **deduced** from fundamental experimental relations for the values of newly defined quantities.
4. Relations which are **generalizations** embodying and based upon general experience or upon extended experimental evidence. Generalizations are not always susceptible of direct proof; witness, the principle of the conservation of energy, and the second law of thermodynamics.

Notwithstanding the brevity of the time which has elapsed since the master minds formulated the principles underlying electrical measurements, the notions of the latest disciples not infrequently exhibit a mythological growth as gross and misleading as that surrounding any ancient rite. The essence of this idolatry[5] is the failure to recognize the part played by definition in so-called demonstrations, and the consequent failure to recognize that certain relations are purely matters of definition or convention. The recognition of these four types is of such importance that the type under which each of the following relations falls has been indicated by appending to the name of

[3] See Pearson, Karl: *Grammar of Science*, Chap. I.

[4] See Royce, Josiah: *The Philosophy of Loyalty.*

[5] Müller, Max: *Science of Language*, Vol. II, Chaps. IX and XIII; also Bacon: *Novum Organum*, Book I, Aphorisms 43, 59, and 60.

the relation either **Definition, Exp. Det. Rel.** (for experimentally determined relation), **Deduction,** or **Generalization.**

9. The Origin of the Systems of Electric Units.—The history of the early studies in electricity and magnetism is of interest because of the light it throws upon the three systems in use at the present time for the measurement of electric and magnetic quantities. The study of electrical phenomena begins with the discovery by Gilbert, about 1570, that many substances, such as glass, sulphur, resin, can be electrified by rubbing. This was followed by the discovery or conception by Gray, in 1729, that some substances are conductors and others are non-conductors of electricity. (These early discoveries are outlined more fully in Chap. II.)

In 1733 Du Fay first noted that there are two kinds of **electrification,** which he called **vitreous** and **resinous electrifications,** and announced the first law of electrostatics, namely: "Bodies charged with like electricity repel, and bodies charged with unlike electricity attract."

In 1747 Franklin proposed his single-fluid theory and proposed to call the two electrifications **positive** and **negative** electrifications. In 1767 Priestley inferred the first quantitative electrical law, namely, that the force between elementary charges varies inversely as the square of the distance between the charges. In 1797 Volta constructed the first chemical generator (voltaic cell).

In parallel with the growth of knowledge relating to electrical phenomena, a growth was taking place in the knowledge relating to magnetic phenomena as exhibited by lodestone, a natural magnet, and by steel magnets which had been magnetized by stroking with lodestone. The Ancients had attributed the orientation of a magnet to the influence of the polestar, but Gilbert discovered that the earth possesses the properties of a magnet and published his remarkable treatise on magnetism, *De Magnete*, in 1600. In 1785 Coulomb discovered that the force between magnetic poles varies inversely as the square of the distance between the poles.

These studies of electric phenomena and of magnetic phenomena were quite independent studies. There was no known

relation between electric and magnetic phenomena until 1820, a century ago. In that year Oersted demonstrated that a current would exert a force upon a magnetic needle. Previous to this demonstration, electrical theory bore no closer relation to magnetic theory than sound bears to light.

10. The Electromagnetic and Electrostatic Units.—Previous to Oersted's discovery both electric and magnetic studies had assumed a very quantitative aspect. The fundamental definition, from which all the units used in electrostatic studies were **derived,** was the definition of the **Unit of Electricity,** namely:

The unit of electricity is that quantity of electricity with which a very small body must be charged so that, when placed at unit distance from a similar body charged with an equal quantity, the force of repulsion between the two will be one dyne. Then, as the relation between electrostatics (electricity at rest) and electric currents (electricity in motion) became fully understood, this system of units was extended to include the theory of electric currents. Likewise, when the quantitative relations between currents and magnetic fields were determined, the system was extended to magnetism. This system of units is known as the **electrostatic system** (E.S.S.) or the **electrostatic units** (E.S.U.).

Quite independently, another system was built up in the reverse order. It started with the study of magnets and a definition of the unit magnetic pole, namely:

A pole is of unit strength, if when placed at unit distance from a similar pole, the force of repulsion between the two is 1 dyne. The system then proceeded in a logical manner to the other magnetic units. It was extended to the field of electric currents and from there to electrostatics. This system of units is known as the **electromagnetic system** (E.M.S.) or the **electromagnetic units** (E.M.U.).

Thus as a result of the independent study of the subject from different starting points, two systems of units, each system complete in itself, arose. Now if it is borne in mind that the only application of electrical theory previous to Oersted's discovery was in the protection of buildings by lightning rods and the only application of magnetic theory was in the study of the earth's

magnetic field for the purposes of navigation, it is not a matter for surprise that the units of both systems were found to be of inconvenient magnitude for measuring the currents and electromotive forces obtained from batteries and dynamos. For example, the unit of electromotive force in the electromagnetic system is so small that the voltage of the common dry cell of everyday use, if expressed in electromagnetic units, is 1.4×10^8 E.M. units. On the other hand, the unit of electricity and of current in the electrostatic system is so small that the current in the ordinary 100-watt lamp, if expressed in electrostatic units, would be approximately 3×10^9 E.S. units.

11. Practical Units.—To meet the needs of everyday practice a committee of the British Association accordingly recommended the adoption of a unit of resistance and a unit of voltage which were defined to be multiples of the corresponding electromagnetic units by certain integral powers of ten.[6] These units were called **practical units.** From these units a third system of units, known as the **practical system,** has evolved. The concrete electrical standards legalized throughout the civilized world by governmental action are intended to represent the practical units.[7]

The practical system includes the familiar units—the volt, the ampere, the ohm, the watt, the joule, the coulomb, the henry, and the farad of everyday use. It will be the only system used in this text.

The question arises as to the best order in which to develop the relations between the practical units: Shall we start with the

[6] Most unfortunately these definitions lead to such a combination of ratios between the practical and the electromagnetic units as 10^{-1}, 10^7, 10^8, and 10^9. As a result, the practical units are not simply related to either the electrostatic or the electromagnetic units.

[7] For a discussion of the practical system of units and a history of the legislative enactments, the following articles may be consulted:

WOLFF, FRANK A.: *The So-called International Units*, Bulletin of the Bureau of Standards, 1904, No. 1, Vol. 1; *The Principles Involved in the Selection and Definition of the Fundamental Electrical Units to be Proposed for International Adoption, Bulletin of the Bureau of Standards*, 1908, No. 2, Vol. V; *Announcement of a Change in the Value of the International Volt*, Circular 29 of the Bureau of Standards, December, 1910; DELLINGER, J. H., *International System of Electric and Magnetic Units*, Scientific Paper No. 292 of the Bureau of Standards, Oct. 11, 1916.

unit pole as in the electromagnetic system or with the unit charge as in the electrostatic system? The prevailing practice is to follow neither of these plans but to treat magnetic theory in terms not of the practical units, but of electromagnetic units derived from the notion of the unit magnetic pole, to treat electrostatics in terms of electrostatic units derived from the notion of the unit charge, and finally to treat electric circuits in terms of the practical units. To reduce units of the first two systems to the units of the latter system—the so-called practical system—the proper multiple must be selected from the following list:

$$10^{-7},\ 9 \times 10^{11},\ 10,\ \frac{1}{3 \times 10^{9}},\ 300,\ 10^{8},\ 10^{9},\ \frac{1}{9 \times 10^{11}}.$$

The result is that the engineer who attaches any quantitative meaning to the theorems of electrostatics and the physicist to whom the data of electrostatics have any work-a-day significance are rare.

In this development[8] of the relations between the units, we propose to avoid this deplorable confusion by starting with the definition of the **practical** unit of electricity in terms of force, and from this unit deriving all the other electric and magnetic units. We start with the unit of electricity rather than the unit magnetic pole, because the present mode of interpreting electrical phenomena is in terms of the properties of the electron, the **natural unit** of electricity. On the other hand, it turns out that the conception of the magnetic pole is not fundamental but extremely artificial. Magnetic phenomena are, in fact, accounted for in terms of the motion of electric charge.

12. Mechanical Units in the Practical System.—In the practical system of units adopted by the British Association, the unit of work or energy (termed the joule) and the unit of power (the joule per second, or watt) are 10^7 times as great as the corresponding units (namely, the erg, and the erg per second) in the C.G.S. system. Since it is highly desirable to retain the centimeter as the unit of length in the practical system, and since

[8] See, also BENNETT, EDWARD: *A Digest of the Relations between the Electrical Units and of the Laws Underlying the Units*, Bulletin 880, University of Wisconsin.

the practical unit of work, the joule, is 10^7 times as large as the erg, it follows that the **unit of force** in the practical system must be, not the dyne, but the **dyne-seven** ($= 10^7$ dynes), otherwise the equation defining work, namely

$$\text{Work} = \text{force} \times \text{distance} \tag{1}$$

would have to be written with some numerical coefficient for use with practical units. For example, if the dyne were taken as the unit of force in the practical system, the work equation would take the form,

$$\text{Work (joules)} = 10^{-7} \times \text{force} \times \text{distance (dynes, centimeters).}$$

Expressed in gravitational units, the **dyne-seven** is a force of 10.2 kilograms, or 22.5 pounds. This is not of inconvenient size for expressing the pull between bus bars, transformer windings, etc.

Another mechanical quantity which appears in a few cases in electrical calculations is mass. This quantity appears in such equations as,

$$f = Ma \qquad \text{Force} = \text{mass} \times \text{acceleration.} \tag{2}$$

$$w = \frac{Mv^2}{2} \qquad \text{Kinetic energy} = \frac{\text{mass} \times (\text{velocity})^2}{2}. \tag{3}$$

These equations hold between the units in the C.G.S. system and, in order to have them hold in the practical system, it is evident that the unit of mass in the practical system must be 10^7 times the C.G.S. unit. This practical unit, the **gram-seven** ($= 10^7$ grams), is approximately 22,000 pounds, or 11 tons, and is, therefore, not of convenient size. It is so seldom, however, that the mass of a charged body, or of a conductor carrying a current, enters into everyday electrical engineering calculations that the inconvenient size of this unit is not a serious matter. Since the three fundamental units in the practical system are the centimeter, the gram-seven, and the second, a convenient abbreviated designation for this system is **C.G.S.S. system** (read C.G. double S system).

The mechanical units in the practical (C.G.S.S.) and in the C.G.S. systems have been listed in the table below.

Mechanical Units in the Practical and in the C.G.S. Systems

Quantity	C.G.S. unit	Practical or C.G.S.S. unit
Length.	Centimeter	Centimeter
Time.	Second	Second
Energy or work.	Erg	Joule or "erg seven"
Power.	Erg per second	Watt or joule per second
Force.	Dyne	Dyne-seven = 10^7 dynes
Mass.	Gram	Gram-seven = 10^7 grams
Velocity.	Centimeter per second	Centimeter per second
Acceleration.	Cm. per sec. per sec.	Cm. per sec. per sec.

13. Nomenclature and Symbols.—Names have been coined for the units in the practical system only.[9] The same names may be applied to great advantage to the corresponding electrostatic and electromagnetic units, provided the units in the latter two systems be designated by attaching suitable prefixes to the names of the practical units. A good practice is to designate the E.S. and E.M. units by prefixing E.S. and E.M. to the names of the practical units.[10] Thus, the unit of electric charge may be designated in the three systems as the coulomb, the E.S. coulomb, and the E.M. coulomb.

For the purpose of having a summary to which reference can be made from time to time as the various electrical units are introduced and defined, the table in Appendix 1 has been compiled. This table lists the names of the practical units in the sequence in which the quantities are introduced and defined in the text, the defining equations, and the symbols which have been adopted by the standardization committee of the American Institute of Electrical Engineers. The relative magnitudes of the corresponding units in the three systems are given in the last three columns of the table.

[9] There are four exceptions to this statement. *Gauss*, *oersted*, *gilbert*, and *maxwell* have been assigned as names to four of the electromagnetic units.

[10] Another practice is to designate the E.S. and E.M. unit by prefixing *stat* and *ab*, respectively, to the names of the practical units. Thus, the coulomb is designated in the three systems as the *coulomb*, the *statcoulomb*, and the *abcoulomb*.

CHAPTER II

FUNDAMENTAL ELECTROSTATIC EXPERIMENTS AND THEIR INTERPRETATION

PART I—QUALITATIVE OBSERVATIONS

14. Electrification by Rubbing Contact. Experiment 1.—Let a dry, glass rod or tube be grasped at one end and let the other end be rubbed with a dry cloth of silk or linen. When the rubbed end of the rod approaches light bodies, such as pieces of straw, pith, or tissue paper, it is found that the bodies are attracted to the rubbed end of the rod, that they cling to it for a short interval of time, and are then repelled from it. If the tissue paper and the pith are slightly moistened, they are instantly repelled after coming in contact with the rod. The study of the forces between the rod and a light body may be facilitated by suspending the body (for example, a small pith ball) by a silk thread from a suitable support, as shown in Fig. 1. It is evident that, by reason of the rubbing, the rod has acquired the property of exerting unusual forces upon other bodies.

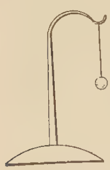

FIG. 1.—Pith ball electroscope.

These observations constitute the historical introduction of the race to electrical phenomena.[1] From at least 600 B.C.,

[1] For the history of the early electrical discoveries and theories, the following references may be consulted:

BENJAMIN, PARK: *The Intellectual Rise in Electricity from Antiquity to the Days of Benjamin Franklin*, 1898 (an entertaining philosophical account).

MOTTELAY, PAUL F.: *Chronological History of Electricity, Galvanism, Magnetism, and the Telegraph from B.C. 2538 to A.D. 1888*, Electrical World, 1891–1892 (abstracts are brief but extremely complete).

PRIESTLEY, JOSEPH: *The History and Present State of Electricity, with Original Experiments*, 1775.

WHITTAKER, E. T.: *A History of the Theories of Aether and Electricity from Descartes to the Close of the Nineteenth Century*, 1910 (a critical exposition).

MOTTELAY, PAUL F.: *Bibliographical History of Electricity and Magnetism*, 1922.

when Thales, a Greek philosopher of Miletus, recorded the fact, it has been known that amber (*elektron* in the Greek tongue) when rubbed acquires the property of attracting light bodies. For over 2000 years, this observation remained an isolated unfruitful fact. The advance in physical science during this period was extremely slow because speculation as to causes and relations was not based upon carefully collected and classified data obtained by observation and experiment, but largely upon assumptions and premises having a remote relation to reality.

About the time of, or shortly after, the discovery of the New World, a marked change was to be observed. Observation and experiment were "in the air." Bacon's *Novum Organum*, which appeared in 1620, was perhaps a reflection of the spirit of the times, rather than a cause of the change. Among the first of the new school of keen observers and experimenters was William Gilbert (1540 to 1603), a contemporary of Shakespeare and Bacon, and physician to Queen Elizabeth. About 1570 Gilbert found that many bodies other than amber and jet, such as glass, sulphur, and the resins, when rubbed acquired the amber-like (elektron-like) property of attracting light bodies. To describe his observations he coined the terms **electrify** and **electrification**: a body which exhibited the amber-like property of attracting light bodies was said to be **electrified** (amberized).

Thus the original or primary meaning of electricity and of electrification is to be given in terms of force as in the following definition.

14a. ELECTRICITY (DEFINITION).—**A body which has acquired the property of attracting and then repelling light bodies is said to be ELECTRIFIED, or to be CHARGED WITH ELECTRICITY.**

14b. THE ELECTRIC FIELD (DEFINITION).—**The region surrounding a charged body, or, in general, any region in which a charged body is subject to a mechanical force by reason of its charge is called a FIELD OF ELECTRIC FORCE or an ELECTRIC FIELD.**

It is worthy of note that about 70 years elapsed after this advance before any observer was critical or analytical enough to call attention to the fact that forces of repulsion as well as attraction may be observed between electrified bodies. This observation was first recorded by von Guericke of Magdeburg.

15. Electrification by Conduction. Conductors and Insulators. Gilbert tried to electrify a wide variety of substances by rubbing them, and found that he could not electrify any of the metals and many other substances. He accordingly divided all substances into two classes: **electrics,** substances which could be electrified by rubbing; and **non-electrics,** substances which could not be so electrified. For a century and a half this classification was a valid, useful, unquestioned classification. In 1729 Stephen Gray, a fellow of the Royal Society, made the discovery or advanced the conception that some substances are **conductors** and others are **non-conductors** of electricity. Gilbert failed to electrify the metals, or the non-electrics, by rubbing because he failed to insulate them, and any charge imparted to them by rubbing was conducted away. These same non-electrics if mounted upon a glass stand or handle could readily be electrified by rubbing. It became evident that Gilbert's classification into electrics and non-electrics was a fortuitous, rather than a basic, classification and that a more fundamental classification of substances is the division into the two groups, **conductors** and **non-conductors,** or **insulators,** of electricity.

Experiment 2.—The following observations bring out the striking differences between conductors and non-conductors of electricity.

a. Let a rod of glass be grasped near the middle and electrified at one end by rubbing. Upon approaching first one end and then the other to the suspended pith ball of the electroscope of Fig. 1, it is found that the rubbed end of the rod is electrified and the other end is not.

b. Let a long rod of metal be provided near its middle with an insulating handle in the form of a rod of glass or hard rubber. Let the metal rod be held by the insulating handle and let it be rubbed or stroked at one end with a piece of fur or flannel cloth. Upon advancing the metal rod to the pith ball, it is found that both ends and all parts of the surface of the rod are electrified. From these observations we conclude that the electric charge imparted to the glass rod remains on the rubbed portions, or spreads from these surfaces very slowly, while the charge imparted to the metal rod **flows** almost instantaneously and

distributes itself over the entire rod. These conclusions are supported by the following observations.

c. Let the electrified metal rod be touched momentarily at one point by a second metal rod which is held at one end by the observer, or let the observer momentarily touch the electrified metal rod with his finger. In either case, the electrified rod becomes completely **discharged**; that is, it gives no evidence of electrification when tested with the pith-ball electroscope. Now let the metal rod be again electrified and let it be touched with a glass rod held by the observer. Tests with the pith-ball electroscope before and after touching show that the state of electrification has not been appreciably affected by touching it with the glass.

d. Following one of Gray's experiments, let a wire of any length (Gray used several hundred feet) be suspended by silk threads. Let an insulated, unelectrified metal rod be momentarily held in contact with one end of the wire while an insulated electrified metal rod is simultaneously touched to the other end of the wire. Tests with the electroscope show that the previously unelectrified rod has acquired an electric charge and that the charge on the electrified rod has diminished. That is to say, the state of electrification has been transferred in part from the electrified rod to the previously unelectrified rod by means of the wire. The unelectrified rod is said to have become **electrified by conduction** through the wire.

These observations and many others led to the conception of electricity as a **fluid** which moves, or spreads or flows, or is **conducted** readily and rapidly from point to point of the metals, but very, very slowly from point to point of materials like glass, silk, and hard rubber. They lead to the division of substances into the two classes, **conductors** and **non-conductors** or **insulators**, defined as follows:

15a. CONDUCTORS AND INSULATORS (Definitions).—**Materials through which the electric charge from a charged body will readily pass to other bodies are called CONDUCTORS. Those materials, which, when brought into contact with a charged body, allow the charge to pass off only at an EXTREMELY SLOW RATE, are called INSULATORS, or NON-CONDUCTORS of electricity.**

It must not be supposed that there is a definite line of division between conductors and insulators. There are no perfect con-

ductors, that is, all substances impede the flow of electricity and there are no perfect insulators, that is, all substances allow electricity to pass, although in vastly different degrees. The metals are all good conductors, and substances like glass, rubber, mica, dry paper, air, and sulphur are good insulators. Frequently a material in itself may be a good insulator but its surface may undergo chemical changes from the action of the elements, or the surface may condense moisture from the air and thus become covered with a film of low insulating value.

16. Two Kinds of Electricity. Experiment 3.—(*a*) Let one end of a rod of resin (sealing wax or hard rubber) be electrified by rubbing it with fur or flannel, and let the rod be so suspended in a stirrup at the end of a string that it is free to rotate in a horizontal plane, as in Fig. 2. If another rod of the same material be electrified in the same manner and approached to the suspended rod, it is found that the suspended rod is repelled. (*b*) If, however, a glass rod be electrified by rubbing with silk and brought near the suspended resin rod, the resin rod is attracted. (*c*) If the suspended resin rod is now replaced by the electrified glass rod, it is found that the suspended glass rod is repelled by another glass rod similarly electrified and is attracted by the electrified resin rod. To summarize:

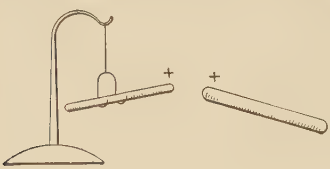

Fig. 2.

1. The similarly electrified glass rods repel.
2. The similarly electrified resin rods repel.
3. The electrified resin attracts the electrified glass.

Observations similar to the above were first made about 1733 by Du Fay, a French scientist. From similar observations of the forces between many different substances each electrified by rubbing with many substances, Du Fay concluded that there are two, and only two, different kinds of electricity. Du Fay named the kind of electricity acquired by the glass rod **vitreous** electricity, and the kind acquired by the resin, **resinous** electricity. He also formulated the first law of electrostatics as stated below. Thirteen years later (1747), Benjamin Franklin in his single-fluid theory proposed to say that a **vitreously** electrified

body is **positively** charged or electrified, and that a **resinously** electrified body is **negatively** electrified. He proposed to assign the algebraic signs + and − to the two kinds of electrification. These designations are in universal use. These experiments led to the formation of the following definitions and conclusions:

16a. LIKE AND UNLIKE ELECTRICITIES (DEFINITION).—Two electrified bodies are said to be charged with LIKE ELECTRICITY if they act in like manner toward a third charged body; that is, if both repel or both attract a small charged test body. They are said to be charged with UNLIKE ELECTRICITIES if one repels and the other attracts the charged test body.

16b. POSITIVE AND NEGATIVE ELECTRICITIES (DEFINITION).—Unlike charges tend to neutralize each other's effects; therefore the two kinds of electricity are named positive and negative electricities, or positive and negative charges. A charge like that acquired by a glass rod which has been rubbed with a silk cloth is arbitrarily called a POSITIVE charge, and a charge like that acquired by the silk is called a NEGATIVE charge.

16c. TWO KINDS OF ELECTRICITY ONLY (EXP. DET. REL.).—Under a classification based upon the nature of the forces (whether attractive or repulsive) exhibited between charged bodies, there are two and only two kinds of electricity.

16d. FIRST LAW OF ELECTROSTATICS (EXP. DET. REL.).—Bodies charged with like electricity repel each other; bodies charged with unlike electricity attract each other.

17. Unlike Charges Are Simultaneously Developed. Electromotive Series. Experiment 4.—By using the silk in the form of a pad at the end of an insulating handle, it may readily be shown by tests of the kind above described that, when a glass rod is rubbed by the silk and becomes positively electrified, the silk becomes negatively electrified. In the same way when the rubber rod is rubbed with fur and becomes negatively electrified, the fur becomes positively electrified. These experiments lead to the following conclusions:

Two bodies that have been electrified by rubbing them together are always oppositely charged.

By rubbing, positive and negative electricities are always developed simultaneously.

A given substance when rubbed with different substances is not always electrified with the same kind of electricity. The

sign of its electrification depends not only upon the rubber used but also upon the condition of its surface. From the results of experiments the following series has been so arranged that if any two of the substances be rubbed together, the one nearer the head is usually positively, and the other negatively, charged.

17a. Electromotive Series.

Substances near the head of the series are electropositive to those below.

1. Lime glass
2. Cat's fur
3. Quartz
4. Mica
5. Lead glass
6. Wool or flannel
7. Cotton
8. Paper
9. Silk
10. Wood
11. Aluminum
12. Zinc
13. Iron
14. Copper
15. Carbon
16. Hard rubber
17. Sealing wax
18. Amber
19. Sulphur
20. Amalgamated leather

18. Electrification by Electrostatic Induction. Experiment 5. (*a*) Let a body A (Fig. 3) be electrified, say positively, and brought near one end of an elongated insulated conductor con-

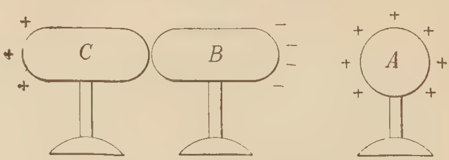

FIG. 3.—Electrification by induction.

sisting of two separable parts B and C which are in contact. The two extremities of the elongated conductor become electrified, as shown by the forces exerted on a small pith ball which is approached to the ends. By bringing the pith ball in contact with A and thus charging it positively we may determine the sign of the charges at the two ends of BC. It is found that the end near A attracts the positively charged pith ball and hence is negatively electrified, while the far end repels the ball and hence is positively electrified. If the charged body A is now moved to a distance all signs of electrification on BC immediately disappear, but immediately reappear when A is brought back again. (*b*) While A is near one end of the elongated conductor let its two parts B and C be pulled apart. Now upon removing

A to a distance, it is found that the near half of *BC* remains negatively charged and the far half positively charged. (*c*) Finally, with the two parts *B* and *C* again in contact, place *A* near one end and momentarily touch the elongated conductor with the finger. The positive charge upon the far end immediately disappears, being conducted through the body to the walls of the room. Upon again removing *A* to a distance, the conductor *BC* is found to be negatively charged. Thus in parts (*b*) and (*c*) of this experiment, conductors *B* and *C* have been electrified by a new process involving neither rubbing nor conduction from a previously electrified body, since by these operations the charge on *A* is not diminished. The conductor *BC* is said to have become charged by **electrostatic induction** or influence.

We are now in a position more fully to describe what occurs when the electrified body of the first experiment attracts the light bodies. These light bodies are conductors of electricity—though very poor conductors indeed. Upon the approach of the electrified body, the nearer portions of the light bodies become oppositely electrified and the more remote portions become electrified with the same sign. The attractive force exerted by the electrified body on the nearby electricity of opposite kind is evidently greater than the repulsive force exerted on the more remote charge of like kind, and the net or resultant force on the light body is a force of attraction.

From these experiments the conclusion is drawn that the forces are directly between the electricities, and that the charged bodies experience the forces only because the charges are in some manner attached to the bodies by **surface forces** which prevent the electricity from leaving the bodies and coming together through the intervening space. This leads to the following restatement of the first law of electrostatics.

18a. FIRST LAW OF ELECTROSTATICS RESTATED (Exp. Det. Rel.).—**LIKE electricities repel each other; UNLIKE electricities attract each other.**

19. Gold-leaf Electroscope for Detecting Charges (invented 1787).—An instrument by which the presence of a charge on a body may be detected and its sign determined is called an **electroscope.** Gilbert devised an electroscope consisting of a straw

balanced at its center on a needle point, after the manner of a compass.

The charged pith-ball electroscope of Fig. 1 is not suitable for detecting weak electrifications. A far more sentitive and useful instrument for detecting and studying weak electrifications is the gold-leaf electroscope devised by Bennett in 1787. As illustrated in Fig. 4, it consists of a long, narrow strip of gold leaf G, attached at its upper end to a vertical metal strip S, against which the gold leaf hangs, face to face. The metal strip S is mounted on suitable insulating members in a glass or partly glass case which serves to shield the gold leaf from air currents. The conducting strip S terminates at its upper end in a metal plate or table T.

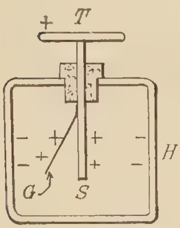

Fig. 4.—Gold-leaf electroscope.

If the table of the electroscope be charged by contact with a charged body, the charge spreads to the leaf G and the strip S, and the leaf is repelled and stands out from the strip.

If now a body with a like charge is gradually moved toward the table, the gold leaf will be seen to rise still higher, since by induction the strip and the gold leaf become more strongly electrified. On the other hand, if a body with an unlike charge or if a large uncharged conducting body is approached toward the table, the gold leaf will be seen to fall.

The gold leaf is so delicate that it is frequently torn away by the large charge which is sometimes imparted in charging by contact. A safer way to charge the electroscope, say with positive electricity, is to charge it by induction in the following manner: Hold a negatively charged body some distance from the table and momentarily touch the table with the finger. A negative charge passes off from the table through the body to ground and the electroscope is left positively charged. The negatively charged body may then be removed from the neighborhood, and the electroscope is left with a positive charge ready for use. If the charge is not large enough, the body may be held nearer to the table and the operations repeated.

20. Electrification by Contact. The Volta Effect (1794). Experiment 6.—When two dissimilar conductors are brought

into intimate contact and afterwards separated, they are, in general, found to be oppositely electrified. For instance, let plates of copper and of zinc having polished plane faces be provided with insulating handles of hard rubber, as illustrated in Fig. 5. Let the two plates be placed in contact, face to face, as illustrated. By means of the hard rubber handles, let the two plates be pulled apart, exercising great care to keep the two surfaces parallel at the moment they separate and start to draw apart. By separately advancing each plate toward the table of a gold-leaf electroscope charged with electricity of known sign, it may be demonstrated that, after contact and separation, the copper plate is always charged with negative and the zinc plate with positive electricity.[2]

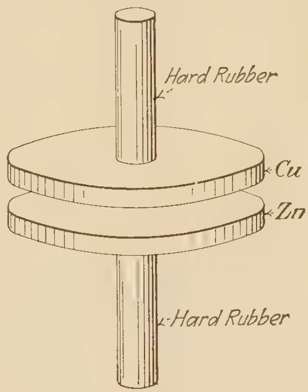

Fig. 5.—Electrification by contact.

The acquirement of opposite charges by (dry) dissimilar metals while in contact is called the **Volta effect.** It was discovered about 1794 by Volta, professor of physics at Pavia, during the course of his experiments to explain Galvani's discovery that muscular spasms occur when a nerve and a muscle of a partly dissected frog are simultaneously touched by two dissimilar metals which are in mutual contact at some point.

After carrying out tests of this kind with different conducting materials, Volta compiled a series in which the materials are so arranged that if a material nearer the beginning of the series is placed in contact with a material nearer the end, the former is found to become positively electrified, and the latter, negatively.

[2] If the charge on the plates is too small to show a pronounced effect on the electroscope, the effect may be multiplied as follows: Let a deep metal can, open at the top, as in the ice-pail experiment of the next section, be placed on the table of the electroscope. Let one of the plates, say the zinc, be earthed by holding it in the hand. Let the copper be placed on the zinc, pulled away by its insulating handle, inserted well into the can, discharged to the can by momentarily touching it to the side or bottom, withdrawn from the can and again placed on the zinc plate. This cycle may be rapidly repeated until a pronounced deflection of the electroscope is obtained.

The voltaic series is—aluminum, zinc, lead, tin, iron, copper, silver, gold, carbon. Materials nearer the beginning of the list are said to be electropositive to those nearer the end.

From these experiments, it appears that, in general, if any two dissimilar materials are placed in contact, a redistribution of their electricities tends to take place. The separation of charges which is brought about by rubbing a glass rod with silk is probably due to the same inherent properties of matter and electricity which lead to this separation in the case of the metals. The difference between electrification by rubbing in the case of insulators and electrification by contact in the case of conductors is possibly this. In both cases, we may assume at the points of contact phenomena of a nature (or forces of a nature) which tend to bring about a redistribution of the electricity of the bodies, the electricity of one body tending to flow to the other across the few points of contact. In the case of the conducting materials, a flow from the points of contact to other parts of the bodies can readily occur, but in the case of the insulators, only the areas in the immediate vicinity of the points of contact become charged. By the operation of rubbing, fresh parts of the insulators are brought into contact and innumerable small areas on the insulators become charged.

PART II—QUANTITATIVE OBSERVATIONS

21. Faraday's Ice-pail Experiment. Experiment 7.—The following experiment is generally known as the "ice-pail" experiment, because it was performed by Faraday (1843), with a pewter ice pail.[3] Let a deep metal vessel P (Fig. 6), with a narrow opening, be mounted upon an insulating stand S and be connected by a wire with an uncharged gold-leaf electroscope. Let a metal ball B be suspended from a silk thread and let it be charged with electricity. Upon lowering the ball into the vessel the following observations may be made:

a. On lowering the ball into P without touching it to P, the outside of the vessel becomes electrified with a charge of the same sign as the ball.

[3] Phil. Mag., 1843.
Experimental Researches, Vol. II, p. 279

b. The ball always causes the same divergence of the gold leaf no matter what its position in the vessel, provided only that it is well below the opening.

c. On removing the ball without contact with P all sign of the electrification of the vessel disappears.

d. Contact of the ball with the inner surface of the vessel produces no greater divergence of the gold leaf than that observed in (*b*).

e. Upon touching the ball to the inner surface of the vessel and then removing it, the ball is found to be completely discharged, and the removal of the ball is without influence upon the divergence of the gold leaf.

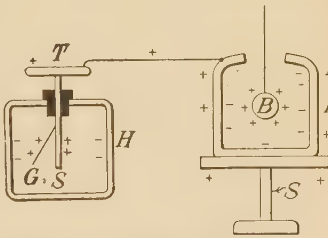

Fig. 6.—Hollow vessel experiment.

f. If, with the charged ball suspended in the vessel, the outside of the vessel is momentarily touched with the finger, the gold leaves collapse and all signs of electrification outside the vessel disappear. If the ball is now removed, the gold leaves diverge to the same position as before, but the vessel is found to have a charge opposite to that of the ball.

g. If the ball is now lowered and touched to the vessel, all signs of electrification disappear and the ball upon removal is found to be completely discharged.

h. If, when the charged ball B is suspended within the vessel P, an unelectrified body C of any shape and material is lowered by a silk thread into the vessel (well below the opening), the divergence of the gold leaf remains the same as with B alone in the vessel. The body C may be moved about in the vessel and even brought into contact with B without affecting the divergence. If the bodies B and C are removed after contact, the charge is found to have divided between them.

i. Let any two materials, such as fur and rubber, be mounted on long insulating handles and rubbed together inside the vessel P. If both materials are left in the vessel or if both are removed, the gold leaf does not rise. But if one material is removed and the other left, there is a deflection of the gold leaf; the deflection is of the same magnitude, no matter which of the

two bodies is left in the vessel, but tests indicate that the electrifications of the bodies are opposite in sign.

While, historically, Faraday's ice-pail experiment is not the first quantitative electrical experiment, it is, because of the astounding simplicity of the above observations and of the conclusions to be drawn from them, a most satisfactory experiment from which to develop the quantitative theory. These conclusions will be developed in the following sections.

22. Equal Quantities of Electricity.—Observations *b*, *d*, *e*, and *h* show that a given charge if placed within a given hollow conducting vessel produces identically the same outside electrification of the vessel, as measured by the divergence of the leaves of a given electroscope which is connected in a given manner with the vessel, no matter where the charge is placed in the cavity, and no matter what the size, shape, or material of the body on which the charge resides. Thus, the electrification of a hollow body which results from carrying a charged body within it, is an easily observed, consistently reproduced effect of electric charge which is not influenced by such accidental matters as the material, size, shape, or position of the body on which the charge resides. We conclude that this effect furnishes a measure of an important property of electric charge, and we use it to define what we mean by **quantity** of electricity. We first frame the following definition of **equal quantities** of electricity:

22a. EQUAL QUANTITIES OF ELECTRICITY (DEFINITION).—Two bodies are said to be charged with equal quantities of electricity of like sign if they produce the same effect on the external electrification of an insulated, hollow, conducting vessel, as a hollow metallic sphere, when they are separately and successively placed within the vessel. Two bodies are said to be charged with equal quantities of opposite sign if they produce no effect on the electrification of the vessel when they are simultaneously placed within the vessel.

23. Multiples of the Unit Quantity of Electricity.—A single charge which alone produces the same electrification of the hollow vessel as that produced by placing two unit charges in the cavity is, by an extension of the above definition, said to be a charge of two units. And so on, for three units, etc. The calibration of an electroscope may be based upon this definition, and its scale

may be graduated in terms of whatever charge may be selected as the **unit charge.** In this way a method of measuring the magnitude of charges may be imagined. To carry out this calibration one must be able to produce any number of charges, each equal in magnitude to the unit. Now observations f and g of the ice-pail experiment indicate a method of readily obtaining any number of charges which are **precisely** equal in magnitude, though opposite in sign, to a given charge. For if a charged body B is inserted within an almost completely closed hollow vessel P without touching B to P, and if P is momentarily connected to earth, then, upon withdrawing B and its charge, P is found to be left with a charge precisely equal and opposite in sign to the charge on B. The charge on P can then be completely transferred to another hollow vessel D, and the process repeated indefinitely. In this manner, any number of units of electricity may be transferred from the earth to the vessel D.

24. Principle of the Conservation of Electricity.—Now that we have defined what we mean by equal quantities of electricity on different bodies, we may deduce from the ice-pail experiment a most fundamental quantitative relation governing the production and disappearance of electricity. From observation i, we conclude that when any two bodies are electrified by **rubbing** or by **contact** the two bodies are electrified with **equal quantities** of electricity of opposite kind. That is, the two kinds have been developed in equal amounts. From observation h, we conclude that in the electrification of C by **induction,** which electrification must vary as the distance between C and B varies, equal quantities of both kinds of electricity must appear, or disappear; also in the electrification of C by **conduction,** when B and C are brought into contact, the quantity of electricity is not altered. Again, if two conductors are charged with equal quantities, but of unlike sign, and the conductors are brought into contact, both charges disappear completely.

All the above observations lead us to conclude that any process of electrification, whether by rubbing, contact, induction, or conduction, always results in the appearance or the disappearance of equal quantities of the two kinds of electricity. Subsequent experience will lead us to conclude that this is also true of the proc-

esses occurring in electric batteries and electromagnetic machines. This leads us to think of the processes as involving not the **creation** of the two electricities, but either the **separation** or the **recombination** (or, in electrokinetics, the **differential circulation**) of the two electricities. We may think of the two electricities as always existent, but without exhibiting any electrical effects when combined in equal quantities. The result of all experience is consistent with the following generalization which is often called the **principle of the conservation of electricity.**

24a. PRINCIPLE OF THE CONSERVATION OF ELECTRICITY (GENERALIZATION).—**Positive and negative electricities always exist in equal quantities. That is, when a separation of charges is brought about by any means, equal quantities of positive and negative electricity always appear. When electrical charges combine, equal quantities of positive and negative electricity disappear.**

25. The Electric Charge on a Charged Conductor Resides on the Outside Surface of the Conductor.—Observation *e* of the ice-pail experiment shows that when a charged conducting body, such as a ball, is touched to the inside surface of a hollow conducting vessel it becomes completely discharged and the hollow vessel is left with a charge. Now, while the charged ball is in contact with the inside surface of the hollow vessel, it may be regarded as a part of the inside surface of the vessel. From this, the conclusion may be drawn that the inside surface of a hollow charged conductor remains uncharged (unless the cavity contains insulated charged bodies).

Faraday confirmed this conclusion by constructing a cubical chamber with 12-foot edges which was covered with tin foil to make the surface conducting. The chamber was insulated from the floor of the laboratory and was charged by a powerful electrostatic generator until "large sparks or brushes were darting off from every part of the outer surface."[4] Meanwhile Faraday, himself, within the cube, was unable, with the most sensitive electroscopes, to obtain the slightest evidence of the intense state of electrification outside the cube. Further experiments on the electrification of solid and of thin-walled bodies of the same shape indicate that a body, though of insulating material, if provided

[4] *Experimental Researches,* Vol. I, Sec. 1173.

with a thin conducting surface film, is the electrical equivalent (to charges at rest) of a solid conductor of the same outside dimensions. From all these experiments, the following conclusion may be drawn:

25a. LOCATION OF THE CHARGE ON CONDUCTORS (EXP. DET. REL.).—**The electric charge on a conducting body resides only on the outside surface of the body, unless, being hollow, the conductor contains insulated charged bodies.**

26. The Screening Effect of a Closed Surface of Conducting Material.—Observations *b*, *d*, and *h* show that charged and uncharged bodies when once within a closed hollow conductor may be shifted to any position and may have the charge divided between them in any manner without altering the electric field outside the conductor. Faraday's experiments in the tin-foil-covered chamber confirm this and show that the inside is screened from the influence of charges outside the chamber. In other words, a closed tin-foil surface, or a closed conducting film, serves, as far as the electrical effects from charges at rest are concerned, to divide the universe into two independent parts—the part within the surface, and the part without. (Caution: It will subsequently be found that the closed conducting surface does not serve to screen the space inside the surface from the electromagnetic effects produced by **moving** charges.) From these experiments the following conclusion may be drawn:

26a. THE SCREENING EFFECT OF A CLOSED CONDUCTING SURFACE (EXP. DET. REL.).—**A sheet or a film of conducting material which forms a closed surface completely screens the space within the surface from the influence of stationary charges (stationary with reference to the surface) outside the surface; and, vice versa, it screens the space outside from the influence of charges enclosed by the surface.**

A metallic screen of mosquito netting or even of wire netting having mesh 1 or 2 centimeters square forms an effective screen, provided the instruments to be screened and the charges setting up the field are not closer to the wire mesh than ten times the width of the opening in the mesh.

27. Distribution of the Charge over the Surfaces of Conductors.—From observations *f*, *g*, *d*, and *e* of the ice-pail experiment the following conclusions may be drawn:

1. When an insulated, charged body is carried within an uncharged, insulated, hollow, conducting vessel, it induces an **equal** charge of the opposite sign on the inside surface of the vessel, and causes an **equal** charge of like sign to appear on the outside surfaces of the vessel and of the conductors connected thereto. This is illustrated in Fig. 6.

2. The distribution of the like charge over the outside surface is independent of the contour of the cavity and of the location of the enclosed charge and of the distribution of the charge on the inside walls. That is, the **distribution** of the charge over the outside surface is entirely independent of conditions under the outer surface; it is dependent, in the manner discussed below, upon the conditions outside the surface. Other experiments show that the distribution of the charge over the inside surface is independent of the conditions outside of the inside surface.

Some conception of the manner in which the surface density of charge (quantity of electricity per unit area of surface) varies from point to point on the surface of a charged conductor of irregular shape may be had by testing with a **proof plane** and gold-leaf electroscope. The proof plane is a small, flexible, metallic disk mounted at the end of a long, thin, insulating rod. The disk may be pressed into contact with the surface at any point, whereby it becomes the surface at that point. It may then be removed by the insulating rod, and the quantity of electricity it carries may be measured by the gold-leaf electroscope. Such tests show that the surface density of the charge on the surface shown in Fig. 7 varies in the manner illustrated by the varying distances from the surface to the dotted curve. That is, the surface density is greatest at the most pointed portions, less at the flatter portions, and least at the **re-entrant portions**. Or we may draw the following conclusion:

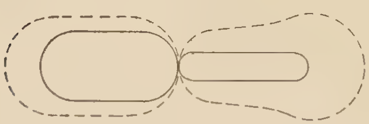

FIG. 7.—Variation of surface density of charge.

27a. VARIATION OF SURFACE DENSITY OF CHARGE (EXP. DET. REL.).—**The surface density of the charge on a charged conductor is greater over those portions of the surface having the smaller radius of curvature, provided these portions are not re-entrant portions of the surface.**

28. Discharge from Points.—Let a needle be mounted (point out) on a charged conductor, and let another insulated but uncharged conductor be brought toward the point of the needle; or let the needle be mounted upon the insulated conductor and let the point be brought toward the charged conductor. In either case a wind from the point of the needle can be detected and upon removing one conductor from the vicinity of the other we find that the charged conductor has lost part of its charge, and the previously uncharged conductor has acquired either part or all of the lost charge. We conclude that the great surface density of charge near the point of the needle must (in a manner to be explained later) cause the air in the vicinity of the point to become conducting, so that electricity can pass from the needle **through the air** to neighboring bodies. This property of points is utilized in **electrostatic generators** to collect electricity from the moving charged carriers without making metallic contact with them. It is possible that the points in which lightning rods are usually terminated may, because of this property, very slightly increase the protective power of the lightning rod.

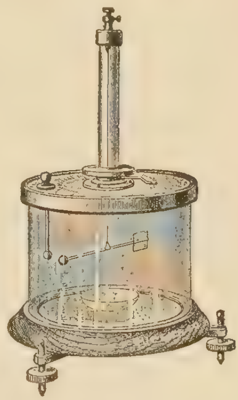

Fig. 8.—Coulomb's torsion balance.

29. The Inverse Square Law of Electrostatic Force.—By measuring with a torsion balance (Fig. 8) the force between small, charged bodies, Coulomb in 1785 arrived at a very simple law stating how the force between two charged bodies depends upon the magnitude of the charges and the distance between the bodies. Coulomb found: first, that the force varies as the product of the charges; and, second, that the force varies inversely as the square of the distance between the centers of the bodies, provided the distance between the bodies is large in comparison with their diameters. In 1837, Faraday discovered that the magnitude of the force between two bodies electrified with given charges depends also upon the insulating medium or **dielectric** in which the bodies are immersed. These results may be

stated in the following form, which is commonly called **Coulombs' Law of Electrostatic Force.**[5]

29a. INVERSE SQUARE LAW OF ELECTROSTATIC FORCE (Exp. Det. Rel. 1785).—**The force of repulsion between two charged bodies in an extended homogeneous medium, far apart as compared with their dimensions, is directed along the line connecting the centers of the bodies, and is directly proportional to the product of their charges and inversely proportional to the square of the distance r between their centers. The magnitude of the force also depends upon the dielectric (insulating medium) in which the bodies are immersed.**

When expressed as an equation, Coulomb's law is generally written in the form:

$$f \text{ (repulsion)} = \frac{Q_1 Q_2}{kr^2}. \tag{4}$$

The constant k, which has been written in the denominator of the expression for the force between charged bodies, is called the **dielectric constant** of the medium, or substance, in which the bodies are immersed. Its value for a given medium will depend upon the units in which force, length, and quantity of electricity are expressed. On the other hand, if the units of length and force alone are specified, any numerical value may be arbitrarily assigned to the dielectric constant of some specified medium.

30. The Principle Governing the Addition of the Forces of Electric Charges.—Coulomb's law relates to the force between charges concentrated upon two small bodies. It does not follow purely as a matter of logical necessity that, if two charged bodies A and B are acting simultaneously upon a third charged test body C, the force exerted by each body A and B upon C when acting simultaneously will be the same as if each were acting alone.

The justification for the following **generalization** is that no experiment has ever refuted the results of calculations based upon it.

[5] Priestley, in 1767, and Cavendish, in 1771, inferred or deduced the inverse square law from the fact that there is no electric force inside a hollow, charged sphere, and that the inverse square law is the only law which will give this result (see Sec. 82). Cavendish also noted the fact that the force depends upon the medium, but this observation remained unknown until Maxwell edited the Cavendish manuscripts for publication in 1879.

30a. THE PRINCIPLE OF SUPERPOSITION OF LINEAR EFFECTS AS APPLIED TO ELECTROSTATIC FORCES (GENERALIZATION).—The force exerted upon a charge at point P, which is at a distance from a number of concentrated charges, or from a distributed charge, is found to coincide with the calculated force obtained by determining the force at P which each concentrated charge, or elementary portion of the distributed charge, would give rise to if it, alone, were in the field, and then, by the polygon of forces, calculating the resultant of all these forces.

In nature, the **effects** of simultaneously acting **causes** (so called) are not always additive in this simple manner. Two or more sets of so-called causes and effects are mutually independent, or mutually non-interacting, or mutually non-interfering, and thus directly additive in the above manner, **only** in those cases in which the relation between the magnitude of the cause x and of the effect y is expressed by a **linear** equation of the form,

$$y = kx.$$

It is for this reason that the principle is called the principle of **linear** superposition.

The relation between the force and the quantity of electricity in Coulomb's law is a linear relation. The following is a case to which the principle of linear superposition does not apply.

If one horse, by exerting a pull of 150 pounds through a tow rope, tows a barge along a canal at the speed of 2 miles per hour, it is found that two horses performing in the same way, while exerting a total pull of (150 + 150) pounds on the barge, will not impart to the barge a speed of (2 + 2) miles per hour, but a speed of only 2.8 miles per hour.

PART III—EXPLANATORY THEORIES AS TO THE NATURE OF ELECTRICITY

31. Facts (Observations) Which Must Be Accounted for by Any Comprehensive Theory.—The electrical phenomena which have been presented thus far are tabulated below:

a. The force of attraction between bodies which have been electrified by frictional contact.

b. The force of repulsion between bodies which have been electrified by frictional contact.

c. The attraction of a neutral body by an electrified body.

d. The force of repulsion between an electrified body and a body electrified from it by conduction.

e. Electrification by rubbing contact.

f. Electrification by metallic contact.

g. Electrification by conduction.

h. Electrification by induction.

i. Conduction and non-conduction or insulation.

j. Two kinds of electricity only.

k. The fact that positive and negative electricities always exist in equal quantities.

l. The fact that electric charge resides only on the outside surface of conductors, unless, being hollow, they contain insulated charged bodies.

m. The variation in the surface density of charge over the surface of conductors.

These phenomena are not accounted for by any of the fundamental conceptions as to the properties of ordinary matter. An explanatory theory is naturally sought, which will account for, or link together, these diverse effects in terms of a few simple relations, in a manner similar to that in which Newton's law of gravitation, for example, accounts for such diverse and apparently unrelated observations as the motion of the moon and planets, the tides, the flow of rivers, and the fall of bodies.

32. The Single- and the Two-fluid Theories of Electricity.—The first comprehensive theory as to the nature of electricity was the **single-fluid theory** advanced by Benjamin Franklin in 1747. Franklin assumed that there is an "electrical matter" or fluid consisting of fine particles so light as to be without appreciable weight but possessing strong powers of attraction and repulsion. That is, a body was assumed to be made up of atoms and electric fluid and these entities were assumed to have the following properties:

Atoms repel atoms.
Electric fluid repels electric fluid.
Atoms and fluid attract.

Now a body containing both fluid and atoms would be at the same time attracted to, and repelled by, an external body of fluid. When

the amounts of fluid and of atoms of the body are so proportioned that the repulsion and attraction are exactly equal, the body is obviously in the state we have called the neutral or uncharged state. The amount of fluid which produces this neutral state is called the **natural** or **normal** amount of fluid for this body.

The fluid moves most readily through conductors and with difficulty through insulators. The process of electrifying a body is conceived to consist in adding to, or subtracting from, its natural quota of fluid. Franklin surmised that the glass rod when rubbed with silk acquires an excess charge at the expense of the silk, and so proposed to say that the glass is positively charged and the silk negatively.

A second theory, the **two-fluid theory,** was advanced by Symmer in England and Du Fay in France about 1759. The two-fluid theory assumed that **ordinary** matter is made up of these constituents—atoms, a positive electric fluid, and a negative electric fluid. A body in its ordinary or uncharged condition contains **equal** quantities of the two fluids uniformly mixed in **unlimited** amounts.

The three entities have the following properties:

Negative fluid repels negative.
Positive fluid repels positive.
Positive fluid attracts negative.
Atoms attract both fluids.

A positively electrified body is assumed to have more positive than negative fluid.

Both the single- and the two-fluid theories, if liberally interpreted, are quite valid theories at the present time, although the former more closely resembles the modern electron theory. In the electric fluid theories, the term **fluid** denoted an agent which was capable of spreading and flowing after the manner in which a fluid is **seen** to flow, and which was susceptible of division into portions of any magnitude, after the manner of a fluid. In these early fluid theories, ideas as to the constitution or structure of the electric fluids were absent, or at any rate played no essential rôle in the application of the theories. Later and more extended

electrical experiments, especially in connection with the flow of electricity through gases, have yielded a somewhat detailed picture of the structure of the electric fluids, a picture which accounts for a vast range of phenomena entirely outside the scope of the fluid theories which were quite adequate to account for the limited number of phenomena recited above. This theory is known as the electron theory of matter and electricity.

While it is impossible at this time to present the evidence upon which the electron theory rests, and while the experiments thus far recited do not warrant ideas as to the structure of electricity any more definite than those contained in the single- and two-fluid theories, yet the outlines of the electron theory are presented at this point in order that the beginner may start at once to carry on his qualitative reasoning in terms of the electron theory. In presenting the theory before the facts upon which it is based, perhaps the ever-present danger attending such a course should be pointed out. The danger is that the theory—the **interpretation** which has been placed upon experience, the mental **image** which has been **constructed**—may be set up in place of experience, and may become the primary object of contemplation and veneration. This is what happened at the base of Sinai when the Israelites set up the golden calf for the thing it signified, thereby committing themselves to years of wandering in the wilderness. The relative parts played by the fundamental quantitative laws and the theory may be put in another way. The laws rest upon experimentally determined relations and not upon theories as to the structure or nature of electricity. In general, theories change, but the quantitative relations remain. The quantitative relations constitute a framework or skeleton, and the theory, the flesh and sinews. One is as essential as the other. The skeleton by itself is a lifeless thing devoid of beauty; the theory alone, a thing without backbone.

33. The Electron Theory of Matter and Electricity.[6]—For a number of years, the atom was an unresolvable, indivisible

[6] See the following books for an account of the electron theory: Fournier d'Albe, E. E.: *The Electron Theory;* Comstock and Troland: *The Nature of Matter and Electricity;* Millikan, Robert A.: *The Electron, Its Isolation and Measurement;* Mills, John: *Within the Atom.*

granule of matter. All the atoms of any one element, as oxygen, were conceived to have identically the same mass and properties, but the mass and the properties of the atoms of one element differed from those of another element. This atom, conceived as unresolvable into constituent parts, may be called the **chemist's atom.**

Many experiments, particularly on the passage of electricity through gases, led to the conception that the formerly unresolvable atoms are complex structures. Each atomic structure is conceived to consist of a positively charged attracting center about which revolve one or more negatively charged particles whose aggregate charge is equal to the positive charge of the attracting center. That is to say, the atoms are conceived to be planetary systems whose proportions resemble in miniature the solar system. The positively charged attracting center is called the **nucleus,** and the negatively charged planetary particles are called **electrons.**

By reason of the interactions between adjacent atomic systems, some electrons may temporarily escape from the control of the attracting nuclei. Such electrons behave like **free** negatively charged gas corpuscles in the interstices between the atoms until they again become attached to some atomic system. Even as atoms or molecules evaporate from the surface of ice or water or iron at a greater and greater rate as the temperature of the evaporating body is raised, so may these free electrons be emitted from the surface of bodies in increasing numbers as the temperature of the bodies is raised to incandescent temperatures. By experiments on these emitted electrons, it is found that, regardless of the material from which they are obtained—carbon, copper, sodium, oxygen, etc.—all the electrons have identically the same negative charge and the same mass. Experiments indicate that the mass of the electron is about $1/1845$ of the mass of the hydrogen atom, and that its radius is of the order of 1/100,000 of that of the atom.

The simplest atomic system is that of the hydrogen atom. It is conceived to consist of a nucleus having but a single planetary electron. Since the mass of the electron is only $1/1845$ of that of the hydrogen atom, 99.95 per cent of the mass of this atom is

associated with the nucleus. It has been proposed to call the nucleus of the hydrogen atom a **proton.** The evidence from the atomic disintegrations which occur in radioactive materials is that the positively charged nuclei of all other atoms consist of a compact arrangement of protons and electrons. Thus the helium atom, having an atomic weight of 4.0, is conceived to consist of two planetary electrons which revolve about a compact nucleus made up of four protons and two electrons. Uranium, the element of highest atomic weight, 239.5, is conceived to contain an excess of 92 protons in the nucleus, and 92 planetary electrons arranged in rings or shells about the nucleus.

Electrons and protons are thus regarded as the common elementary constituents out of which all atomic systems are built up. The chemical properties of the elements are conceived to be determined by the number of planetary electrons revolving about the nucleus, and by the readiness with which the different atomic systems may lose or may acquire one or more than the normal number of electrons. The periodicity in the chemical properties, as shown by the periodic table of the elements, is attributed to a periodicity in the filling of successive rings or shells of planetary electrons. The atom, so conceived, may be called the physicist's atom.

The electrons and protons are endowed with the following properties:

a. Every electron repels every other electron.

b. Every proton repels every other proton.

c. Every electron attracts every proton.

(These forces exist between all electrons and protons, both the free electrons and those held in planetary systems.)

d. The law of force between these elements is the inverse square law. This law is conceived to apply to distances between centers which are a small fractional part of the atomic radius.

We thus have the following picture of the structure of a body, such as a copper wire. The body is made up of planetary copper atoms. Each atom, as a unit, is violently colliding with its neighbors (thermal agitation). Each atom may be said to have a certain volume (occupy a certain volume), in the sense that an army occupies a given territory not by filling it, but because

its influence keeps other systems out of the region. As a result of the overlapping of the fields of force of the thermally agitated atomic systems, the more easily detachable planetary electrons are continually escaping from the attractive influence of a particular atomic center. These electrons eventually become attached to other atomic systems either as excess electrons or by taking the place of electrons which the other system has lost. Such an interchange is continually occurring.

An atom which has lost one or more electrons acts as a small, positive charge equal in magnitude to the charge of the one or more electrons. If the number of free electrons in a body is just sufficient to supply all atoms with the proper number, the body is electrically neutral, even though some of the electrons are free at any instant, for at the same instant there is an equal number of atoms which lack one electron and therefore act as positive charges. The small negative charges and an equal number of small positive charges mixed indiscriminately produce no effect which can be measured outside the body. But if the body has more than the proper number of electrons there is a surplus negative charge on the body. If the body has a deficit of electrons there is a surplus positive charge on the body.

The electrons which are free at any instant of time—those which have escaped from one atomic system and have not yet become attached to a second system—constitute the electric fluid of the fluid theories, or the **electric gas** of the electron theory. Every material—solid, liquid, or gaseous—contains an **atmosphere** of free electrons in the interstices between the atoms or molecules. For example, air contains from 1000 to 5000 free electrons per cubic centimeter, or one electron per 10^{16} molecules, while the number of free electrons in copper is thought to be of the order of 10^{19} per cubic centimeter or one electron per 3000 atoms. A transfer of electricity from one body to another is a transfer of electrons to the body which is said to receive the negative charge. An electric current in a wire is a drift of the electron atmosphere through the wire. The difference between a good conductor and an insulator is partly that the former has a great number of free electrons per unit volume and the latter very few.

At the surface of bodies not only the molecules but also the free electrons of the electron atmosphere experience forces of the

kind illustrated by surface tension phenomena—forces which keep all but the more violently moving molecules and electrons from shooting off into space. The magnitude of the unbalanced restraining forces which an electron experiences as it approaches the surface depends upon the atomic structure and the surface arrangement. The more electropositive materials, such as glass with reference to silk, or zinc with reference to copper, are the materials from the surface of which the electrons can escape more readily. When two materials are brought into contact, the more electropositive material gives off more electrons than it receives and so becomes positively charged. This is electrification by contact or by rubbing. Section 33*a* contains data (mainly for future reference) relating to atoms and to electrons.

33a. Atomic and Electronic Data.

Atoms and molecules:	
Molecules per gram-molecule (0 = 16)	$(6.062 \pm .006)10^{23}$
Volume of a gram-molecule of gas (0 = 16) under standard conditions	22.42 liters
Molecules per cubic centimeter of gas (standard conditions)	$(2.705 \pm .003)10^{19}$
Mass of hydrogen atom	$(1.662 \pm .002)10^{-24}$ grams
Mass of 1 cubic centimeter of H (standard conditions)	8.99×10^{-5} grams
Radius of hydrogen molecule	Order of 1×10^{-8} cm.
Mean distance between centers of gas molecules (standard conditions)	3.3×10^{-7} cm.
Atoms per cubic centimeter of copper	8.47×10^{22}
Mean distance between centers of copper atoms	2.27×10^{-8} cm.

(Standard conditions for a gas are 0°C., 76 centimeters, g = 980.6, or 0°C. and a pressure of 1.0132 megadynes per square centimeter.)

Electrons:	
Electronic charge	1.591×10^{-19} coulombs
Electronic mass	9.00×10^{-28} grams
Electronic mass	$\frac{1}{1845}$ of H atom
Ratio of charge to mass	$1.768 \times 10^{15} \frac{\text{coulombs}}{\text{gram-sevens}}$
Radius of electron	Order of 10^{-13} cm.

34. Engineering Problems and Applications.—The electrical phenomena which have been presented have the following useful applications, and have given rise to the following engineering problems.

All electrical generators are devices in which electrons are forced through the generator from the positive to the negative terminal against the repulsive forces of the electrons previously transferred and the attractive forces of positive nuclei left behind. Work must be done to force or carry the electrons against these forces. This work is said to be converted into an electrical form of energy. It may be converted back into other forms—mechanical, thermal, or chemical—if the electrons are permitted to travel back from the negative terminal to the positive through suitable circuits outside the generator. The first application of the relations which have been brought out in this chapter is in the devising of electrostatic generators and in arriving at an understanding of the causes of the inherent limitation upon the power output of electrostatic generators.

The second application is in the devising of **meters** for measuring the amount of work which will be done when electrons travel from one point of an electric field to another.

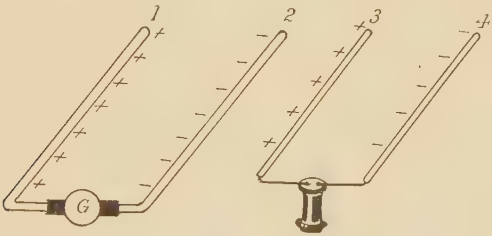

FIG. 9.—Electrostatic induction from power lines.

The puncture and the destruction of the insulating materials used on high voltage cables and in the insulating bushings (tubes) needed to carry high-voltage wires through the walls of buildings occur when the force upon the electrons in the insulation materials exceeds certain values for each material. The design of high-tension insulating members involves the calculation of the relative values of the forces which electrons would experience at different points, and the proportioning of the parts so that these forces will not exceed the safe values. This is a third application.

The alternating-current generators which supply power-transmission lines rapidly reverse the sign of the charges on the power wires. At a given instant, the power wire 1 of Fig. 9 is positively charged and wire 2 negatively. In $\frac{1}{50}$ or $\frac{1}{120}$ second later, wire 1 is negatively and wire 2 positively charged. Suppose a telephone line parallels the power line (as indicated in Fig. 9) either on the same poles or across the road, or 100 meters away. At the instant power wire 1 is positively charged and wire 2 negatively, electrons are induced to flow from the near telephone wire 3 through the telephone receiver (if the line is in use) to the far telephone wire 4. When the polarity of the power line reverses, the induced polarity of the telephone wires also reverses. The resultant flow of electrons back and forth through the telephone receiver produces noises which interfere with the use of the

telephone system. A fourth application is in the mitigation of this inductive interference between power circuits and telephone and telegraph circuits. The induced currents for different arrangements of the wires may be predetermined by calculation, and the lines laid out to reduce the interfering currents to a minimum.

The Cottrell process of removing and recovering either injurious or valuable mists and dusts from the discharge gases of manufacturing processes is to pass these gases through long flues in which a strong electric field is set up between the walls of the flue and a wire of small diameter strung along the axis of the flue. The field is made strong enough to cause a brush discharge to occur from the wire. The particles of mist or dust thereby become charged and are driven to, and deposited upon, the walls of the flue.

34a. Exercises.

1. What is the fundamental significance of the statement that a body is electrified, or is electrically charged? In other words, how would you proceed to determine whether any given body is electrified?

2. What is meant by an electric field?

3. Enumerate the methods of electrifying a body. Describe each method enumerated. State the precautions it is necessary to observe in the attempt to electrify bodies by each of the methods described above.

4. What is meant by "like electricities," or "like electrical charges"? By "unlike electricities"?

5. What is the significance of the designations + (positive) and − (negative) as applied to electrical charges? Is there anything appropriate about the use of these designations? Why was the designation + assigned to the kind of electricity which is developed on a glass rod by rubbing it with silk?

6. How many kinds of electricity are there? What is the basis of classification upon which your answer is based? Would it have been reasonable on the part of the first experimenters to have expected to find either a greater or a smaller number of kinds of electricity?

7. State the first law of electrostatics as originally framed with reference to the forces between electrified bodies.

8. From the experiments on induced electrification, can a statement other than the first law, as framed above, be made with reference to the forces between charges? If so, give the statement and point out the difference between this statement and the law as previously stated.

9. Using the above reframed statement as a basis, describe the process by which an electrified body attracts a body that is apparently unelectrified or neutral.

10. State the principle of the equality of the quantities of the two electricities (generally known as the principle of the conservation of electricity). Recite the experimental basis for the principle.

11. Is the electrification of a body by friction essentially an "abrading" operation, analogous to the removal of "dirt" by rubbing or polishing?

That is, do you rub or abrade electricity from a body as you remove dirt? If not, what is the purpose of the rubbing? What is the evidence upon which your statements are based?

12. Under what conditions are two quantities of electricity said to be equal? Is it necessary to define what we mean by equal quantities of electricity or is the meaning of **equal quantities** self-evident?

13. State Coulomb's law of force between small charged bodies. Express this law by means of a formula. How was this law first determined?

14. *a.* A body moving through the air is acted on by the force of gravity and by a wind pressure. Is it allowable to calculate separately the accelerations produced by these forces and then to combine these vectorially (or to superpose them)? Why?

b. The velocity of water from a nozzle varies approximately as the square root of the pressure inside. In the pipe line supplying a certain nozzle, there are two pumps inserted, each one increasing the pressure by 50 pounds per square inch. Is it allowable to calculate separately the nozzle velocity produced by each pump and then to add these to get the total velocity? Why?

15. What is meant by the principle of the linear superposition of the electric fields of charges? What is the evidence, or the nature of the evidence, upon which is based the principle of the superposition of electric fields?

16. Assume that a straight conducting wire about 100 centimeters in length and at a great distance from other objects has been given a negative charge. How does the quantity of electricity per centimeter of length vary as the unit length under consideration is taken nearer and nearer to the end of the wire? Why? Explain in terms of the fundamental properties attributed to electrons. A qualitative rather than an exact quantitative statement of the variation is desired.

17. Assume that the ends of the charged wire in exercise 16 are very sharply pointed. Account for the brush discharge which occurs at the points.

18. When the question "What is electricity?" is raised by the layman, what kind of an answer does the questioner invariably want? Is it reasonable to say that, since the type of answer desired cannot be given, our knowledge of electricity is inadequate or is in an unsatisfactory condition? If not, support your contention by citing four or five other queries relating to common everyday affairs which we rarely have the intellectual keeness or curiosity to raise, but which, if raised, would be answered in terms far less comprehensive than the answer which can be made to the query "What is electricity?"

CHAPTER III

THE CALCULATION OF THE FORCES OF THE ELECTROSTATIC FIELD

THEME: The primary meaning of the electrical quantities defined in this chapter is in terms of **force**.

35. In the explanation of the experimental facts presented in the preceding chapter and in the working out of the engineering applications outlined at the end of the chapter, we are concerned very largely with the forces which act upon electric charges at various points in the field and with the work which these forces do as electricity moves or circulates in the field. The constant necessity of dealing with **force** and **work** in a precise quantitative way leads to the use of the terms and methods which we now proceed to introduce.

36. The Unit of Electricity—the Coulomb.—It has been previously stated that the law of force between two charges r centimeters apart in an extended homogeneous medium, when expressed in the form of an equation, takes the form

$$f \text{ (repulsion)} = \frac{Q_1 Q_2}{kr^2}, \tag{4}$$

in which the value of the constant k (which has arbitrarily been written in the denominator rather than in the numerator of the expression) depends upon the medium in which the charges are immersed, and upon the units in which force, length, and quantity of electricity are expressed.

A very natural proceeding would be to select some medium as a standard medium, and to define the unit of electricity to be that quantity with which a very small body must be charged so that, when placed in the standard medium at a distance of 1 centimeter from a similar body charged with an equal quantity, the force of repulsion between the two will be 1 dyne. This is

the manner in which the original unit of electricity was defined. This is the unit of electricity in the **electrostatic** system of units—evacuated space being selected as the standard medium. For the unit quantity so defined, Coulomb's law may be written:

$$f \text{ (repulsion in dynes)} = \frac{Q_1Q_2}{k_1r^2} \; \frac{\text{(E.S. units)}}{\text{(cm.)}}, \tag{5}$$

in which $k_1 = 1$ for evacuated space.

As previously stated (Sec. 10), this electrostatic unit was subsequently found to be inconveniently small for measuring the quantities delivered by generators and batteries, and the so-called **practical units** were defined to be certain multiples of the original units. These definitions were equivalent to defining the **unit of electricity** in the practical system (subsequently named the **coulomb**) to be approximately 3×10^9 times as great as the electrostatic unit defined above, or were equivalent to defining the coulomb to be that quantity which will repell a like quantity at a distance of 1 centimeter with a force of approximately $(3 \times 10^9)^2$ dynes or 9×10^{11} dyne-sevens.[1]

Thus the first electrical unit of the practical system, the unit of electricity, is defined **in terms of force** in the following manner.

36a. UNIT OF ELECTRICITY: THE COULOMB (DEFINITION).—**The unit of electricity, called the COULOMB, is that quantity of electricity with which a very small body must be charged so that, when placed in an evacuated space at unit distance (1 centimeter) from a similar body charged with an equal quantity, the force of repulsion between the bodies will be 8.9892×10^{11} dyne-sevens (approximately 9×10^{11} dyne-sevens).**

Coulomb's law for the force between charged bodies when written in the units of the practical system takes the form,

$$f \text{ (repulsion in dyne-sevens)} = \frac{Q_1Q_2}{kr^2} \; \frac{\text{(coulombs)}}{\text{(cm.)}}. \tag{6}$$

[1] The practical unit of electricity (the coulomb) was defined to be one-tenth as great as the electromagnetic unit. Since the ratio of the E.M. unit to the E.S. unit of electricity has been found by a long series of precise measurements to be 2.9982×10^{10} (which value has also been obtained as the velocity of light, s, in centimeters per second), it follows that the coulomb is 2.9982×10^9 times as great as the E.S. unit, and that the force of repulsion between two charges, each of 1 coulomb and 1 centimeter apart, would be 8.9892×10^{11} dyne-sevens.

37. Dielectric Constant and Specific Inductive Capacity (DEFINITIONS).—The constant k appearing in the denominator of the expression for the force between charged bodies is called the **dielectric constant** of the medium or substance in which the bodies are immersed.

The value of k for the standard medium, evacuated space, when all the quantities are expressed in **practical units** is

$$k = \frac{10^9}{s^2} = \frac{1}{8.9892 \times 10^{11}}. \tag{7}$$

This value is to be regarded as a value which was arbitrarily assigned in the defining of the practical units in terms of the original electrostatic and electromagnetic units.

The relative dielectric constant k_r of a substance—also called its **specific inductive capacity**—is defined as the ratio of its dielectric constant to the dielectric constant of the standard medium, evacuated space. (For air, k_r has the value 1.00059.)

In this text, the dielectric constant will not be used but will be discarded and will be replaced by the quantity, the **permittivity** of the medium, which we proceed to define.

38. Permittivity of a Medium.—If we proceed to use Coulomb's law as expressed by Eq. (6) to derive formulas for such quantities as the pull between charged plates, etc., we find that the factor $k/4\pi$ appears in many of the formulas. Unfortunately, the factor 4π appears in many of the most frequently used formulas in a manner which has been characterized as puzzling or irrational. For example, the pull between two parallel plates oppositely charged with Q coulombs per square centimeter is found to be

$$\frac{4\pi}{k}\frac{Q^2}{2} \text{ dyne-sevens per sq. cm.}$$

This looks irrational because we expect π to appear only in problems dealing with circles, cylinders, or spheres. An examination of the derivation of these formulas shows that the irrational 4π factors creep into the important and most frequently used equations because of the manner in which the unit quantity of electricity and the subsequent units have been fixed by Eq. (6). It is evident that a simple expedient will suffice to banish the 4π

from those equations involving the factor k. This expedient is to discard the dielectric constant k and to use a new constant, which we will call the **permittivity** of the medium, and will represent by the symbol p, and will define by the equation,

$$p = \frac{k}{4\pi}. \tag{8}$$

This means that Coulomb's law will be written in the form:

$$f \text{ (repulsion in dyne-sevens)} = \frac{Q_1 Q_2}{4\pi p r^2} \; \frac{\text{(coulombs)}}{\text{(cm.)}} \quad . \tag{9}$$

in which the numerical value of the permittivity p_o for the standard medium, evacuated space, must be

$$p_o \text{ (evacuated space)} = \frac{10^9}{4\pi s^2} = 8.85 \times 10^{-14}. \tag{10}$$

In other words, the irrational and objectionable 4π may be banished from the formulas most frequently used in engineering calculations by shifting it to a formula which is little used. The attempt to explain the appearance of the 4π in Eq. (9) can never be perplexing in the same sense that it is perplexing in the case of the formulæ to be subsequently developed. The 4π appears in Eq. (9) because we deliberately put it there, and we put it there to escape what Heaviside has termed an eruption of 4πs in subsequent equations. For the present, then, the permittivity may be defined as follows:

38a. PERMITTIVITY (DEFINITION).—**By the permittivity of a medium is meant the value of the constant p, appearing in the denominator of Eq. (9) for the force between two charged bodies immersed in the medium—p_o, the value for free space, being 8.85×10^{-14}.**

In the subsequent pages we propose to express Coulomb's inverse square law by Eq. (9), and we propose to use the **permittivity** p rather than the dielectric constant k in all formulas.

38b. RELATIVE PERMITTIVITY (DEFINITION).—**The relative permittivity p_r of a substance is defined as the ratio of the permittivity of the substance to the permittivity of the standard medium, free space. (For air, p_r has the value 1.00059.)**

The relative permittivity of any substance is, of course, equal to its relative dielectric constant or its specific inductive capacity.

39. Electric Intensity at a Point.—The region surrounding charged bodies, or, in general, any region in which a charged test body experiences a mechanical force because of its charge, is called an **electric field.** A method of describing and specifying the properties of an electric field, or of comparing two fields, is necessary. The method suggests itself of exploring the field with a small test body containing a test charge Q_t, of known magnitude and sign, thereby obtaining the magnitude and direction of the force f which the test body experiences from point to point. This force f is proportional to the charge Q_t on the test body, provided the test body and the charge Q_t on it are so small that their introduction into the field does not (by induction) cause any appreciable alteration of the original distribution of the charges on the conductors in the field.[2] If, therefore, the

[2] The reason for this proviso that the test charge shall be small is as follows: If a charged test body is brought into a field in which there are conducting bodies, the test body experiences forces from two sets of charges; first, from the set of charges giving rise to the original field; second, from the set of charges which are induced on the conductors by the charge of the test body. For example, if the small, positively charged test sphere S of Fig. 9a is brought into the vicinity of the large conducting body A, the test sphere S experiences a pull toward A even though all tests show A to be quite neutral when the test sphere S is removed. This is because the charge on the test body (by induction) brings about a redistribution of the electrons of the large conductor A, by attracting the electrons toward the end of A nearest to the test body. In describing the electric field we desire to eliminate from consideration the forces arising from the second group and to specify the forces due to the original set of charges. This is readily accomplished by imagining the small test body to have a very small (infinitesimal) charge, because the forces experienced by the test body vary in the following way with the size of the test charge Q_t;

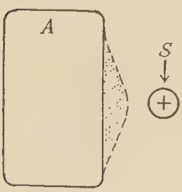

FIG. 9a.—Charge induced by the test charge.

$$\text{Force from original charges} = KQ_oQ_t$$
$$\text{Force from induced charges} = K_1Q_t^2$$

in which, K and K_1 represent constants whose values depend upon such factors as the configuration of the conductors and the location of the test charge.

Therefore,
$$f = KQ_oQ_t + K_1Q_t^2$$
and
$$\frac{f}{Q_t} = KQ_o + K_1Q_t \tag{11}$$

By making the test charge small enough, the value of the second term in the

force f is divided by Q_t, a quantity is obtained whose value is independent of the magnitude of the test charge used on the exploring body. The quantity, or the quotient f/Q_t, which represents the force which would be exerted by the original charges of the field upon unit charge at a point, occurs so frequently in calculations and discussions that it is a great convenience to have a name for it. It is called the **electric intensity at the point,** and is defined as follows:

39a. ELECTRIC INTENSITY AT A POINT (Definition). —**The electric intensity at a point P in an electric field is defined to be a vector quantity F whose direction and magnitude are the same as the direction and magnitude of the force which the original distribution of charges giving rise to the field would exert on a unit positive charge if it were placed at the point P.**

The above definition when written in the form of an equation takes the form:

$$\mathsf{F} \quad \text{(dyne-sevens per coulomb)} = \frac{\mathbf{f}}{Q_t} \quad \frac{\text{(dyne-sevens)}}{\text{(coulombs)}} \qquad (12)$$

$$\mathsf{F} \quad \text{(volts per cm.)} = \frac{\mathbf{f}}{Q_t} \quad \frac{\text{(dyne-sevens)}}{\text{(coulombs)}}. \qquad (12)$$

Briefly, the electric intensity at a point is the force in dyne-sevens exerted upon the test body per coulomb of electricity upon the test body. To visualize the properties of an electric field, we think of the magnitude and direction of the electric intensity at numerous points throughout the field. This leads us to think of the electric intensity as a vector quantity whose magnitude and direction are functions of the position of a point which we move about in the field, and in abbreviated fashion we then speak of the electric intensity as a **vector point function**—(a vector whose magnitude and direction is a function of the position of the point in the field).

To visualize the direction of the vectors from point to point in the field, it is very helpful to map out the field by means of a system of **lines of electric intensity** in the manner described and illustrated in Sec. 94.

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right member of Eq. (11) can be made negligibly small in comparison with the first term; the value of the first term is independent of the value of the test charge.

39b. LINES AND TUBES OF ELECTRIC INTENSITY (Definitions).—A line of electric intensity is a curve in the electric field such that the tangent at every point is in the direction of the force which would be exerted upon a small test charge at that point. The positive direction along a line of intensity is defined to be that direction in which a positive test charge would move.

A tube of electric intensity is the tubular surface formed by all the lines of electric intensity which can be passed through points of the boundary of any small area in the field.

39c. Unit of Electric Intensity (Definition).—*The electric intensity at a point is said to be unity if a test body when placed at the point experiences a force of one dyne-seven per coulomb of charge upon the test body. The natural name for the unit is the "dyne-seven per coulomb." Another name for this same unit is the "volt per centimeter."*

The name universally used for the unit of electric intensity is the **volt per centimeter.** The origin and the significance of this name are explained in Sec. 52. The names **volt per centimeter** and **dyne-seven per coulomb** mean the same thing.

40. Mechanical Force Acting upon a Charge in a Field (Deduction).—From the manner in which the electric intensity has been defined, it follows that the mechanical force $\mathbf{f}$ exerted by the original charges of the field on a concentrated charge Q situated at a point where the electric intensity is F is expressed by the equation

$$\mathbf{f} \text{ (dyne-sevens)} = Q\mathsf{F} \text{ (coulombs, volts per cm.)}. \tag{13}$$

41. The Electric Intensities in the Field of a Single Concentrated Charge (Deduction).—If the field is due to a single concentrated charge, the electric intensity at any specified point P in the field may be calculated directly from the definition of electric intensity, for if a small, positively charged test body is placed at the point P the force acting on it may be calculated from Coulomb's law. Let the concentrated charge be represented by Q and the charge on the test body by Q_t, and let r represent the distance from the charge to the point P, then,

$$f = \frac{QQ_t}{4\pi p r^2},$$

The electric intensity F is the force per unit charge and is equal to f/Q_t, or

$$F \text{ (volts per centimeter)} = \frac{Q}{4\pi p r^2} \quad \frac{\text{(coulombs)}}{\text{(cm.)}} \cdot \qquad (14)$$

F is directed radially away from the charge Q if Q is a positive charge.

No electric charge that we know of is concentrated at a point. This equation can be used, however, with very small errors for charges on small bodies when the distance r is large compared with the dimensions of the body. It may be demonstrated by the methods outlined in succeeding sections that, when the charge is uniformly distributed over the surface of a sphere, the electric intensity at any point outside the sphere is the same as though the charge were concentrated at the center.

42. The Calculation of Electric Intensities in the Field of Two or More Concentrated Charges.—From the principle governing the addition of electric forces (Sec. 30a), namely, that the force on a test body due to two or more concentrated charges is found to coincide with the force which is calculated by assuming that each concentrated charge exerts the same force as if it alone were in the field, it follows that the electric intensity at a point P which is at the distance r_1, r_2, r_3, etc. from the charges Q_1, Q_2, Q_3 is

$$\mathsf{F} \text{ equals the vector sum, } \frac{1}{4\pi p}\left[\frac{Q_1}{r_1^2} + \frac{Q_2}{r_2^2} + \frac{Q_3}{r_3^2} + \right] \cdot \qquad (15)$$

In this equation, the quantities within the bracket are directed quantities, having the directions Q_1P, Q_2P, Q_3P, etc. The sum is the vector sum, to be obtained by the geometrical addition of the individual intensities or vectors. A simple method of addition is the triangle construction or the parallelogram construction. These constructions may be carried out to scale on a drawing board, or the required vector may be calculated by use of trigonometry. Either of these methods becomes tedious, however, when a large number of vectors are to be added, and the following method may then be used to advantage:

Choose three mutually perpendicular axes, such as the familiar X, Y, and Z axes of the Cartesian coordinate system. The following work can often be made very simple by choosing the

axes carefully. Next resolve each vector which is to be added into three components, one parallel to each axis. Then all the components along the X axis may be added algebraically to get the X component of the resultant. Likewise, all the Y components are added to get the Y component of the resultant, and the same with the Z components. Then having the three components of the resultant, we may easily find the resultant itself. This method is illustrated in the next paragraph.

43. The Calculation of Electric Intensities in the Field of Charges Which Are Distributed in a Known Manner (DEDUCTION).—When it is desired to calculate the electric intensity at a point, and the distance from the point to the charged body is not large compared to the dimensions of the body, it cannot be assumed without serious error that the total charge is concentrated at some one point, and the following method must be used. Let the total space containing the charge be divided up into small volumes, each one so small that the charge within it may be considered as concentrated at some

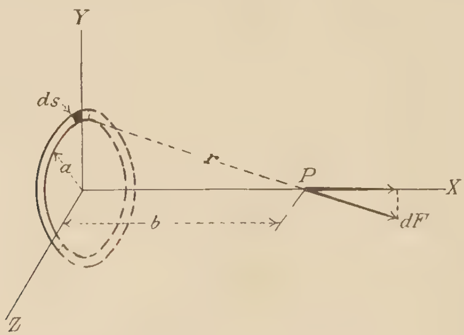


FIG. 10.—Intensity due to the charge on a ring.

point within that volume. Then we have a number of concentrated charges to be dealt with as in the preceding paragraph. Instead of a small number of charges, however, we have a very large number and the additions of the components are usually carried out by integration. This is illustrated by the following problem:

Assume that a charge of q coulombs per linear centimeter is distributed along a fine wire bent into a circle of radius a (Fig. 10). The electric intensity at a point P which lies on the axis of the circle at a distance b centimeters from the plane of the circle is to be calculated. When the distance b is not large compared to a, all of the charge cannot be considered as concentrated at one point. Let s represent the distance around the circle measured from some arbitrarily chosen point, and let ds represent a very short section of the

wire. The charge on this section is $q(ds)$. It may be considered as a concentrated charge. It produces an electric intensity at P equal to

$$dF = \frac{q(ds)}{4\pi p(a^2 + b^2)}$$

and the direction is along the line through P and ds. To add this vector to the intensities due to the other short segments of the wire the method previously outlined is used. Choose the axis of the circle as the X axis of the three-axis system and let the Y and Z axes then be laid off perpendicular to this and to each other. Add the X components of the intensities first. The X component of the intensity dF is

$$dF_x = \frac{q(ds)}{4\pi p(a^2 + b^2)} \cdot \frac{b}{\sqrt{a^2 + b^2}},$$

since $\frac{b}{\sqrt{a^2 + b^2}}$ is the cosine of the angle between dF and the X axis. The X component of the resultant is then the summation of these small components for all sections of the wire from $s = o$ to $s = 2\pi a$, or

$$F_x = \int_0^{2\pi a} \frac{bq(ds)}{4\pi p(a^2 + b^2)^{3/2}} = \frac{2\pi abq}{4\pi p(a^2 + b^2)^{3/2}} = \frac{Q}{4\pi p} \frac{b}{r^3}. \tag{16}$$

It is evident from the symmetry of the arrangement that the Y component and the Z component of the resultant will both be zero. Therefore, no integrations need be carried out to determine the Y and Z components. The advantage gained by making one of the axes coincide with the axis of the circle is evident. The resultant electric intensity at P is equal to the X component and is given by the expression above.

44. The Electric Intensities at Which Insulating Materials Fail (to Insulate).—In the continued building up of the charges giving rise to the field (as by the continued operation of a static machine), a stage is eventually reached at which the insulating material (air, oil, wood, paraffin, glass, etc.) separating the charged conductors fails to insulate. The failure, or breakdown, or puncture, of the insulating medium is indicated by the formation of a **brush discharge** or by the passage of an **electric spark.** This failure is explained in terms of the electron theory as follows: As the charges upon the conductors become larger and larger, the forces acting on the free electrons in the insulating materials become greater and greater. Under these forces the free electrons experience an acceleration (to their normal velocity of thermal agitation) which is proportional to the force. As this acceleration becomes greater and greater, the velocities acquired by the electrons in the short path between collisions with molecules becomes greater and greater. Finally, at an

electric intensity which is fairly definite for any insulating material, such as glass, the few free electrons acquire such high velocities that upon colliding with neutral molecular or atomic structures they tear away the more easily detachable electrons, thus giving rise to an increased number of free electrons. This destructive process rapidly increases the supply of free electrons, and thus a conducting channel or path is formed through the insulating material. The intense destructive bombardment results in sufficient heat to char a path through wood and melt a channel through glass.

44a. DISRUPTIVE ELECTRIC INTENSITY OR DIELECTRIC STRENGTH (DEFINITION).—The electric intensity at which an insulating material fails or breaks down is called the DISRUPTIVE ELECTRIC INTENSITY, or the DIELECTRIC STRENGTH, of the material. Dielectric strength is expressed in VOLTS PER CENTIMETER.

The electric intensities at which a few insulating materials fail are listed in the following table.

44b. Dielectric Strength of Insulating Materials.

Material	Dielectric strength (peak), kilovolts per centimeter
Air	30
Air (liquid)	210
Glass	90–300
Porcelain	130–200
Gutta percha	80–200
Vulcanized rubber	150–300
Mineral oils (transformer)	150–350
Mica	400–800
Impregnated paper	120–200
Varnished cambric	280–500
Fused quartz	300

45. The Importance of a Scheme for Specifying the Potential Energy of a Field Due to the Presence of a Charge.—Electric machines and appliances are devices in which work is done by reason of the motion of electric charges under the action of the forces of an electric field. It is evident, therefore, that the calcu-

lations of electric intensities play an important rôle, not only in connection with the development and design of electric insulators, but also in the explanation of the properties of machines and appliances, and in problems relating to the work done by machines. Now in every transaction of work there are, under the Newtonian mechanics, two forces involved—Newton's action and reaction. We will find that we can regard the electrostatic force between segregated charges as one of the two forces involved in every movement of electric charge. If a charge moves in a direction **against** the electrostatic force of the field—for example, if two unlike charges are pulled away from each other, or if two like charges are pushed toward each other—some external agent is doing work against the electrostatic force. The work so done against the electrostatic force is conceived to become the energy of configuration, or the potential energy, of the system of charges, because if the charge is permitted to move in the reverse direction (or **with** the electrostatic force), the electrostatic force will do an equal amount of work against other forces.

We now propose to describe a scheme for measuring and specifying the amount of work a system of charges is capable of doing as the system changes from its actual state to any other, due to the movement of charges in the field; or, in other words, a scheme for specifying the energy which the system possesses. It will be found that one of the great merits of the scheme is the manner in which it facilitates the calculation of electric intensities.

46. Propositions Relating to the Work Which Is Done in Moving a Body from One Point to Another in the Field of Attracting Centers Having Spherically Symmetrical Properties.—In the gravitational field of the sun, or in the electric field of a concentrated charge, the force on a test body at any point P is along a line drawn through the point P and the attracting or repelling center O. **The attracting center, moreover, has spherically symmetrical properties.** That is to say, the magnitude of the force on a given body at a given distance from the center is the same for all lines radiating from the center.

Let us compute the work which must be done against such a system of forces by an external agent in moving a small, positive test charge Q_t as the test charge moves from a point A to a point

B in the field of a concentrated positive charge Q located at the fixed point O of Fig. 11. Let the radial distances OA and OB be represented by r_1 and r_2, respectively. Let the test charge move from A to B along any path whatsoever.

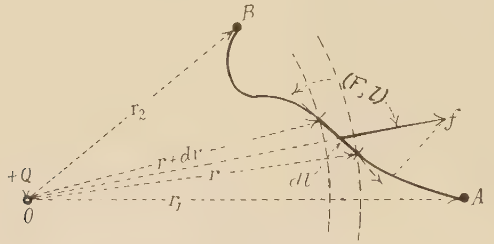


Fig. 11.—Centrally directed forces.

Let l represent distance measured **along the path in the direction of motion** from some arbitrarily chosen point in the path.

Then dl represents the increment in this distance as the charge moves over a short segment of the path.

Also, let f represent the magnitude of the radially directed electrostatic force at the element dl.

(f,l) represent the angle between f and dl in the **direction of integration.**

r and $r + dr$ represent the distance of the test body from O at the beginning and end of dl.

Then

$$\cos(f,l) = \frac{dr}{dl}$$

$$f = \frac{QQ_t}{4\pi pr^2}.$$

The component of this force in the direction of movement along dl is

$$f \cos(f,l) = \frac{QQ_t}{4\pi pr^2}\frac{dr}{dl}.$$

The work done by an external agent **against** this force as the body moves the length dl is

$$-f\cos(f,l)\,dl = -\frac{QQ_t}{4\pi pr^2}\frac{dr}{dl}\cdot dl = -\frac{QQ_t}{4\pi pr^2}\,dr.$$

That is, the work done by the agent as the test charge moves from the surface of a sphere of radius r (described about O as a center) to the surface of a sphere of radius $r + dr$ is the same whether the test charge moves from one spherical surface to the other along the short radial path in the direction of the centrally directed force, or along some more lengthy straight path making any angle with the force.

The total work W done by the external agent against the electrostatic force f on the test charge Q_t as it moves from A to B is the value of the following line-integral:

$$W = -\int_{l_A}^{l_B} f \cos (f,l)\, dl. \tag{17}$$

To evaluate this integral, imagine a great number of spherical surfaces described about O as a center, each sphere of slightly greater radius than the preceding, and consider the following propositions relating to the total work done as the test charge moves from A to B.

1. Any path from A to B splits up into elementary lengths each one of which leads from a spherical surface to a surface of slightly larger or slightly smaller radius.

2. The work done as the test charge moves from any one of these spherical surfaces to the next surface is independent of the path traversed.

3. By sketching in a few paths, it may be seen that any path which leads from A to B crosses from each surface to the next smaller surface an odd number of times for surfaces of radius between r_1 and r_2. If the path passes between any surfaces of radius larger than r_1 or smaller than r_2, it crosses between these surfaces an even number of times.

From the above propositions it may be seen that, no matter what path the test charge may traverse in getting from the point A on the sphere of radius r_1 to the point B on the sphere of radius r_2, the net amount of work done by the agent against the forces of the field is the same as if the test charge had moved radially inward from the spherical surface of radius r_1 to that of radius r_2.

Therefore, Eq. (17) yields the following result:

$$W \text{ (joules)} = -\int_{r_1}^{r_2} \frac{QQ_t dr}{4\pi p r^2} = \frac{QQ_t}{4\pi p}\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \tag{18}$$

The following proposition, therefore, characterizes the mode of space variation of the forces of the field.

46a. WORK PROPOSITION 1.—In a field of CENTRALLY DIRECTED FORCES HAVING SPHERICALLY SYMMETRICAL PROPERTIES, the work done in moving a given body from a point A to a point B, or the line-integral from A to B of the force on the body, is independent of the path followed by the body in moving from A to B, and depends only upon the distances r_1 and r_2 of the end points of the path from the attracting (or repelling) center.

Imagine the body in Fig. 11 to move from a point A_1 to a point B_1 along one path and then back to A_1 along any other path, thereby traversing a closed loop. Since the work done in moving the body from B_1 to A_1 is equal but of opposite sign to that in moving the body from A_1 to B_1, the net work done in traversing the closed loop is zero. The following is therefore, an alternative way of stating the above proposition:

46b. Work Proposition 1A.—*In a field of centrally directed forces having spherically symmetrical properties, the work done in moving a given body around any closed loop, or the line-integral of the force on a body as it moves around any loop, is zero.*

It should be noticed that this property of the electrostatic and gravitational fields, namely, that the work done upon a given test body as it moves from one fixed point to another is independent of the path, is due not to the fact that the force varies inversely as the square of the distance from the attracting center, but to the fact that the attracting center has spherically symmetrical properties. The work done between two points would be independent of the path, even if the force varied inversely as the third power, or any other power, of the distance.

47. The Line-integral of a Distributed Vector. Circuitation (Definitions).—The integrand in Eq. (17) of Sec. 46 consists of the product of an elementary length of the line multiplied by the component of a vector quantity (the force) parallel to the path at the point—the tangential component of the force. Integrals of this kind are called **line-integrals of the vector quantity,** or simply **line-integrals.** Since many of the more important laws and calculations relating to vector fields pertain to two

integrals—(*a*) **line-integrals** of the distributed vector, and (*b*) **surface-integrals** of the distributed vector—we proceed to describe and illustrate in some detail the operations involved in taking a line-integral. (Surface-integrals will be discussed in Sec. 84.)

By a vector field is meant a region in which some vector quantity V has a definite direction and magnitude at every point in the field. A line-integral deals with a **given line** in a vector field. Not only must the line be definitely laid out, but one direction along this line must be arbitrarily specified as the direction in which the line is to be traversed in taking the integral. The specified direction may be indicated by an arrow, and may be referred to either as the **arrow direction,** or as the **specified direction,** or as the AB direction.

The sum which results from the following mathematical operations is called the (value of the) **line-integral of the distributed vector V in the arrow direction along the specified line,** and the process is called the operation of finding the line-integral of the vector V.

Operation 1.—Divide the line into segments each so short that the component of the vector V which is tangential to the segment (namely, $V\cos(V,l)$) has substantially the same value at all points of any one segment. Let dl represent the length of such an elementary segment (see Fig. 12).

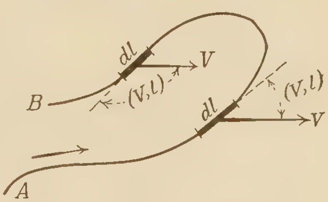


FIG. 12.—Line-integral.

Operation 2.—Compute the value of the product $V\cos(V,l)dl$ for each segment of the line. In this product,

V represents the value of the vector at points of the segment.

(V,l) represents the angle between the direction of the V vector and the specified direction of integration along the segment.

It is to be noted that $\cos(V,l)$ may be positive or negative, depending on whether the angle (V,l) is less than, or greater than, a right angle. The product $V\cos(V,l)$ gives the component of V which is tangential to the line of integration (the tangential component). This product is positive when the tangential

component points in the arrow direction along the line, and negative when it points oppositely.

Operation 3.—Take the algebraic sum S_1 of all the products thus formed. This sum is the (value of the) **line-integral** of the distributed vector V in the specified direction over the specified line.

In those cases in which the value of the **line-integral** may be computed by the methods of the calculus, its value is expressed by the following integral equation:

$$S_1 = \int_{l_A}^{l_B} V \cos (V,l) dl \tag{19}$$

in which

V and (V,l) have been defined above.

l represents distance measured along the path of integration in the specified direction of integration from some arbitrarily chosen origin in the path.

dl represents the increment in this distance as a short segment of the path is traversed in the specified direction.

l_A, the lower limit, and

l_B, the upper limit (namely, the coordinates of the end points of the path), are so taken that the arrow direction is from point A to point B; that is, l_A has the lower algebraic value and l_B the higher.

Illustrative examples of the application of Eq. (19) to specific problems in which the equation of the path of integration is known have been worked out in Sec. 53.

When we think of a body as moving along the line of integration, and regard the driving force which acts on the body at each point of the line as the distributed vector, then the line-integral of the force gives the work done on the body in moving it from one terminus of the path to the other. Another illustration of the application of the line-integral of a vector quantity may be taken from the less familiar vector field obtained by imagining the vectors which represent the velocities of the fluid at innumerable fixed points within a body of moving water. If the velocity is uniform in direction and magnitude, as indicated in Fig. 13, the **line-integral** of the velocity vector around any complete circuit may be seen to be zero. If, however, the velocities are

as indicated in Fig. 14, in which we have represented the velocities in a whirlpool or in a whirling bucket of water the line-integral around any closed paths (circuit) is not zero. The line-integral of velocity around a closed path in a body of water appears to measure the whirlpool effect, or it partly describes the character of the flow.

In the statement of propositions relating to line-integrals, the following term, introduced by Heaviside, will be found very useful.

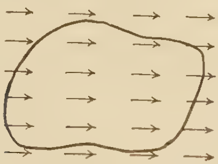


FIG. 13.—Uniform flow in water.

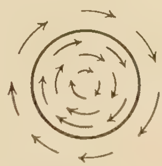


FIG. 14.—Whirling flow in water.

47a. CIRCUITATION OF A VECTOR (DEFINITION).—By the circuitation of a distributed vector is meant the (value of the) line-integral of the vector around a CLOSED curve. By the operation of circuitation is meant the operation of finding the value of this line-integral.

Thus, Proposition 1*A* of the preceding section may be stated in the form,

47b. LINE-INTEGRAL PROPOSITION 1A.—In any field of centrally directed vectors having spherically symmetrical properties, the CIRCUITATION of the vector is zero for all closed loops.

48. The Potential at a Point.—The work done against the forces of the field per unit charge carried by the test body of Sec. 46 is W/Q_t. From Eq. (18) it is

$$\frac{W}{Q_t}\text{ (joules per coulomb)} = \frac{Q}{4\pi p}\left[\frac{1}{r_2} - \frac{1}{r_1}\right]. \tag{20}$$

The work done by the external agent against the electrostatic forces of the field per unit charge upon a test body is seen to depend upon only three things, the magnitude of the charge Q which sets up the field, and the distances from the charge Q of the beginning A and the end B of the path. It is quite independent of the path followed by the charge in traveling from A to B.

Now the expression for the work done against the electrostatic forces takes a very simple form, if the test body is conceived to

start from a point at an infinite distance from the charge Q and to move up to a point whose distance from Q is r. In this case, $1/r_1$ in Eq. (20) is zero, and the expression for the work done against the electrostatic force per unit charge on the test body becomes

$$\frac{W}{Q_t} \text{ (joules per coulomb)} = \frac{Q}{4\pi p r}. \tag{21}$$

That is to say, the work done against the electrostatic force per unit charge carried by a test body which starts from a region so remote that the forces are negligibly small and moves up to any point in the field depends only upon the magnitude of the charge Q and the distance r of the end point of the path from the charge Q.

The quantity (or the quotient) W/Q_t, which appears in Eq. (21) and which represents the work done in bringing a test charge into the field from a region infinitely remote, has a definite numerical value for every point in the field of a given charge or a given distribution of charges. This quantity occurs so often in calculations and discussions pertaining to the work which is done when charges move in an electric field and in calculations pertaining to the potential energy of the field by reason of the location of the charges, that it is a great convenience to have a short name for it. It is called **the (value of) the potential (function) at the point.**[3] This is shortened to **the potential at the point,** and it is invariably represented by the symbol E. The definition of potential will then be as follows.

48a. POTENTIAL AT A POINT. (DEFINITION).—**The POTENTIAL at a point P in an electric field is, defined to be equal to the algebraic value of the work which would be done AGAINST the electrostatic forces of the original distribution of charges giving rise to the field, in bringing a unit positive charge to the point P from a region infinitely remote from the charges.**

Written as an equation, this definition takes the form:

$$E \text{ (volts)} = \frac{W}{Q_t} \frac{\text{(joules)}}{\text{(coulombs)}}. \tag{22}$$

or

$$E \text{ (volts)} = -\int_{l_A}^{l_P} \frac{f}{Q_t} \cos (f, l) dl. \tag{23}$$

[3] The name potential was applied to this point function in 1828 by GEORGE GREEN in his paper *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.*

From this definition, it follows that points at a great distance from the charges giving rise to the field are said to be at zero potential. If positive work has to be done against the electrostatic forces in bringing a positive test charge up to the point (that is, if the test charge must be pushed up by an external agent), the potential at the point is a positive quantity. On the other hand, if negative work is done against the electrostatic forces in bringing the test charge up to a point (that is, if the electrostatic forces will pull the test charge up to the point against resisting forces), the potential of the point is negative. This is the case when the charges of the field are negative, or it is generally the case for points in the immediate vicinity of the large negative charges of the field.

To visualize the properties of an electric field, we may represent, by the equipotential surface diagrams described in Sec. 94, the value of the potential at numerous points throughout the field. We accordingly speak of the potential as a **scalar point function**—a quantity having magnitude but not direction, whose value is a function of the position of the point in the field.

48b. Unit of Potential (DEFINITION).—*The potential at a point is said to be unity (or one volt) if the work done against the electrostatic forces in bringing a positive test charge to the point from an infinitely remote point is one joule per coulomb of test charge. The descriptive name of this unit is the "joule per coulomb," but a short name, the "volt," has been applied to it.*

49. Potential Increase: Potential Difference.—The terms **potential increase, potential difference,** and **potential drop** denote quantities which differ in no respect from **potential,** save that the first three terms are defined in terms of the work done between end points both of which are in the field, while for the fourth term one of the end points is infinitely remote.

49a. POTENTIAL INCREASE (DEFINITION).—**The POTENTIAL INCREASE ALONG A PATH FROM A POINT A TO A POINT B is defined to be equal to the algebraic value of work which would be done AGAINST the electrostatic forces due to the original distribution of charges giving rise to the field, in moving a unit positive charge from A to B along the path.**

The above definition, when written as an equation, takes the form

$$\Delta E(\text{ volts}) = \frac{W}{Q_t}\frac{\text{(joules)}}{\text{(coulombs)}} \tag{24}$$

or

$$\Delta E = -\int_{l_A}^{l_B} \frac{f}{Q_t} \cos(f, l)dl. \tag{25}$$

In this defining equation, W represents the work done against the electrostatic forces of the field as the test charge moves from A to B, and ΔE represents the potential increase from A to B.

It has been shown (Sec. 46) that the work done against the electrostatic forces in carrying a charge from A to B is the same along one path as along any other. Now the path from an infinitely remote point to point B may be made to consist of a path from the remote region to A plus any path from A to B. Obviously, then, the potential increase from A to B is equal to the potential at B minus the potential at A. The potential increase from A to B is an algebraic quantity, that is, it may have a positive or a negative value.

$$\Delta E(A \text{ to } B) = E_B - E_A. \tag{25a}$$

49c. Potential Difference (DEFINITION).—*When it is not necessary to specify which point is at the higher potential, the absolute value of the result of subtracting one potential from another is called the "potential difference" between the two points.*

"Potential drop from A to B" as defined and used in much of the literature is the negative of the potential increase as defined above.

50. Physical Significance of Potential.—The work which a system is capable of doing by reason of a change in the configuration of its parts is called its energy of configuration, or its **potential** energy. There is, therefore, an appropriateness in **potential** as a name for the quantity W/Q_t, since a knowledge of the values of W/Q_t for points in a field enables us to compute the work which would be done by the electrostatic forces if the configuration of the system of charges changes by reason of the movement of a charge from one point to another.

The expression, **the potential at a point,** is frequently misleading to a beginner. It frequently conveys the notion that the potential at a point is something which is inherent in, or whose value is determined by, the conditions peculiar to the region (medium) in the immediate vicinity of the point in question. The impression frequently prevails that the value of the potential at a point may be determined by tests conducted at or in the immediate vicinity of the point. Such is not the case. The value of the potential at a point can only be determined by traveling (in imagination) with a test body from the point in question to a region in which the forces become vanishingly small, in order to determine the work done by the forces during the journey. The operation involved in finding the potential is in striking contrast to that involved in finding the electric intensity. The electric intensity at a point is the force per coulomb which a test charge experiences at the point.[4] It may be determined by tests made at the point. On the other hand, to determine the value of the potential at the point, the values of the force at many points along a line must be determined by test or by calculation, and from these data the work done against these forces must be calculated. The work is calculated by integrating or summing up (as in Sec. 46) the amounts of work done upon unit charge as it moves by small steps along the line.[5]

51. Potential is the Negative of the Line-integral of Electric Intensity.—The potential at a point is the work which would be done against the electrostatic forces of the field in bringing a unit

[4] If we are attempting to account for the forces experienced by bodies in terms of the state of a medium (an ether), we conceive that a statement as to the value of the electric intensity at a point conveys information as to the state of the medium at the point. On the other hand, a statement of the value of the potential gives absolutely no information about the state of the medium **at the point,** but it conveys information as to the state of affairs along a line in the medium.

[5] An inspection of this operation will show that it also has the following significance. Imagine a belt charged with 1 coulomb of positive electricity per centimeter of length to extend from any end point in the field to an infinitely remote point along any path in which it is constrained to move (but without friction). Then the potential at the end point is equal to the force with which the belt must be held in order to keep the electrostatic forces from starting it to slide along the path.

positive charge from a remote region to the point. Now the work done **against** the electrostatic forces in carrying the charge along a path is the negative of the line-integral along the path of the electrostatic force on a unit positive charge. But the electrostatic force on a unit positive charge at a point has been called the electric intensity at the point. Therefore, the following relation exists between potential and electric intensity.

51a. The potential at a point P in the field of specified charges is the negative of the line-integral of the electric intensity along a path extending from a point A which is infinitely remote from the charges to the point P.

$$E \text{ (volts)} = -\int_{l_A}^{l_P} F \cos (F, l) dl, \tag{26}$$

in which

F represents the electric intensity over the elementary length dl, and

(F,l) is the angle between the direction of integration along the length dl and the direction of the electric intensity.

The line-integral relation between potential and electric intensity which is expressed in Eq. (26) is obtained at once by noting that in the defining equation for potential (Eq. (23) the factor f/Q_t is the electric intensity.

In like manner, it follows from Eq. (24) defining potential increase that:

51b. The potential increase from a point A to a point B is the negative of the line-integral of the electric intensity along a path from A to B.

$$\triangle E \text{ (volts)} = -\int_{l_A}^{l_B} F \cos (F,l) dl. \tag{27}$$

52. Potential Gradient and Electric Intensity.—Since potential increase is the negative of the line-integral of electric intensity, then, inversely, the component of the electric intensity along any line is the negative of the rate of increase of the potential along the line. In other words, upon removing the integration sign from Eq. (27) it becomes

$$dE = -F \cos (F,l) dl$$

or

$$F \cos (F,l) = -\frac{dE}{dl}. \tag{28}$$

Now the rate of increase with distance of any specified quantity which is a function of distance is always called the **gradient** of the quantity. Potential gradient, for example, is defined as follows:

52a. (DEFINITION).—**By the POTENTIAL GRADIENT IN A SPECIFIED DIRECTION at a point** P **in an electric field is meant the rate at the point** P **at which the potential INCREASES with distance measured in the specified direction.**

The potential gradient in a given direction at a point, for example, along a line parallel to the X axis, is represented by the symbol $\mathbf{grad}_x\, E$.

$$\text{grad}_x\, E \text{ (volts per cm.)} = \frac{dE}{dx} \text{ (defining grad}_x\, E). \tag{29}$$

From Eqs. (28) and (29),

$$F_x = -\frac{\partial E}{\partial x} = -\text{grad}_x\, E. \tag{30}$$

That is to say,

52b. At a point in the electric field, the component of the electric intensity in a specified direction is the negative of the potential gradient in that direction.

When the specified direction is directly opposite to the direction of the intensity vector at the point, the gradient is a maximum, being equal in value to the electric intensity at the point. To distinguish this maximum value from its components, we may call it the **vector potential gradient,** and may represent it by the symbol **grad E,**—without subscript.

52c. (DEFINITION).—**By the VECTOR POTENTIAL GRADIENT at a point** P **is meant vector drawn in the direction in which the potential INCREASES at the maximum rate, and having a value equal to this maximum rate of increase of potential with distance.**

$$\text{grad } E = \left[\frac{dE}{dl}\right]_{\max} = -\mathbf{F}. \tag{31}$$

This equation is a vector equation; all the previous equations have dealt with the components, and are to be regarded as algebraic equations.

The negative sign appears in Eq. (31), or the potential gradient is a vector oppositely directed to the electric intensity vector, because the potential increases (that is, work is done **against**

the electrostatic forces), when the test charge is moved **against** the electric intensity.

It is evident that the potential gradient in a specified direction (as previously defined) is the component of the vector (grad E) in that direction.

$$\text{grad}_x\, E \text{ is the } X \text{ component of grad } E. \tag{32}$$

52d. Unit of Potential Gradient (DEFINITION).—*The potential gradient at a point is unity if the potential at the point increases with distance at the rate of 1 volt per centimeter. The name of the unit is the "volt per centimeter."*

The manner in which the name for the unit of electric intensity was derived should now be clear. The two quantities electric intensity and potential were defined years before special names were coined for the units in which these quantities are measured. When special names were coined for the units, the name, volt, was assigned to the unit of potential. From this it follows that the name of the unit of potential gradient would be the **volt per centimeter,** and then, since the electric intensity is numerically equal to the potential gradient, the same name was applied to the unit of intensity. These relations are summarized in the following table:

52e. Relation between Potential and Electric Intensity.

Quantity	Definition	Names of the units	
		Descriptive	Specially coined
Potential	Work done on unit charge	Joules per coulomb	Volt
Potential gradient	Work done per cm. on unit charge	Joules per cm. per coulomb	Volts per cm.
Electric intensity	Force on unit charge	Dyne-sevens per coulomb	Volts per cm.

53. Attention to Signs in Taking Line-integrals.—In taking line-integrals, the proper attention to directions and to algebraic signs is a matter of much importance. In the simpler cases, the proper sign for the final result can

often be obtained by an inspection which is independent of the mathematical work, but in the more complicated cases the mathematical work must be depended upon for the correct sign. We will, therefore, illustrate the method of applying Eq. (26) for the potential to a few paths whose equations in a system of rectangular coordinates are known. The paths are shown in Fig. 15. In the following examples the electric field is that of a single concentrated charge of Q coulombs which is at the origin of the system of coordinates. The specified direction of integration along the path of integration, as indicated in the definition and by the limits, is towards the point P whose potential is desired. The limits of integration are from the starting point (lower limit) to the end point P (upper limit). These limits have algebraic signs in any specific problem.

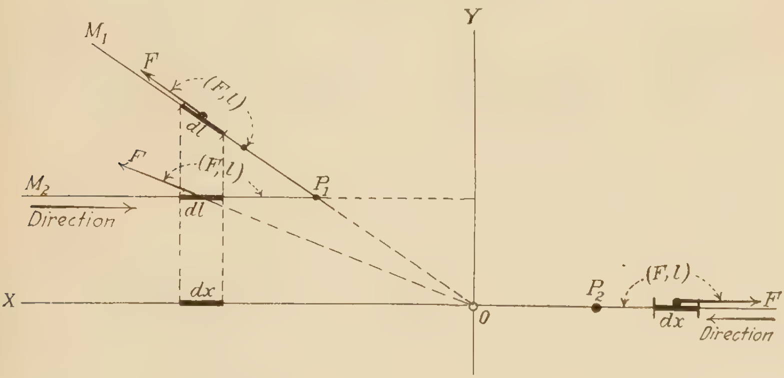


FIG. 15.—Signs in line-integrals.

Example 1.—Let us calculate the potential of a point P_2 on the X axis whose coordinates are $(a, 0)$, by taking the line-integral along the X axis from the right toward the left. Then in the formula

$$E = -\int_{l_A}^{l_P} F \cos(F, l)\, dl \tag{26}$$

$$F = Q/4\pi p x^2$$

$$(F, l) = \pi, \qquad \cos(F, l) = -1$$

$$dl = -dx.$$

Therefore $$E = -\int_{\infty}^{a} \frac{Q(-1)(-dx)}{4\pi p x^2} = \left[\frac{Q}{4\pi p x}\right]_{\infty}^{a} = \frac{Q}{4\pi p a}.$$

Example 2.—Let us calculate the potential of the point P_1 whose coordinates are $(-a, b)$ by taking the line-integral along the straight line M_1P_1 which, when extended, passes through Q. Let us travel to P_1 from a point infinitely remote to the left. Then the positive direction along M_1P_1 is as

indicated by the arrow, and the quantities in Eq. (26) have the following values for all elementary lengths,

$$F = \frac{Q}{4\pi p x^2} \frac{a^2}{(a^2 + b^2)}$$

$$(F, l) = \pi, \qquad \cos (F, l) = -1$$

$$dl = \frac{\sqrt{a^2 + b^2}}{a} dx.$$

Therefore $E_p = -\int_{-\infty}^{-a} \frac{Q}{4\pi p x^2} \cdot \frac{a^2(-1)\sqrt{a^2 + b^2}}{(a^2 + b^2)} \frac{dx}{a}$

$$= \left[\frac{-Q}{4\pi p x} \frac{a}{\sqrt{a^2 + b^2}}\right]_{-\infty}^{-a} = \frac{Q}{4\pi p\sqrt{a^2 + b^2}} = \frac{Q}{4\pi pr},$$

in which $r = \sqrt{a^2 + b^2}$ is the distance from the charge Q to the point P_1.

Example 3.—In like manner, in calculating the potential of P_1 $(= -a,b)$ by taking the line-integral along the line M_2P_1 whose equation is $y = b$, we have

$$F = \frac{Q}{4\pi p\,(x^2 + b^2)}$$

$$\cos (F, l) = \frac{x}{\sqrt{x^2 + b^2}}$$

$$dl = dx$$

$$E_p = -\int_{-\infty}^{-a} \frac{Qxdx}{4\pi p(x^2 + b^2)^{3/2}} = \left[\frac{Q}{4\pi p\sqrt{x^2 + b^2}}\right]_{-\infty}^{-a}$$

$$= \frac{Q}{4\pi p\sqrt{a^2 + b^2}} = \frac{Q}{4\pi pr}$$

54. Calculation of the Potential by Means of the Potential Formula. The Potentializing Operation (DEDUCTION).—The value of the potential at any point in the electric field of charges at rest may be readily calculated by the following method, provided the location and the magnitude of each charge are known.

Case 1. Potential Due to Single Concentrated Charge.—In previous sections, it has been demonstrated that the potential at a point P which is at a distance of r centimeters from a single concentrated charge of Q coulombs is given by the formula

$$E \text{ (volts)} = \frac{Q}{4\pi pr} \frac{\text{(coulombs)}}{\text{(cm.)}}. \qquad (21)$$

Case 2. Potential Due to Two or More Concentrated Charges. The force on a test body due to two or more concentrated

charges coincides with the force which is calculated by assuming that each concentrated charge exerts the same force as if it alone were the cause of the field (Sec. 30). Under these conditions, it may be seen (after sufficient study) that, if the test charge moves from one point to another, the work done by the forces of the field is the algebraic sum of the amounts of work which would be done by the forces of the individual charges. From this it may be seen that the potential at a point is the algebraic sum of the potentials calculated from the individual charges.

That is to say, the potential at a point P which is at the distances r_1, r_2, r_3, etc. from the concentrated charges Q_1, Q_2, Q_3, etc. is given by the expression

$$E \text{ (volts)} = \frac{1}{4\pi p}\left[\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \right] \frac{\text{(coulombs)}}{\text{(cm.)}}. \qquad (33)$$

$$E \text{ (volts)} = \frac{1}{4\pi p} \sum \frac{Q}{r} \qquad \frac{\text{(coulombs)}}{\text{(cm.)}} \qquad (34)$$

Since the work and the potential are scalar quantities, this addition is far simpler to make than is the vector addition which is required in calculating, by means of Eq. (15), the electric intensity at the point.

Case 3. Potential Due to a Distributed Charge.—If the charge giving rise to the field is distributed in a known manner over an extended surface or throughout an extended volume of space, the potential at a point P is calculated as follows. The surface or the volume is divided into elementary portions, each so small that the change it contains may be considered as concentrated at a point. If dq represents any of these elementary portions of the total charge, and if r represents its distance from P, then the potential at P, due to this charge, is $\frac{dq}{4\pi pr}$, and, as in the previous case, the potential at P due to the entire charge is

$$E \text{ (volts)} = \frac{1}{4\pi p} \sum \frac{dq}{r} \quad \frac{\text{(coulombs)}}{\text{(cm.)}}. \qquad (34)$$

If equations can be found which will express the values of both dq and r as functions of the coordinates of the volume or surface containing the charge dq, the summation required in

Eq. (34) may frequently be carried out by integration. Accordingly, Eq. (34) is frequently written in the form

$$E \text{ (volts)} = \frac{1}{4\pi p} \int \frac{dq}{r} \frac{\text{(coulombs)}}{\text{(cm.)}}. \qquad (35)$$

The elementary portions of charge dq appearing in Eq. (35) may be located either upon the surface of conductors or within a certain volume of a non-conducting material. In discussing the distribution of charge, the terms defined below are used.

54a. SURFACE DENSITY OF CHARGE (DEFINITION).—**The surface density of charge (symbol σ) at any point on the surface of a charged body is the quantity of electricity per unit area at that point.**

If q is the number of coulombs on a very small area (a) taken around the point, then the surface density (σ) at the point is

$$\sigma \text{ (coulombs per sq. cm.)} = \frac{q}{a} \text{ (as } a \text{ approaches zero).} \qquad (36)$$

54b. VOLUME DENSITY OF CHARGE (DEFINITION).—**The volume density of charge (symbol ρ) at a given point is the quantity of electricity per unit volume at the point.**

$$\rho \text{ (coulombs per cu. cm.)} = \frac{q}{v} \text{ (as } v \text{ approaches zero).} \qquad (37)$$

Equation (35) for the potential at a point due to a distributed charge may now be written in the form

$$E \text{ (volts)} = \frac{1}{4\pi p} \int_{\substack{\text{over the} \\ \text{charged} \\ \text{surface}}} \frac{\sigma(da)}{r} + \frac{1}{4\pi p} \int_{\substack{\text{throughout} \\ \text{the charged} \\ \text{volume}}} \frac{\rho(dv)}{r}. \qquad (38)$$

54c. Potentializing Operation.—Equations 33, 34, 35, and 38 all call for the same four-step operation, which is called the **potentializing operation,** while the formulas are known as the **potential formulas.**

The **potentializing operation,** or the operation involved in calculating the potential at any point due to a known distribution of electricity, is as follows:

Step 1.—Divide the total charge giving rise to the field into elementary quantities each on a small surface or in a small volume.

Step 2.—Find the distance r of each elementary quantity of charge from the point P whose potential is desired.

Step 3.—Divide each elementary quantity of electricity by its distance r from the point P.

Step 4.—Take the algebraic sum of all the quotients which can be so formed. This sum is the value at P of the potential function due to the charge Q.

$$E \quad \text{or} \quad \text{Pot}\, q = \frac{1}{4\pi p} \sum \frac{dq}{r}. \tag{39}$$

55. The Applications of the Potential Function.—Since electric machines and appliances are devices in which work is done by reason of the motion of electric charges in an electric field, and since the terms "potential," "potential increase," and "potential difference" designate the work done per unit charge, these terms will be constantly used in the study of the properties of machines and of electric circuits.

In the engineering problems outlined at the end of the preceding chapter and in the following chapter on electrostatic appliances, we are concerned largely with the forces which the electrons experience at various points. Many observations must be accounted for, and the precise quantitative answers to many questions must be sought in terms of these forces. Since the force on electrons is determined by the electric intensity, the solution of these problems requires the determination of electric intensities. Now one of the great merits of the concept of the potential function is that in many cases the easiest way to obtain the electric intensity is from the potential function. It is, moreover, impracticable to devise instruments for measuring intensity directly, whereas the measurement of the potential difference between two conductors by means of so-called **voltmeters** turns out to be a measurement which can be made with extreme precision in a standards laboratory and with extreme facility under the conditions of everyday practice.

The remaining sections of this chapter indicate several of the general applications of the potential function. Other applications to machines and appliances will follow in succeeding chapters.

56. The Work Done When Electricity Moves through Electric Fields (Deduction).—Suppose electricity is allowed to flow

from a point P_2 on one conductor along some path to a point P_1 on another conductor, and that P_2 is maintained at a potential E volts higher than P_1. For example, the conductors may be the receptors or terminals of an electrostatic generator. In the steady operation of the machine, electricity is transferred back from P_1 to P_2 **against** the electrostatic forces along some internal path through the generator at the same rate as it flows from P_2 to P_1 **with** the electrostatic forces in the external path, and the potential of each point is maintained constant.

From the manner in which potential increase along a path has been defined, it follows that if the potential increase E along a path remains constant while a total charge of Q coulombs flows over the path in the positive direction, the work done **against** the electrostatic forces in that path is

$$W \text{ (joules)} = QE \text{ (coulombs, volts).} \qquad (40)$$

Along the internal path directed from P_1 to P_2 the work **against** the electrostatic forces is positive. Along the external path directed from P_2 to P_1 it is the same in absolute value but negative, that is, work is done **by** the electrostatic forces.

57. Field Mapping by Equipotential Surfaces.—As an aid in visualizing the features of any specified electric field, we have, in Sec. 39, referred to the possibility of mapping the field by a system of lines and tubes of electric intensity. Another aid, equally powerful, in the visualization of the properties of the field is to map out the field by means of a system of equipotential surfaces in the manner described and illustrated in Sec. 94.

57a. EQUIPOTENTIAL SURFACE (DEFINITION).—**An equipotential surface is the locus of all points at which the potential has a given value.**

58. Calculation of the Electric Intensity from the Potential Function.—The two names, **the dyne-seven per coulomb,** and **the volt per centimeter,** are alternative names for the same unit of electric intensity. By their forms they suggest two quite different ways of calculating the value of the electric intensity at a point P.

The first name suggests the method used in Eq. (15) in the calculation of electric intensities, namely, the calculating of the force which would be experienced by a charged test body per unit charge. This involves the addition of vector quantities, which addition may or may not be difficult.

The second name suggests that the potential, or the expression for the potential, at the point be first calculated, preferably in terms of the coordi-

nates, x, y, and z, of the point P. The potential will be calculated by means of the potential formulas (33), (34), or (38). The X, Y, and Z components of the potential gradient, namely, $\frac{dE}{dx}$, $\frac{dE}{dy}$, and $\frac{dE}{dz}$, may then be calculated, and from Eq. (30) we may set

$$F_x = -\frac{dE}{dx}, \quad F_y = -\frac{dE}{dy}, \quad \text{and} \quad F_z = -\frac{dE}{dz}.$$

From these the value and the direction of the electric intensity may be computed. The value of the intensity is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}. \tag{41}$$

The directional cosines of the vector F with the axes are

$$\cos(F, x) = F_x/F, \quad \cos(F, y) = F_y/F, \quad \text{and} \quad \cos(F, z) = F_z/F. \tag{42}$$

This method of calculating the electric intensity is in many cases far easier to apply than is the previous method of adding many vector quantities.

As an illustration, the charge on the ring of Fig. 10 (Sec. 43) is all at the distance $\sqrt{a^2 + b^2}$ from the point P on the axis. Therefore the potential E of the point P is

$$E = \frac{Q}{4\pi p\sqrt{a^2 + b^2}}.$$

The electric intensity along the axis is

$$F_b = -\frac{dE}{db} = \frac{Q}{4\pi p(a^2 + b^2)^{3/2}},$$

which agrees with the result obtained in Sec. 43.

58a. Exercises.

1. Using Coulomb's law to define the unit quantity of electricity, what, in your estimation, is a very natural definition of the unit of electricity?

2. In the Practical System of Electric Units, what is the unit of length, unit of work, unit of force? The unit of force is equal to a force of how many pounds?

3. Write the "rationalized" formula which expresses Coulomb's law for the force between charged bodies in a vacuum (or air) when all the quantities in the formula are expressed in Practical Units.

4. What is the unit quantity of electricity in the practical system called? From the above formula define the practical unit of electricity.

5. What is meant by the permittivity of a medium? What is the value of the permittivity of a vacuum? What is meant by the relative permittivity of a medium? What is its value for air?

6. Note.—In any problems which deal with charges uniformly distributed over spherical surfaces, the following proposition may be used. (This proposition may be demonstrated as an exercise in elementary integral calculus (see Sec. 90a).)

"The force exerted upon a charge at a point P by a charge Q which is uniformly distributed over a spherical surface is zero if the point P lies

anywhere within the spherical surface; if the point P lies outside the spherical surface, the force exerted upon the charge at P is the same as it would be if the charge Q were all concentrated at the center of the spherical surface."

7. Two small metal balls are placed 20 centimeters apart, center to center. One is charged positively with 10^{-9} coulombs, and the other is charged positively with 8×10^{-10} coulombs. Calculate the force on each sphere. State this force in dyne-sevens, pounds, grams, and dynes.

8. Two small balls each weighing 0.05 gram are suspended from a common point by strings 12 centimeters long. The balls are charged with equal quantities of electricity and the force of repulsion between the charges holds them 6 centimeters apart. Find the charge on each ball.

9. Write the **defining** formula for the electric intensity at a point. Write several **derived** formulas for computing the values of the electric intensities at specified points in fields of specified types.

10. Find the value of the electric intensity (magnitude and direction) at a point 15 centimeters from a concentrated positive charge of 4.5×10^{-10} coulombs.

11. A metal ball 2 centimeters in diameter has a positive charge of 3×10^{-8} coulombs. Draw a curve showing the value of the electric intensity at points along a radial line extending from the center O to a point A 10 centimeters from the center.

12. Two metal balls, A and B, 1 and 3 centimeters in diameter respectively, are placed 80 centimeters apart, center to center, and are each charged with 10^{-9} coulombs. Calculate the electric intensity at the point C midway between them for the two cases: (a) both charges positive; (b) A positive and B negative.

13. An imaginary plane surface is marked off with a rectangular coordinate system using the centimeter as the unit of length. Imagine charges to be located as follows: $+10^{-9}$ coulombs at (0, 10); $+5 \times 10^{-9}$ coulombs at (8, 0); and -10^{-9} coulombs at (-12, 0). Calculate the electric intensity at the origin.

14. Find the potential at a point 15 centimeters distance from a concentrated positive charge of 3×10^{-9} coulombs. Repeat for an equal negative charge.

15. Plot points and draw a curve showing the potential at points along the line OA of exercise 11.

16. Find the potential at point C, exercise 12, for the two cases. Also calculate the potential at the origin, exercise 13.

17. Calculate the potential of ball A in exercise 12 for the case in which A is positively charged and B negatively charged. Calculate the potential of ball B and the potential rise from B to A.

18. Two other balls, C and D, each 4 centimeters in diameter, and both insulated and carrying no charge, are placed in line with the balls of exercise 17, so that the distances, center to center, are as follows: A to C, 140 centimeters; A to D, 180 centimeters; B to C, 60 centimeters; B to D, 100 centimeters. Calculate the approximate potentials of C and D, and the difference of potential between them.

a. Suppose now that C and D are connected by an extremely fine wire (its surface being so small that any charge on the wire is negligible). Compute the induced charges which appear on the spheres C and D, and the new potential of these spheres.

19. Derive a formula for the potential at the point whose coordinates are (x, y) in the plane of exercise 13. Let the charges be located as in exercise 13. Now for the point (x, y) derive the formula giving the components of the potential gradient parallel to the x axis and the y axis respectively. Let the point (x, y) be the origin and compare results with the results obtained in exercise 13.

20. The potential difference between the electrodes of an X-ray tube is 10,000 volts. If an electron starts with zero velocity at the negative electrode, what will be its velocity when it strikes the positive electrode if no collisions occur during its passage from the negative to the positive electrode. If the electron has an initial velocity of 6×10^9 centimeters per second, what will be its velocity upon reaching the positive electrode?

CHAPTER IV

ELECTROSTATIC APPLIANCES

59. This chapter deals with:

a. Electrical machines (electrostatic generators), for the continuous separation and delivery of the two electricities.

b. Electrical condensers or accumulators, arrangements of conductors which permit of the accumulation of the separated electricities in **larger quantities** than is possible on bodies in their customary space dispositions.

c. Electric meters, for measuring quantities of electricity and differences of potential.

These appliances are classed as **electrostatic** appliances because their properties depend upon the forces between electric charges which, taken as a whole and without reference to the constituent electrons, are **at rest** with reference to each other and to the bodies between which the forces are measured. It is true that the charges, as a whole, are in motion in the electrostatic generator, yet the motion is so slow that for all practical purposes the only forces requiring consideration are those forces between charges which are expressed by Coulomb's inverse square law.

60. Methods of Separating the Electricities in Large Quantities.—The two methods of separating the electricities, namely (*a*) by **rubbing contact** and separation; and (*b*) by electrostatic **induction** or **influence,** led to the development of two types of machines for separating the electricities in large quantities. The two types are: (*a*) **rubbing-contact** (frictional) electric machines; and (*b*) **induction** or **influence** machines.

The earliest machines were of the frictional type, the first being von Guericke's machine of 1640, in which the rotating element was a sphere of sulphur. Newton used a rotating glass-sphere machine in 1670. The essential elements of all these frictional machines are illustrated in the plate machine of Fig.

16. Frictional machines were used in all the experiments requiring the separation of considerable quantities of electricity at high differences of potential until they were entirely superseded by the more satisfactory machines of the induction type, which were developed about 1860.

The first appliance of the induction type was the **electrophorus** devised by Volta in 1775. The principal of the **electrophorus** was embodied in rotating machines by Nicholson in 1788 and by Belli in 1831, but these machines did not come into use. The first induction machines to come into general use were the rotating glass-plate machines invented between 1860 and 1883 by Voss, Holtz, Toepler, and Wimshurst.

61. Rubbing-contact or Frictional Electric Machines.—The rotating element of the machines of the frictional type assumed the form of a sphere, a cylinder, or a plate of insulating material,

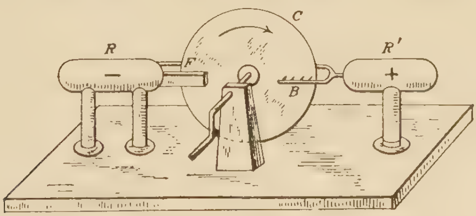


FIG. 16.—Frictional electric machine.

generally of glass. The essential elements of all these forms are illustrated in the plate machine of Fig. 16. It comprises:

1. A rotating glass plate C mounted on a horizontal axis.
2. Rubbing pads F.
3. Receptors R and R'.
4. Collecting brushes or combs B.

The rubbing pads F were generally in the form of leather-faced cushions, the surface of which was smeared with an amalgam of mercury, tin, and zinc, mixed with lard. The two surfaces of the rotating plate as they leave the pads are highly charged with positive electricity and the pads are negatively charged. The negative charge flows to ground if the pads are grounded, or it charges the negative **receptor** or **prime conductor** R if this conductor is not grounded. The positively charged

surfaces of the glass plate pass between the sharp points of the metal combs B'. A negative charge is induced on the points, and by the brush discharge a negative charge passes from the points to the plate, thereby partly neutralizing the charge on the plate and leaving the receptor or prime conductor R' positively charged. Electricity continues to accumulate on the two receptors and the potential difference between the two continues to increase until the rate at which the electricity leaks off by brush discharge and by surface leakage just balances the rate at which it is added to the receptors. As previously stated, these frictional machines have been entirely superseded by induction machines.

62. The Electrophorus.—The electrophorus, devised by Volta in 1775, is useful for the production of a series of small charges of approximately equal magnitude. It consists of a plate I (Fig. 17), of resin or of hard rubber, generally mounted for mechanical strength on a metal plate S, called the sole plate, and a metal carrier disk C provided with an insulating handle H. The results are the same whether the sole plate rests upon insulating pillars or is connected to earth. The sequence of the operations in obtaining a series of charges from the electrophorus is as follows:

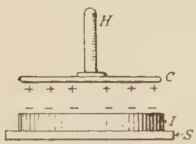


FIG. 17.—Volta's electrophorus.

Electrifying Operation. Separation of the Electricities by Rubbing Contact.—The upper surface of the resin inductor I is electrified (negatively) by beating it with fur or flannel. If the sole plate S is insulated from earth, care should be taken in electrifying the resin to keep from communicating the positive charge of the fur to the sole plate.

Charging Operation. Separation of the Electricities by Induction and Conduction.—The metal carrier C is placed upon the resin and is momentarily connected to earth by touching it with the finger. Since the metal and resin surfaces touch at only a few points and since the resin is such an excellent non-conductor, only those parts of the resin are discharged which are extremely close to the few points of contact of the two surfaces. The charge on the remaining portions of the resin acts to induce a

positive charge on the carrier disk C, approximately equal in magnitude to the negative charge on the resin; that is to say, negative electricity passes to earth from the disk while it is momentarily connected to earth.

Carrying Operation. Separation of the Electricities by Moving Parts.—The metal carier C with its positive charge may now be **pulled** away from the negatively charged resin by its insulating handle H, and its positive charge may be completely imparted to any hollow conducting vessel by inserting C within the vessel and touching it to the vessel.

The uncharged carrier disk may then be removed from the hollow vessel and any number of nearly equal charges may be obtained in succession by induction from the original electrification of the resin by merely repeating the **charging** and **carrying** operations. Of course the negative charge on the resin gradually decreases in magnitude, due to the continued slow leakage to ground and to the intermittent leakage to the carrier disk during the brief intervals the carrier rests upon the resin. The source of the energy which is being stored in an electropotential form, while the positive charge on the carrier is being pulled away from the negative charge on the resin and inserted in the hollow vessel, is the muscular energy expended in moving the charge against the forces of the field.

All induction machines are appliances for rapidly and automatically carrying out the above charging and carrying operations by means of a rotating carrier. The machines have the additional feature of automatically generating the **inducing** charge. Before describing a rotating-plate machine embodying these automatic features, we will describe **a ball and can** electric doubler and a **water-dropper** induction generator. These are described mainly because they illustrate the features of the induction generator in such a striking fashion that the principles to be used in calculating the power output and power limitations of induction generators are easier to grasp.

63. Ball and Can Electric Doubler.—We venture to call the device described below an **electric doubler** because it is similar in principle to an induction device invented (by Bennett in 1786) in a form which made it possible approximately to double,

quadruple, octuple, etc. a given charge by repeating a certain cycle of operations once, twice, three times, etc. These early devices were accordingly called **electric doublers.** The **ball and can doubler** consists of two metal balls (about 5 centimeters in diameter, each provided with a long glass handle, and two open-topped metal cans, each mounted upon a glass beaker for insulation. Thin-walled aluminum tubes (or pith balls) are suspended against the outer sides of the cans to serve as electroscopes. These parts are shown in Fig. 18*a*, one ball being drawn larger than the other in order to distinguish between the two. The sequence of operations in building up a charge on the cans

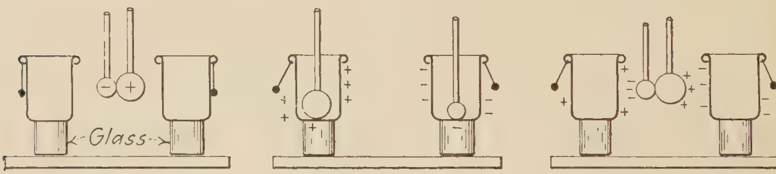


FIG. 18.—Ball and can doubler.

is as follows: (The glass handles and glass beakers must be clean and dry; the balls must be handled by grasping the glass handles at the extreme end.)

Preliminary Contact Electrification.—Momentarily bring the two balls into contact in the space between the cans, as in Fig. 18*a*. Let us assume that the cans and balls are initially uncharged, but that by contact the left ball acquires a slight negative charge and the right an equal positive charge, as indicated in Fig. 18*a*.

Regular Carrying Operation.—Separate the balls and momentarily touch the ball **carriers** to the inside of the can **receptors** as shown in Fig. 18*b*. (Note that the right ball of Fig. 18*a* is now in the left can, and the left ball in the right can.) The balls give up their charges to the cans, and after contact the cans have extremely small charges of the signs shown in Fig. 18*b*.[1]

[1] It will be noted that electrification by contact, not only between balls but also between balls and cans, plays an essential rôle in bringing about the initial charges on the cans. As the charges gradually build up from the combined effect of electrification by contact and by induction, the latter becomes the all-important phenomenon in the operation of the doubler.

Regular Charging Operation.—Momentarily bring the balls into contact in the space between the charged can **inductors**, as in Fig. 18*c*. While the balls are in contact, the slight charges on the cans **induce** a slight separation of the electricities between the two balls of the signs shown in Fig. 18*c*.[1]

If these regular charging and carrying operations are rapidly repeated, the electroscopic indicators will start to diverge from the cans after from 20 to 50 repetitions. Further repetitions will then build up the charge on the cans and the difference of potential between the cans at a rapidly increasing rate. This is indicated by the rapid rate at which the divergence of the electroscopic indicators increases. Finally a point is reached at which no further increase of charge or of potential difference is possible, because the charge leaks off over the glass and by brush discharge as rapidly as it is added to the receptors.

64. The Water-dropper Induction Generator.—A water-dropper induction generator devised by Kelvin in 1867 is illustrated in Fig. 19. Water from a reservoir *S*, or directly from the water mains, is conveyed by pipes which terminate near the centers of each of the two insulated cylindrical metal **inductors** *I*. Each pipe terminates in a rounded cloth plug through which the water seeps at such a rate that the falling streams break into drops before emerging from the inductors. If by any agency the inductors receive opposite charges of the signs shown in the figure, the falling streams by induction become charged as indicated. The charged drops fall on cloth diaphragms in the hollow **receptors** *R*, give up their charges to these hollow bodies, and then drip as uncharged drops from the ends of the funnels within the receptors. As shown in the figure, the receptors are so cross-connected to the inductors that the charge conveyed by the falling drops of water adds to the original charge on the inductors. If the receptors and inductors are well insulated, the

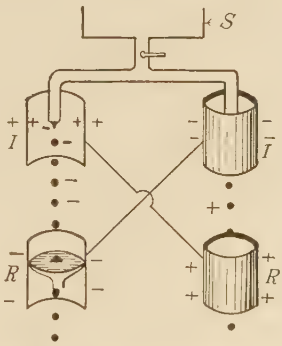


Fig. 19.—Water-dropper generator.

charge and the potential build up as exponential functions of time until finally the forces of the field become strong enough to divert the falling drops from their course into the receptors. The source of the energy which is being stored in the electropotential form as the charges build up is evident; it is the work done by the gravitational forces against the attractive forces of the inductors and the repulsive forces of the receptors for the charged water-drop carriers.

In this device the functions of the essential elements of induction machines are clearly evident. These elements may now be listed as:

1. Inductors, for inducing charges on the carriers.
2. Carriers, for conveying charges to the receptors.
3. Receptors, for receiving the separated charges.
4. Charging contactors, for conveying the charges from earth to the carriers.
5. Collecting contactors, for delivering the charges from the carriers to the receptors.
6. Replenisher contactors, for imparting a part of the separated charge to the inductors, thereby increasing their small initial charge.
7. Framework, for supporting and insulating the above elements and permitting of the necessary carrier movements.

In the rotating-plate machines described below, the charging, collecting, and replenisher contactors take the form of small brushes with fine wire bristles or of sharp-pointed combs which make electrical contact with the moving parts through the brush discharge from the points.

65. Induction Machines of the Rotating-plate Type.—The only induction generators in use at the present time are in the form of rotating-plate machines. These machines were devised in various forms in the period between 1860 and 1883 by Varley, Toepler, Holtz, Kelvin, Voss, and Wimshurst. They are all identical in general principle with the Nicholson doubler of 1788. We will describe only one of these forms, that frequently known as the Toepler-Holtz machine.

The Toepler-Holtz machine consists of two circular plates of varnished glass or of vulcanite, as illustrated in Figs. 20 and 21.

The larger plate L is stationary, and the smaller plate S (30 to 60 centimeters in diameter) is mounted on a horizontal axis so that it may be rotated at high speed in a plane parallel to, and about 1 centimeter distant from, the large plate.

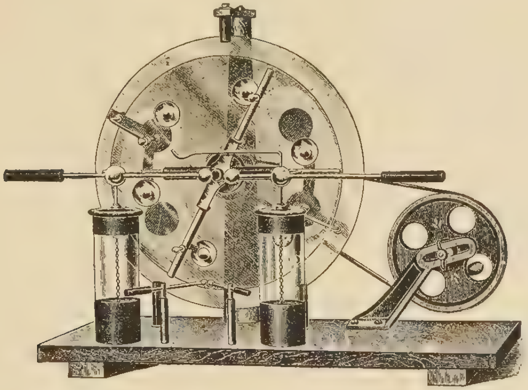


FIG. 20.—Toepler-Holtz machine.

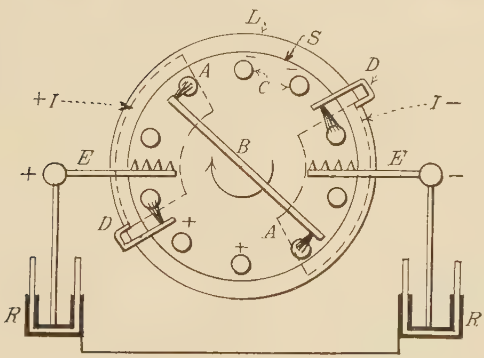


FIG. 21.—Toepler-Holtz machine.

The two inductors I, I are segments of tin foil which are cemented to the large plate, generally on its outside face.

The carriers C are small tin-foil disks with projecting metallic buttons cemented at regular intervals to the outside face of the rotating disk.

A, A represent the two charging contactors in the form of small wire brushes connected together through the metal arm B.

D, D represent the two replenisher contactors in the form of wire brushes mounted on the stationary plate and in contact with the inductors I, I.

E, E represent the two collecting contactors in the form of sharp-pointed combs which by brush discharge collect charge from the moving carriers and deliver it to the receptors in the form of Leyden jars R, R.

If we assume that the plate is rotating in the direction shown by the arrow and that the inductors have acquired charges of the signs indicated, then the machine will function as follows: By induction and by conduction along the bar B, the carriers acquire charges of the signs indicated as they pass under the charging brushes AA while they are still in the field of the inductors. They are carried out of the field of the previous inductor into that of the second. While in the field of the second inductor, they first pass under the replenisher brush D and give up part of their charge to it, thereby adding to the charge on the inductors. They then pass under the collecting comb E and by brush discharge give up a part of their charge to the receptors, which thereby become charged with electricity of the sign shown. They finally pass under the second charging brush, acquire a charge of the opposite sign to that carried during the half cycle above described, and then start on the second half cycle of the complete revolution.

66. Output and Limitations of Electrostatic Induction Generators.—At the present time, the main application of the electrostatic generator is to supply the small quantities of electricity (or small continuous currents) at the **high differences of potential** required in some experimental work. By running machines of the above type having vulcanite plates about 50 centimeters in diameter at peripheral speeds of the order of 3 kilometers per minute, it is possible to separate electricity at the rate of 0.00015 coulomb per second per rotating plate, with a potential difference of 120,000 volts between receptors. This is an energy output of 18 joules in each second, or a power output of 18 watts. This output is supplied at an efficiency of the order of 15 per cent, that is, to obtain the power output of 18 watts, it requires a power expenditure of about 120 watts to drive the machine.

This question is frequently raised: Since the difference of potential at which electricity is delivered from electrostatic generators is far higher than it is feasible to obtain in electromagnetic machines, why is it that the power output of electrostatic generators is such a small fractional part of that

obtainable from electromagnetic generators of equivalent cubical contents? The immediate answer is that the quantity of electricity delivered per second (the current) from electrostatic machines is very small as compared with the current from the electromagnetic generator. The question remains, Why is the quantity of electricity per second so limited in the one case? Is it not possible (as frequently proposed) to get large quantities of electricity at high differences of potential by using a high-velocity blast to blow charged particles through a glass tube in a machine of the water-dropper type? It is not feasible to get large quantities per second through an electrostatic generator at high efficiency for the following reason: Let us imagine a cylindrical stream of charged particles moving through a glass tube in a generator of the water-dropper type, the stream being 1 centimeter in diameter. We will find in the following chapter that this stream must contain a charge of less than 10^{-7} coulombs per linear centimeter. A greater charge than this would give rise to electric intensities from the stream to the inductor which would puncture the glass tube and render the machine inoperative. Now if this stream were to move with the extremely high velocity of 1 kilometer per second, it would mean the delivery of only 0.01 coulomb per second. With a difference of potential of 200,000 volts, this would be a power output of only 2 kilowatts. This power output would be obtained at an extremely low efficiency. To obtain high efficiency in the water-dropper generator, the repulsive forces must be large enough to bring the carriers substantially to rest as they enter the receptor.

In the electromagnetic generator, by forcing a wire 1 centimeter in diameter through a magnetic field, it is possible to force electrons to move through the wire at a rate 10,000 times as great as the 0.01 coulomb per second. The outward electric intensities from the wire may be less than one one-thousandth as great as from the moving stream in the water-dropper type. The inherent difference between the two types is this: In the electrostatic generator the only way to move one electricity without the other is to separate them and to put them on different carriers; the limitation arises from the fact that if very much charge is placed on a carrier the electric intensity becomes high enough to break down the surrounding insulation. In the magnetoelectric generator, by pushing a wire broadside on through a magnetic field, we have a method of forcing the free electrons to move along in the interstices between the atomic structures from which they are for the moment free. In other words, we obtain a differential movement of the electricities without segregating them on different carriers in different parts of space.

67. Electrostatic Voltmeters or Electrometers.—By an **electrometer,** or an **electrostatic voltmeter,** is meant an instrument in which the attractive and repulsive forces between the charged fixed and movable elements of the instrument are utilized to measure the difference in the potentials of the elements, and of any bodies to which these elements may be connected. The

term **electrometer** is now applied to only two types—the absolute or guard-ring type, and the fiber suspension quadrant type of high sensibility. The term **electrostatic voltmeter** is generally applied to the more rugged types designed for measuring differences of potential in excess of 10 volts. A gold-leaf electroscope may be converted into an electrometer by suspending the gold leaf in any definite position in a **metallic** housing, and providing a scale against which the deflection of the gold leaf may be measured. To measure the difference of potential between two bodies, the housing is connected to one body and the gold leaf to the other. The absolute and the quadrant electrometers, and the multicellular and vertical vane electrostatic voltmeters, were all developed by Kelvin, the first two in the period 1855–1860.[2] The absolute electrometer is a refinement of the attracted disk electrometer used by Snow-Harris in 1834.

68. The Absolute or Guard-ring Electrometer.—The guard-ring electrometer is illustrated in Fig. 22. It consists of two circular metal plates mounted with their parallel faces a short distance apart in two horizontal planes. The lower plate is in one piece and is mounted on a micrometer screw S so that its distance from the upper plate may be set at any value. The upper plate is in two parts—a movable central disk in the form of a circular **trap door** which is suspended from one end of a balance, and a fixed circular **guard ring** which encircles, but is separated from, the trap door by a narrow circular slit. The trap door and its guard ring are in electrical contact.

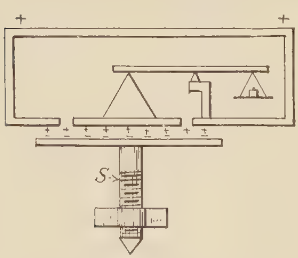


FIG. 22.—Guard-ring electrometer.

To measure the difference of potential between two bodies, the bodies are connected by wires to the upper and lower plates, respectively, and the force with which the suspended disk is attracted is measured on the balance. In the following chapter

[2] For an excellent account of the refinements in the development and use of these instruments, see the paper on "Electrometers and Electrostatic Measurements," pp. 263–314 of WILLIAM THOMSON'S *Papers on Electrostatics and Magnetism.*

it will be shown that the surface density of the charge acquired by the opposing faces of the plates will be practically uniform over the entire central portion occupied by the trap door. It will be shown (Sec. 88) that under these conditions the electric intensities at points in the space between the plates are perpendicular to the planes of the plates and are uniform in value from one plate to the other. The value of the electric intensity will be shown to be:

$$F \text{ (volts per cm.)} = \frac{\sigma}{p}, \tag{43}$$

in which

σ represents the surface density of charge (in coulombs per square centimeter) over the suspended disk.

p represents the permittivity of the medium separating the plates.

$$p = 8.85 \times 10^{-14} \text{ for air.}$$

One-half of this value is contributed by the positive charges on the upper plate and the other half by the negative charges on the lower.

If a represents the area of the lower face of the suspended disk (reckoned to the middle of the slit), the charge on it is:

$$Q = a\sigma$$

and since this charge is in a field of intensity $\sigma/2p$, it follows that the force of attraction on the suspended disk is

$$f \text{ (dyne-sevens)} = \frac{a\sigma^2}{2p} = \frac{Q^2}{2ap}. \tag{44}$$

If the distance between the parallel plates is b centimeters, the difference of potential E between the disks is:

$$E \text{ (volts)} = \frac{b\sigma}{p}. \tag{45}$$

By eliminating σ between Eqs. (44) and (45), the force of attraction may be expressed in terms of the difference of potential, thus:

$$f \text{ (dyne-sevens)} = \frac{apE^2}{2b^2} \text{ (volts, cm.)}. \tag{46}$$

From Eqs. (44) and (46) it will be seen that both the quantity of electricity on the suspended disk and the difference of potential

between the plates may be computed from the measured pull on the disk and the dimensions of the instrument. The instrument, therefore, may be regarded as an **absolute** coulombmeter or as an **absolute** voltmeter, because it enables us to measure these quantities directly by means of their so-called absolute definitions in terms of the fundamental standards of length, time, and mass. While the guard-ring electrometer is primarily a coulombmeter, it is generally regarded as a voltmeter and is so used. At low differences of potential the forces are very small, and the instrument cannot be regarded as an instrument of precision for the measurement of potential differences much lower than 200 volts. The following is an example of the magnitude of the quantities. With a difference of potential of 1000 volts and a separation of 0.15 centimeters between plates, the pull on a circular disk 16 centimeters in diameter is 3.95×10^{-4} dyne-sevens, or 4.03 grams ($g = 981$).

69. The Quadrant Electrometer.—The quadrant electrometer (Figs. 23 and 24), was devised by Kelvin for measuring extremely

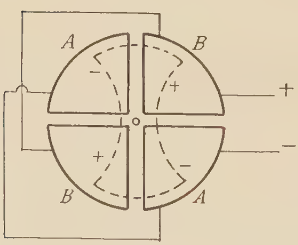


FIG. 23.—Quadrant electrometer.

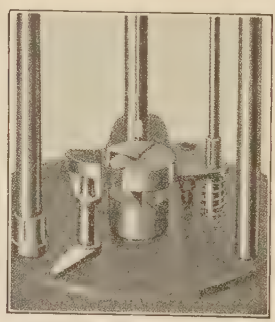


FIG. 24.—Electric system of quadrant electrometer with one quadrant pushed aside.

small differences of potential. The fixed element is a hollow, flat, circular, metallic box, which has been divided into four quadrants by narrow saw slots along two diameters. The quadrants are insulated, and alternate quadrants are electrically connected. The movable element is a very light figure-of-eight-shaped vane of aluminum foil, attached at its center point to a wire stem at right angles to the plane of the vane. This hori-

cental vane is suspended by the stem within the box from a long, fine (quartz) fiber. The fiber has sufficient torsional rigidity to hold the vane in a position which is symmetrical with respect to the A and B pairs of quadrants when there are no electrical forces acting on the vane. The angle through which the vane is turned from this zero position by the electrical forces is indicated by the deflection of a beam of light, which, after reflection from a small mirror mounted on the stem of the vane, falls on a graduated scale about 1 meter distant from the mirror. Electrical contact is made with the vane, either through the suspension or through a fine wire extension of the stem which passes down through the box and dips into a cup of conducting liquid.

If large differences of potential are to be measured, the instrument may be used as an **idiostatic** instrument (see Sec. 70) in the following manner: Let the vane be electrically connected to one pair of quadrants (say the A pair) and let the two pairs of quadrants be connected to the respective points between which the difference of potential is to be measured. The vane and the A quadrants acquire a charge of one sign (say $+$) and the other quadrants acquire a charge of the opposite sign. As a result of the forces between the charges on the vane and the quadrants, the vane is subject to a couple which deflects it toward the quadrants to which it is not connected. The theory of the instrument indicates that, **for small angles of deflection,** the torque of the electrical forces is directly proportional to the square of the difference of potential E between the quadrants, and that the relation between the angular deflection θ of the vane and the difference of potential is expressed by the equation

$$\theta = KE^2, \tag{47}$$

in which K is a constant of the instrument whose value is to be experimentally determined.

The more usual method of using the quadrant electrometer is as an **heterostatic** instrument, as described in Sec. 70. The latest forms of these instruments when used as heterostatic instruments with an independent voltage of 100 volts applied between one pair of quadrants and the vane, give a deflection of 1 millimeter on a scale 1 meter distant for a difference of

potential between quadrants of only 0.0005 volt. (These instruments have a period of about 5 seconds and a capacitance of only 1.3×10^{-11} farads.)

70. Idiostatic versus Heterostatic Instruments.—The electrostatic instruments described above may be used in one of two ways:

1. As **idiostatic** instruments (the Greek "idios," individual), in which the forces depend only upon the **one** difference of potential which is being measured.

2. As **heterostatic** instruments (the Greek "heteros," other), in which **another** difference of potential, independently maintained, is used to increase vastly the forces and the sensibility of the instrument.

The method of connecting and of using the instruments as idiostatic instruments has been described in Secs. 68 and 69. As idiostatic instruments, the relation between the force on the moving member and the difference of potential is given by the equation

$$f = KE^2 \tag{47}$$

The method of using the guard-ring electrometer as a heterostatic instrument to measure a small, unknown difference of potential e is as follows: The source of the small difference is connected in series with a source of large

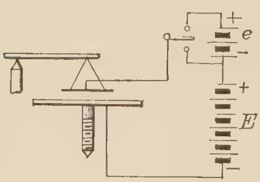


FIG. 25.

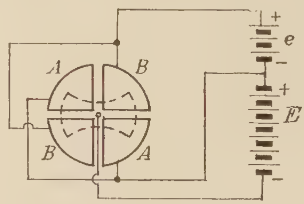


FIG. 26.

Connections for heterostatic use.

unvarying difference of potential of known value E, as shown in Fig. 25. Two readings of the force on the disk are taken, one with the voltage $E + e$ between the plates, and the other with the voltage E. The forces will be:

$$\begin{aligned} f_1 &= K(E + e)^2 = K(E^2 + 2Ee + e^2) \\ f_2 &= KE^2 \\ f_1 - f_2 &= K(2Ee + e^2) \end{aligned}$$

The force due to the small potential e alone would have been $f_3 = Ke^2$.

Thus, by using the heterostatic method, the force to be measured, namely, $f_1 - f_2$, is greater than the force f_3 in the ratio of $\frac{2E}{e} + 1$ to 1.

To use the quadrant electrometer as a heterostatic instrument, it is connected as shown in Fig. 26. The two sources of potential are connected in series; the quadrants are connected across the small unknown potential e and the vane is connected to the outer terminal of the source of known

potential E. The torques τ_1 and τ_2 on the vane in the directions of the A and B quadrants, respectively, are:

$$\tau_1 = K(E + e)^2 = K(E^2 + 2Ee + e^2)$$
$$\tau_2 = K(E)^2 = KE^2$$
$$\tau_1 - \tau_2 = K(2Ee + e^2)$$

With the idiostatic connection, the torque would have been

$$\tau_3 = Ke^2.$$

Hence by the heterostatic connection, the resultant torque on the vane has been increased in the ratio of $\frac{2E}{e} + 1$ to 1. For example, if $E = 100$ volts and $e = 0.0005$ volt, the torque and the sensibility of the instrument have been increased in the ratio of substantially 200,001 to 1.

Since the deflection is proportional to the torque, the relation between the deflection and the unknown difference of potential is

$$\theta = K_1(2Ee + e^2)$$

or

$$\theta = K_2e \text{ (substantially if } \frac{E}{e} > 100). \tag{48}$$

That is, the deflection is proportional to the first power of the unknown potential e, provided the heteropotential E is many times greater than e.

71. Commercial Electrostatic Voltmeters.—As previously stated, the more rugged types of commercial electrostatic instruments for measuring

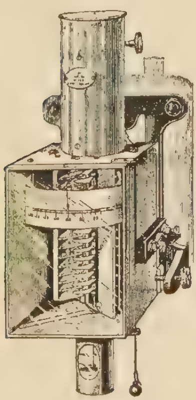


Fig. 27.—Kelvin multicellular voltmeter.

Fig. 28.—Kelvin vertical vane voltmeter.

differences of potential in excess of 10 volts are called electrostatic voltmeters.

The Kelvin **multicellular** type, illustrated in Fig. 27, consists of a number of parallel vanes (10 or more) mounted on the same vertical stem and each swinging between its own quadrants. The pointer consists of an aluminum

needle moving in front of a graduated scale. By increasing the number of vanes the sensibility of the instrument is increased, so that full-scale deflection of the idiostatic instrument is obtained with differences of potential as low as 80 volts.

In the Kelvin **vertical-vane** type, illustrated in Fig. 28, a slightly unbalanced aluminum vane is mounted in a vertical plane on knife-edges. By the electrostatic forces the vane is drawn within a single pair of vertical quadrants. A pointer attached to the vane moves in front of an experimentally calibrated scale. This type is made to measure voltages between 1000 and 20,000 volts.

FIG. 29.—Mechanism of oil immersed voltmeter.

Figure 29 illustrates the mechanism of a type used to measure alternating differences of potential up to 250,000 volts. The moving and fixed elements are immersed in a grade of oil having a dielectric strength about eight times as great as air.

72. The Relation between the Potential Difference between Two Conductors and the Quantity of Electricity Which Has Been Transferred from One to the Other (EXP. DET. REL.).—Imagine any two **insulated** conductors A and B **in specified surroundings.** These conductors and a few features of the surroundings are crudely illustrated in Fig. 30. A and B may represent the two insulated cans of the "ball and can electric doubler."

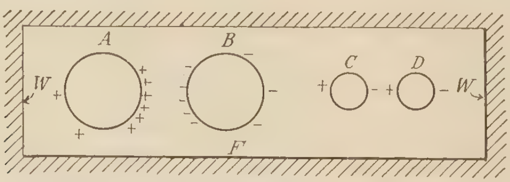


FIG. 30.—Conductors having capacitance.

In this case F and W may represent the conducting floor and the walls of a room, and C and D may represent two other insulated conducting bodies within the room. Or A and B may represent (in cross-section) the two insulated wires of a power line each of which is 100 kilometers in length. In this case, C and D may represent two other insulated wires, as the wires of a telephone circuit on the opposite side of the highway. F will then represent the surface of the earth, and W should be removed.

Suppose electric charges (electrons) are **transferred** from A to B, charging A positively and B negatively. The electrons may be transferred from A to B across the intervening space on carriers, as in the ball-and-can electric doubler; or they may be transferred from A to B by conduction through a wire containing a plate machine, a voltaic battery, or an electromagnetic generator. We are not concerned with the manner in which the charge is transferred, but simply with **the state of affairs which results after the transfer of a quantity** (q). Imagine that the charge is transferred in small amounts, and that the potential differences e between A and B corresponding to gradually increasing quantities of electricity q on A or B are experimentally determined. If the potential differences e are plotted against the quantities q, the relation is found to be a straight-line relation, as illustrated in Fig. 31.[3] This relation may be expressed as follows:

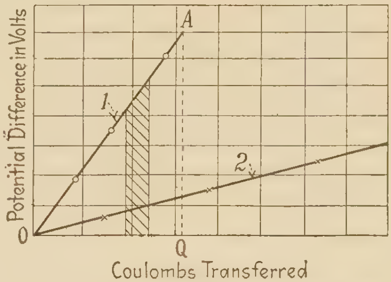


FIG. 31.—Relation between the charge transferred and the resulting potential difference.

[3] If we accept the principle that the force effects of elementary charges may be superposed in the linear manner stated in Sec. 30, this straight-line relation may be deduced by the following argument: Imagine that a charge q has been transferred from A to B and that the charges on A and B and all surrounding conductors have assumed the distribution resulting in equilibrium. Corresponding to this distribution, we conceive of a definite set of electric intensities and a definite force-integral or potential difference between A and B. Now imagine a second distribution in which all charges are n times as great. This will be an equilibrium distribution because any element of charge which was formerly in equilibrium will still be in equilibrium, since all intensities are exactly n times as great and will again balance out. Since the electric intensities are all n times as great, the potential difference between A and B will be n times as great as with the first distribution. That is, the potential difference e is directly proportional to the transferred quantity of electricity q. At this early stage in the development of electrical theory, the student ought to regard the principle of linear superposition for electrostatic fields as an assumption not rigorously demonstrated. The determination by experiment of the straight-line relations given in Eqs. (49a) and (50a) is a part of the evidence justifying the principle.

72a. RELATION BETWEEN THE QUANTITY TRANSFERRED AND THE RESULTING POTENTIAL DIFFERENCE (EXP. DET. REL.).—**When electricity is transferred from one insulated conductor A to another B, the resulting potential difference between A and B is directly proportional to the quantity transferred.**

The straight-line relation between the two variables is expressed by either of the following equations:

$$q = Ce \tag{49a}$$

$$e = Sq \tag{50a}$$

73. Capacitance, or Permittance, and Elastance.—The proportionality constants, C and S, between the potential difference and the quantity of electricity, which appear in Eqs. (49a) and (50a), are used so frequently that names have been coined for them. They are called the **capacitance** and the **elastance** of the conductor A relative to the conductor B. These terms may be defined as follows:

73a. CAPACITANCE, OR PERMITTANCE[4] (DEFINITION).—**By the CAPACITANCE or PERMITTANCE of one insulated conductor, A, with respect to another, B, is meant the constant ratio between the quantity of electricity transferred from A to B and the potential difference between A and B which results. Capacitance is invariably represented by the symbol C.**

$$C\ (\text{farads}) = \frac{q\ (\text{coulombs})}{e\ \ (\text{volts})}. \tag{51}$$

73b. FARAD (DEFINITION).—**The unit of capacitance is named the FARAD.**[5] **Two conductors are said to have a capacitance of 1 farad if a potential difference between the conductors of 1 volt requires the transfer of 1 coulomb.**

With the unit defined as above, it follows that the capacitance of two conductors relative to each other is numerically equal to

[4] In the older texts, the quantity, which is herein called the **capacitance**, is called the **capacity**. The new term is recommended by the American Institute of Electrical Engineers. The use of the term **capacity** is objectionable because it contributes to the erroneous notion that the capacitance or capacity of one conductor to another is analogous to the capacity of a bucket to hold water.

[5] As a tribute to Faraday, the unit of capacitance was named the **farad**. The farad is such a large unit that it is customary to express capacitances in microfarads (μ.f).

$$1 \text{ farad} = 10^6\ \mu\text{.f.}$$

the number of coulombs which must be transferred to cause a potential difference of 1 volt between the conductors.

73c. ELASTANCE (Definition).—By the ELASTANCE of one insulated conductor, A with respect to another, B, is meant the ratio of the potential difference between A and B to the quantity of electricity which has been transferred. Elastance is represented by the symbol S.

$$S \text{ (darafs)} = \frac{e}{q} \quad \frac{\text{(volts)}}{\text{(coulombs)}}. \tag{52}$$

73d. Daraf (Definition).—The unit of elastance is named the DARAF.[6] Two conductors have an elastance of 1 daraf if the transfer of 1 coulomb causes a potential difference between the conductors of 1 volt.

Equations (49a) and (50a) may now be rewritten with all the units specifically named.

$$q \text{ (coulombs)} = Ce \quad \text{(farads, volts)} \tag{49}$$

$$e \text{ (volts)} = Sq \quad \text{(darafs, coulombs)} \tag{50}$$

It is evident that the elastance is the reciprocal of the capacitance.

$$S \text{ (darafs)} = \frac{1}{C} \; \frac{1}{\text{farads}}. \tag{53}$$

74. Relation between the Capacitance and the Dimensions of Two Conductors.—Let us form some general ideas as to the effect of the dimensions and geometrical shapes of two conductors A and B upon their capacitance and elastance. The elastance of A to B is numerically equal to the potential difference between A and B which results from the transfer of 1 coulomb of electricity from A to B. The difference in potential, in turn, is the work which would be done by the forces of the field per unit charge upon a small positively charged test body which is imagined to move from A to B (after the coulomb has been transferred). To form definite ideas, therefore, as to the elastance of two conductors, the following calculations or estimates must be carried on:

1. Imagine that 1 coulomb of electricity has been transferred from A to B, charging A positively and B negatively.

2. Determine the manner in which the positive and negative charges are distributed over the surface of the two conductors. If other conducting bodies are in the field, determine the distribution of the charges which are induced on their surfaces.

3. Pick out the path from A to B along which it is easiest to visualize the magnitude of the electric intensities due to the above charge distributions. Calculate the electric intensities at many points along this chosen path.

[6] **Daraf** was coined by reversing the spelling of farad.

4. Imagine a small, charged test body to move along this chosen path and calculate the work done by the forces per unit of charge on the test body. This line-integral is the potential difference between A and B caused by the transfer of 1 coulomb. By definition it is called the elastance of A to B.

Let us make these estimates (very roughly) for the case of the two spheres A and B of Fig. 32, and for the case of the two parallel plates of Fig. 33.

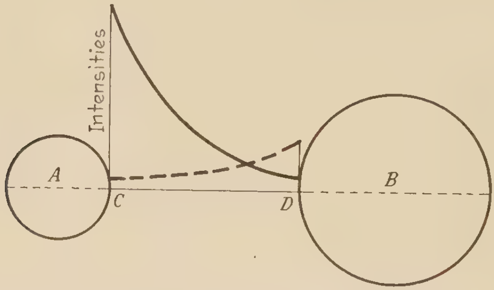


Fig. 32.—Electric intensities between two spheres.

If the distance between the spheres is large in comparison with their radii, the surface density of charge is fairly uniform over the surface of each sphere. The path along which it is easiest to estimate the intensities is the path CD connecting the centers. The electric intensities at points in this path are directed along the path in the direction from A to B. The positive charge on A and the negative charge on B give rise to electric intensities whose magnitudes are illustrated by the height of the ordinates from CD to the full curve and to the dotted curve, respectively. The work done by these forces on a test charge which moves along CD from A to B will be proportional to the sum of the areas included between each curve and the base line CD. The elastance of A to B therefore, will be proportional to this area, and the capacitance will be inversely proportional to it. With such a diagram in mind, one can readily see the effect of a change in the dimensions upon the elastance. For example, if the proportions are as in Fig. 32, the elastance will be increased (or the capacitance decreased) very slightly by pulling the spheres twice as far apart. Again, the capacitance will be increased very little by doubling the diameter of the large sphere (keeping CD constant); on the other hand, the capacitance will be almost doubled by doubling the diameter of the small sphere.

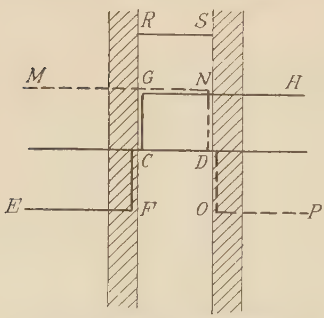


Fig. 33.—Electric intensities between parallel plates.

In striking contrast to the above is the case of the two parallel plates shown in Fig. 33. If the distance between the plates is small in comparison with the length of a side, the surface density of the positive and negative charges will be rather uniform over the opposing faces (rising somewhat near the edges). The easiest path for the estimate is the path CD perpendicular to the plates at their center. The electric intensities at points in this path are in a direction parallel to the path. The magnitudes of the intensities in a direction to the right which are due to the positive charges on A and to the negative charges on B are, respectively, indicated by the heights of the ordinates to the full curve $EFGH$ and to the dotted curve $MNOP$. For all points in the region between the plates these intensities aid, giving the curve RS; for all points in the region outside of the opposing faces the two sets of intensities substantially neutralize. The elastance of plate A to B is proportional to the area $CRSD$, and the capacitance is inversely proportional to this area. It may be seen that, if the plates are close together, the intensity along CD depends only upon the surface density of the charge near the center of the plates. Any element of charge which is at a distance from the center greater than ten times the distance between the plates exerts at points on CD an intensity which is approximately at right angles to CD. This intensity will be substantially neutralized by the intensity due to the corresponding element of charge on the other side of the center. Since the surface density of the transferred unit charge is inversely proportional to the area of the plates, it follows that electric intensities along CD and, therefore, the work-integral or potential difference are inversely proportional to the area of the plates. They will moreover, be directly proportional to the distance CD between plates. Therefore, the capacitance of one plate to the other will be directly proportional to the area of the plates and inversely proportional to the distance between the plates.

In the following chapter, the potential differences between charged bodies of the simpler geometrical forms are calculated by the methods of the calculus outlined in Chap. III. From these calculations, formulae are then derived for the capacitance of parallel plates, concentric spheres, and coaxial cylinders.

75. THE ELECTRIC CONDENSER (Restricted Definition).—**An APPLIANCE consisting of two insulated conductors A and B, each having an EXTENDED surface which is separated from the surface of the other by a thin layer of insulating medium, and which is so arranged and used in an electric circuit that the charge on A is equal but opposite in sign to that on B, is called an ELECTRIC CONDENSER. The two conductors are called the ELECTRODES, and the insulating medium between the extended surfaces of the conductors is called the DIELECTRIC of the condenser.**

By the CAPACITANCE of the condenser is meant the constant ratio between the quantity of electricity transferred from A to B and the potential difference between A and B which results.

By the CHARGE in the condenser at any instant is meant the quantity of electricity which has been transferred from one electrode to the other.

In a condenser, the two insulated conductors are arranged to have a large capacitance relative to each other. The purpose of the arrangement is twofold: (*a*) to reduce to a minimum the cost of the conductors and insulating material necessary to store a specified quantity of separated electricity **at a specified potential difference**; and (*b*) to make the space occupied by these materials as small as is practicable.

The principle of the condenser was independently discovered by Kleist, a German monk, in 1745, and by Musschenbrock, of the city of Leyden, a year later. Musschenbrock was testing the notion that water enclosed in a glass bottle would not lose its charge as rapidly as an exposed conductor. The bottle was partly filled with water, an electric charge from a frictional machine was conveyed to the water by a nail thrust through the cork. The bottle was held in one hand during the charging. The hand thus constituted one electrode and the water the other electrode of a condenser, the glass being the dielectric. Upon touching the nail with the free hand, the experimenter discharged the condenser, obtaining a shock of such severity as to arouse a widespread interest in the phenomenon. This experiment led to the Leyden-jar condenser illustrated in Fig. 34. It consists of a glass jar coated on the bottom and to within 5 to 10 centimeters of its top, both inside and outside, with tin foil. Contact with the inside coating is made through a suitably supported metal rod which passes out through the top of the jar.

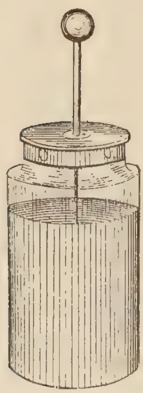


FIG. 34.—Leyden jar condenser.

The term "condenser" is defined above in its original and restricted sense as denoting an appliance having extended conducting surfaces separated by a thin insulating layer. The application of the term has been extended, however, and we frequently speak of any two conductors **between which a transfer of charge takes place** as constituting a condenser, even though the conductors may be small and far apart—as, for example, the two wires of an open-air telephone line.

75a. ELECTRIC CONDENSER (EXTENDED DEFINITION).—**When two conductors are so arranged and used in an electric circuit that the charge on one is always equal but opposite in sign to that on the other, the two conductors are said to constitute an ELECTRIC CONDENSER.**

76. Dielectric Properties. Relative Permittivity.—To Faraday, the method of regarding electric forces as a direct action of electric charges at a distance was repugnant. He sought to explain the force between two distant charges by the action between contiguous particles in the medium separating the charges. Taking this view of electric induction, "there seemed reason to expect some particular relation of it to the different kinds of matter through which it would be exerted, or something equivalent to **specific electric induction** for different bodies, which, if it existed, would unequivocally prove the dependence of induction on the particles."[8]

This line of thought led Faraday to compare the capacitances of two spherical condensers which were identically the same, save that in one the concentric spheres were separated by air, and in the other by dielectrics like shellac and glass. These experiments resulted in the discovery that the capacitance of one conductor to another depends upon the dielectric between the conductors.[9] This implies that the force between two charges depends upon the medium through which it acts.

76a. RELATIVE PERMITTIVITY, SPECIFIC INDUCTIVE CAPACITY.—Let two condensers have equal dimensions, and let the dielectric in the standard S be an evacuated space, while in T it is some other substance. The ratio of the capacitance of T to that of S is the RELATIVE PERMITTIVITY p_r of the dielectric in T. (Faraday called this ratio the SPECIFIC INDUCTIVE CAPACITY of the dielectric.)

The relative permittivities of a few insulating materials are listed in the table on page 102.

[8] FARADAY: *Experimental Researches* (November, 1837), Vol. I, Series XI, p. 363.

[9] Cavendish was familiar with this fact about 1776, but his discoveries were not published until Maxwell edited and published the Cavendish manuscripts in 1879.

76b. Relative Permittivities of Dielectrics.

Substance	Relative permittivity
Evacuated space (standard)	1
Air, 760 millimeters	1.00059
Glass	5.0 – 9.0
Dry paper	1.7 – 2.6
Porcelain	4 – 7
Mineral transformer oils	2 – 2.7
Mica	2.5 – 5.5
Paraffin wax	1.9 – 2.3
Pure rubber (vulcanized)	2 – 3
Loaded rubber (vulcanized)	3 – 6
Hard rubber	2 – 3.5
Distilled water	76
Alcohol	26

All substances have a higher permittivity than evacuated space. We attribute this to the fact that substances consist of atomic structures which are built up of positive nuclei and negative electrons. In the electric field of the charges on the electrodes of the condenser, the nuclei and electrons of the atomic structures of the dielectric suffer a shift in opposite directions from the normal distribution. This shift gives rise to layers of **concealed** charges in the dielectric. These layers of concealed charges partially neutralize the forces due to the charges on the metal electrodes and hence lead to an increased capacitance between electrodes. The location of the layers of concealed charge in a glass-plate condenser is illustrated in Fig. 35. This question is discussed at greater length in the next chapter.

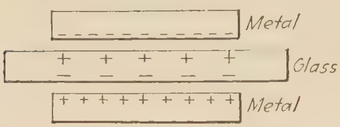


FIG. 35.—Location of layers of concealed charge.

The desirable properties of dielectrics for condensers are four in number:

1. The dielectric should have **high dielectric strength,** in order that it may not be punctured and destroyed if it is subject to high differences of potential.

2. **High relative permittivity** is desirable in order that the capacitance of a condenser of given volume may be as great as possible.

3. The material should be an extremely poor conductor of electricity or should have a **high electrical resistivity** (the precise definition of this term

will be given later), in order that the charge may not leak through the dielectric, or over its surface from electrode to electrode.

4. If the charges on the plates are to be rapidly alternated in sign, the **molecular frictional** energy loss accompanying the elastic displacements in the dielectric must be low, in order to avoid undue heating of the dielectric.

It should be realized that these properties are not related one to the other in any way; high permittivity does not imply either high dielectric strength or high resistivity.

77. Forms of Condensers and Their Applications.—Condensers are extensively used for the following purposes:

1. To suppress the sparking, or to mitigate the burning of the contacts, which occurs when circuits are opened and closed with great frequency for long periods, condensers are connected across the "break" in the circuit.

2. In communication circuits, condensers are essential elements which make it possible to construct electrical **filters** to select certain desired currents and to reject others.

3. Charged condensers discharge through properly proportioned circuits in an oscillatory manner, and are extensively used to furnish high-frequency currents for radio telegraphy.

4. **Standard** condensers whose constants are accurately known are used in a number of electrical measuring operations.

The materials extensively used as the dielectrics of the commercial forms of condensers are air, oil, glass, mica, and paraffined paper.

In the small variable condensers used in radio receiving circuits and in some standard condensers, the clearance between the metal plates of opposite polarity is from 1 to 2 millimeters and the dielectric is air. These condensers spark between plates at voltages of the order of 1000 volts. They are made with capacitances between 0.0001 and 0.005 microfarad.

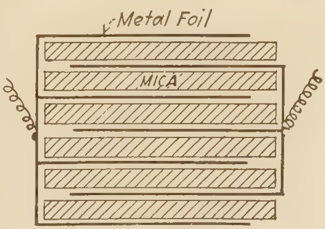


FIG. 36.—Mica condenser.

The condensers extensively used in telephone filter circuits are made up of long tin-foil strips separated by two strips of paraffined tissue paper having a thickness of 0.0025 centimeter each. The strips are rolled into a compact form and after drying in a vacuum are impregnated and sealed in metal containers. These units have capacitances between 0.05 and 5 microfarads. They puncture at voltages of the order of 300 volts.

Power condensers take the form of metal plates mounted 0.5 to 2 centimeters apart in oil, or 0.3 to 0.5 centimeter apart in compressed air under 8 to 16 atmospheres pressure, or they may take the form of glass plates or jars with metal foil or copper plating on opposite faces.

Standard condensers of the larger capacitances (0.1 microfarad) are made by piling up alternate sheets of tin foil and mica and connecting the tin foil as illustrated in Fig. 36. The sheets are pressed together under heavy pressure and are dried in a vacuum and impregnated.

78. Energy of a Charged Condenser (Deduction).—When positive and negative charges are separated by conveying a charge from one electrode of a condenser to the other, the work which is done in conveying any elementary portion of the charge against the forces of the charges previously separated is converted into energy of the electropotential form. This energy is said to be stored in the condenser. An expression for this energy may be derived by imagining that the total charge Q, with which the condenser is charged, has been conveyed from one electrode to the other in a very large number of small charges of magnitude dq.

Let S represent the elastance of the condenser.
q represent the quantity of electricity in the condenser at any instant.

The relation between the potential difference e between the electrodes and the quantity q is the straight-line relation shown by Fig. 31 and is expressed by Eq. 50.

$$e \text{ (volts)} = Sq \text{ (darafs, coulombs).} \tag{50}$$

If this charge q is increased by the infinitesimal amount dq, the work dW which is done in transferring the charge dq, or the energy which is thereby stored in the condenser, is the product of the potential difference e between the electrodes times the quantity transferred.

$$dW \text{ (joules)} = e(dq) = Sq\,dq.$$

The total energy W stored in a condenser of elastance S having a charge Q in the condenser may be found by integrating the above expression between the limits 0 and Q.

$$W \text{ (joules)} = \int_0^Q Sq(dq) = \frac{SQ^2}{2} \text{ (darafs, coulombs)} \tag{54}$$

$$W \text{ (joules)} = \frac{Q^2}{2C} \; \frac{\text{(coulombs)}}{\text{(farads)}}. \tag{55}$$

If the potential difference between the conductors with the

charge Q in the condenser is represented by E, the expression for the energy may be also written in the forms

$$W \text{ (joules)} = \frac{CE^2}{2} \text{ (farads, volts)} \tag{56}$$

$$W \text{ (joules)} = \frac{E^2}{2S} \begin{matrix}\text{(volts)} \\ \text{(darafs)}\end{matrix} \tag{57}$$

$$W \text{ (joules)} = \frac{QE}{2} \text{ (coulombs, volts).} \tag{58}$$

These same conclusions may be arrived at from geometrical relations by noting that the total work done will be represented by the area OAQ under the straight line of Fig. 31.

79. Condensers in Parallel and in Series.—There are two simple or elemental ways of connecting a **number** of condensers to receive charge from the same electric machine (or battery, or magneto-electric generator). These elemental connections are illustrated in Figs. 37 and 38. They are known as the **parallel** and **series** (or cascade) connections, respectively. These parallel and series arrangements may be combined in any manner into **complex** networks of condensers of the type illustrated in Fig. 39.

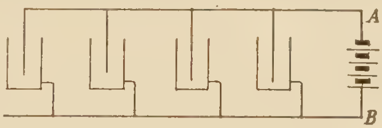


FIG. 37.—Condensers in parallel.

Condensers in Parallel.—When two or more condensers each have their electrodes connected to two common terminals A and B as in Fig. 37, they are said to be connected in **parallel**. Let E represent the potential difference between the terminals A and B, and let Q_1, Q_2, Q_3, etc. represent the charges in the respective condensers. Then, by definition, the capacitance between the two systems of conductors connected to A and B is

$$C_{(A \text{ to } B)} = \frac{Q_1 + Q_2 + Q_3 + }{E} = \frac{Q_1}{E} + \frac{Q_2}{E} + \frac{Q_3}{E} + \text{etc.}$$

But, by definition, Q_1/E, Q_2/E, Q_3/E are the capacitances C_1, C_2, C_3 of the respective condensers. Therefore

$$C_{(A \text{ to } B)} = C_1 + C_2 + C_3 + \text{ (farads).} \tag{59}$$

The relation is expressed in the following rule:

79a. CAPACITANCE OF CONDENSERS IN PARALLEL (DEDUCTION). The capacitance between the common terminals of two or more condensers connected in parallel is equal to the sum of the individual capacitances.

The total charge delivered to a parallel combination of condensers divides among the condensers in proportion to their respective capacitances.

Condensers in Series.—When two or more condensers are so connected between two terminals A and B that the charge on the negative electrode of the first condenser comes (through the source) from the positive electrode of the last, the charge on the negative electrode of the second comes from the positive electrode of the first, the charge on the negative electrode of the

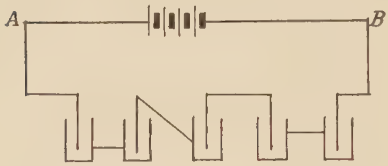


FIG. 38.—Condensers in series.

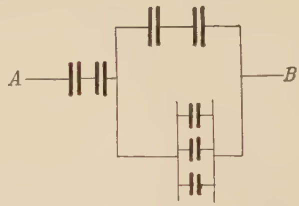


FIG. 39.—Network of condensers

third comes from the positive electrode of the second, and so on through the series, these condensers are said to be connected in **series** (see Fig. 38).

Let Q represent the charge which has been so transferred from electrode to electrode, and let E_1, E_2, E_3, etc. represent the potential differences between the electrodes of the respective condensers. Then, by definition, the elastance from the terminal A to B is:

$$S_{(A\ to\ B)} = \frac{E_1 + E_2 + E_3 +}{Q} = \frac{E_1}{Q} + \frac{E_2}{Q} + \frac{E_3}{Q} + \cdot$$

But, by definition, E_1/Q, E_2/Q, E_3/Q are the elastances S_1, S_2, S_3, etc. of the respective condensers. Therefore,

$$S_{(A\ to\ B)} = S_1 + S_2 + S_3 + \text{(darafs).} \tag{60}$$

This may be expressed as follows:

79b. ELASTANCE OF CONDENSERS IN SERIES (DEDUCTION).—The elastance between the (end) terminals of two or more condensers connected in series is equal to the sum of the individual elastances.

The total potential difference across a series combination of condensers is divided among the condensers in proportion to their respective elastances.

80. Mechanical Force between Two Charged Conductors Constituting a Condenser (DEDUCTION).—When charge is transferred from one conductor to another, the attractive forces between the charges tend to draw the conductors together. If we can compute, or can experimentally determine, the increase in the capacity which will be caused by a given displacement of one of the conductors, the following argument enables us to compute the component of the force in the direction of the displacement. Let the capacity of the conductors with reference to each other be represented by C, and let the quantity in the condenser be Q. Imagine the conductors to be isolated so that the quantity Q remains constant, and suppose one of the conductors to be displaced by the infinitesimal amount dx. This displacement is to be a pure translation without rotation. (The distance dx is measured in the direction of translation, and is always taken as a positive quantity.)

Imagine the displacement of the conductor to cause an **increase** in the capacitance by the amount dC.

Before the displacement, the energy stored was $\frac{Q^2}{2C}$.

After the displacement, the energy is $\frac{Q^2}{2(C + dC)}$.

The decrease in the stored energy is $\frac{Q^2}{2C} - \frac{Q^2}{2(C + dC)} = \frac{Q^2}{2C^2}\,dC$.

By the principle of the conservation of energy, this decrease in the stored electrical energy must equal the mechanical work done by the electrical system when the conductor moves over the distance dx. If f_x represents the **component-of-the-force** on the displaced conductor (due to the charges) tending to move the conductor **in the direction of the displacement** dx **from the initial to the final position,** the mechanical work done upon the conductor is $f_x(dx)$. Therefore,

$$f_x(dx) = \frac{Q^2}{2C^2}\,dC.$$

From which,

$$f_x \text{ (dyne-sevens)} = \frac{Q^2}{2C^2}\frac{dC}{dx} \quad \frac{\text{(coulombs)}}{\text{(farads, cm.)}}. \tag{61}$$

$$f_x \text{ (dyne-sevens)} = \frac{E^2}{2}\frac{dC}{dx} \quad \frac{\text{(volts, farads)}}{\text{(cm.)}}. \tag{62}$$

If, instead of keeping a constant charge Q in the condenser, we hold the potential difference between the conductors at the constant value E during the movement, the argument is as follows:

Before the displacement, the energy stored was $\frac{CE^2}{2}$.

After the displacement, the energy stored is $\frac{(C + dC)E^2}{2}$.

The increase in the stored energy is $\frac{E^2(dC)}{2}$.

The quantity of electricity which must be added to the charge to hold the voltage constant is

$$dQ = E(dC).$$

The work done in adding this charge is

$$dW = E(dQ) = E^2(dC).$$

The mechanical work done by the electrical forces on the displaced conductor must be the difference between the work which is done in transferring the additional charge dQ, and the increase in the stored energy. This difference is

$$E^2(dC) - \frac{E^2(dC)}{2} = \frac{E^2(dC)}{2}.$$

That is, the work done by the forces on the conductor is the same as in the case in which the charge is kept constant. Therefore, the component-of-the force on the conductor in the direction x is given by the Formulas (61) and (62).

If one of the two charged conductors is imagined to turn about a fixed axis through a small angle $d\theta$, and if dC represents the resulting increase in the capacity of the conductors, an argument similar to the above will yield the following formulas for the magnitude of the torque τ about this axis of the forces on the conductor.

$$\tau(\text{dyne-seven, cm.}) = \frac{Q^2}{2C^2}\frac{dC}{d\theta} \quad \frac{(\text{coulombs})}{(\text{farads, radians})}. \tag{63}$$

$$\tau\,(\text{dyne-seven, cm.}) = \frac{E^2}{2}\frac{dC}{d\theta} \quad \frac{(\text{volts, farads})}{(\text{radians})}. \tag{63a}$$

80a. Exercises.

1. Explain in detail the operation of an electrostatic generator of the "water dropper" induction type. Sketch and name the essential elements, and state the functions of each element. Discuss and account for the movements of charge and the energy transformations.

2. What is meant by the expression, "the electrical capacitance of two conductors relative to each other"?

3. Does the expression, "the capacitance of a conductor," have any meaning? If not, why not? If so, give the full meaning.

4. What is meant by an electric condenser? Describe a condenser from two viewpoints: from the viewpoint of the definition in purely electrical terms, and from the visual viewpoint of the relative disposition of the parts constituting the condenser.

5. What is meant by the expression "the charge in a condenser"? Is it the charge on one conductor or the sum of the charges on the two? That is, what is involved in the operation of charging a condenser?

6. Assume that two thin parallel metal disks, as in the parallel disk electrometer, each of radius R, are spaced b centimeters apart, and that their adjacent surfaces are charged, one positively and the other negatively. Assume further that the charges are uniformly distributed over the adjacent surfaces.

a. If the charge on each surface is q coulombs per square centimeter, derive the expression for the electric intensity at any point P on their common axis, and at a distance h from the charged face of one of the disks.

b. If R is 100 or more times as great as b, what simple formula expresses the electric intensity at any point between the disks with reasonable accuracy?

c. Calculate the value of the electric intensity at two points, O and P, on the axis of the disk for the following dimensions: $R = 10$ centimeters, $b = 0.4$ centimeter, $q = 1.3 \times 10^{-9}$ coulombs per square centimeter. Point O is midway between the two disks, and point P is in the space outside at a point 0.1 centimeter from the nearest disk.

7. *a*. Derive an expression for the difference in the potentials of the two parallel disks specified in exercise 6*b*. Solve by two methods: first, by making use of the expression for the electric intensity which has been derived in problem 6; second, by making use of the fundamental potential formula.

b. What is the difference of potential between the plates for the conditions specified in exercise 6*c*?

8. *a*. Derive the expression for the force of attraction per square centimeter between the parallel charged disks of exercise 6.

b. What is the force of attraction, in practical units and in grams, between the disks for the conditions specified in exercise 6*c*?

c. Make use of the information from exercise 7 to write the expression for the force of attraction per square centimeter in terms of the difference in the potentials of the disks, instead of in terms of charges on the disks.

9. Deduce the expression for the capacity between two adjacent portions of the parallel disks of exercise 6, each of area a. Calculate the approximate capacity of the two disks 10 centimeters in radius specified in *c*.

10. A metal ball of radius r_1 is placed inside of, and concentric with a hollow conducting sphere of radius r_2. A negative charge of Q coulombs is transferred from the ball to the hollow sphere.

a. Find the potential of the hollow sphere.

b. Find the potential of the metal ball.

c. Determine the difference of potential between the two.

d. Derive the expression for the capacitance of these two concentric spheres with reference to each other.

e. What does this expression for the capacitance reduce to when the radius of the outer sphere becomes very great compared with the radius of the inner one?

f. What does the capacitance become when the distance separating the two spheres is small compared with the radius of either sphere?

11. Two metallic spheres, having radii of 3 and 5 centimeters, are placed in air 60 centimeters apart from center to center. With the spheres initially uncharged, 2×10^{-8} coulombs of negative electricity are conveyed from one to the other. Calculate:

a. The approximate value of the potential of each sphere.

b. The capacitance of one sphere with reference to the other.

12. Two Leyden jars with capacitances of 0.002 and 0.003 microfarad, respectively, are connected in parallel, and a difference of potential of 20,000 volts is impressed across the combination. The thickness of the glass in each jar is 0.4 centimeter. Find (*a*) the capacitance of the combination, (*b*) the ratio of the charges on the coatings of the two jars, (*c*) the ratio of the potential differences across each condenser, and (*d*) the electric intensity in the dielectric of each jar.

13. Carry out the same calculations as in exercise 12 when the jars are connected in series across the same difference of potential.

14. Measure the necessary dimensions and calculate as closely as you can the capacitance of a given multiple-plate air condenser.

15. Measure the necessary dimensions for determining the capacitance of a given Leyden jar. Taking the relative permittivity of glass as 7, calculate the approximate value of the capacitance of the jar.

CHAPTER V

MATHEMATICAL TREATMENT OF DISTRIBUTED AND CONCEALED CHARGES

81. General Statement of the Problem.—The laws and the methods by which it is possible to compute: (*a*) the electric intensities at points in the field of a **known distribution of charges** and (*b*) the potential differences between the conductors in such a field have been presented in Chap. III. There are, however, no instruments by which the distribution of charge over the surface of conductors may be easily and directly measured. On the other hand, the potential difference between two conductors may be most readily obtained from a single reading on the electrostatic voltmeters described in the previous chapter. As a result, the electrostatic problems to be solved in engineering practice are almost invariably of the following type:

The potential differences between the conductors in the field are known. The problem is **to compute** either

(*a*) the electric intensity at any point in the field; or

(*b*) the surface density of the charge at any point on the conductors.

When the solution of either of these problems has been obtained, the solution of the other may be obtained from it by straightforward methods.

The problem of computing the distribution of the charges over the conductors is the inverse of computing the intensities and the potential differences from a known distribution of charge, and its solution may be far more difficult. This chapter is devoted to the development of the mathematical methods of treating electrostatic problems which deal with the distribution of charge over the surface of conductors, or in space, or throughout the volume of dielectrics.

The application of this mathematical treatment is in the rendering of a more precise account of electrostatic phenomena and of the properties of the appliances previously described, and in

such engineering applications as the design of insulators to withstand high differences in potential, and in the determination of the electrostatic properties of electric power and communication lines.

82. The Inverse Square Law of Force.—The inverse square law for the force between charged bodies, as it has been formulated in Sec. 29, contains the **restriction** that the dimensions of the bodies must be small relative to their separation and that they must be in an **infinitely extended homogeneous** medium. This restricted form of the law has thus far been regarded as embodying no observations other than Coulomb's torsion-balance measurements (of 1785) of the force upon small charged spheres. In the explanations of electrostatic phenomena advanced in the previous chapters, it has been implicitly assumed that the force between the particles of the postulated electric fluids (or between the electrons and the protons) follows the inverse square law, and that the principle of linear addition applies to these forces. With the object of requiring the explicit statement of these postulates and of directing attention to their consequences, we now raise two general questions, namely:

Question 1.—By what postulates and laws may the electric intensities and the distribution of the charges be calculated when

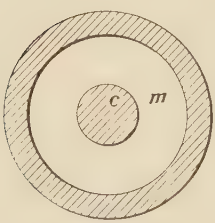


FIG. 40.—Lead sheathed wire.

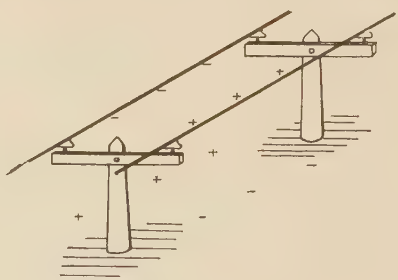


FIG. 41.—Power line.

the system does not consist of point charges in an infinitely extended homogeneous medium, but when it contains large conducting bodies as illustrated in Figs. 40 and 41? In Fig. 40, the problem is to compute the electric intensities throughout the insulating medium m surrounding the conductor c of the lead-

sheathed cable. In Fig. 41, two long wires are in proximity to the surface of the earth, and the problem is to compute the relation between the charges on the wires and the difference in potential between the wires.

Question 2.—By what postulates and laws may the electric intensities and the distribution of the charges be computed when the dielectric surrounding the conductors is not homogeneous but is made up of different materials as illustrated in Fig. 35? In this case, two charged conducting plates P are separated by layers of air and a plate of glass G.

As an historical introduction to these laws and principles it may be well first to point out that Coulomb's torsion-balance measurements of 1785 cannot be regarded as a determination that the value of the exponent in the force formula is precisely -2, since the measured forces are so small that the errors of the method are large. A far more rigorous demonstration that the force varies inversely as the precise square of the distance between the charges is the argument by which Joseph Priestley in 1767 and Henry Cavendish in 1771 deduced the inverse square law. In 1767 Priestley confirmed an observation to which his attention had been directed by Benjamin Franklin, namely, that when a hollow conducting vessel is charged there is no charge on the inner surface (except near the opening) and no force is exerted on uncharged bodies inside the vessel. From this observation he reasons:

> May we not infer from this experiment that the attraction of electricity is subject to the same laws with that of gravitation, and is, therefore, according to the square of the distances; since it is easily demonstrated that were the earth in the form of a (spherical) shell, a body in the inside of it would not be attracted to one side more than the other.[1]

In his remarkable paper of 1771 before the Royal Society,[2] Cavendish imagines a spherical shell of uniform matter (as in

[1] PRIESTLEY, J.: *The History and Present State of Electricity with Original Experiments*, London, 1767, p. 711.

[2] CAVENDISH HENRY: *An Attempt to Explain Some of the Principal Phenomena of Electricity, by Means of an Elastic Fluid*, Phil. Trans. Roy. Soc., 1771.

Fig. 42), whose particles repel with a force inversely as the square of the distance. Newton had given the following simple demonstration that the resultant force on a particle placed anywhere within such a shell, as at P, is zero.

Imagine a great number of straight lines to be drawn through P in such a manner as to divide the whole volume of the sphere into a great number of double cones of small angular opening. One of these double cones is illustrated in Fig. 42. It cuts out on the surface of the sphere, bases whose areas a_1 and a_2 are to each other as the squares of the heights of the cones.

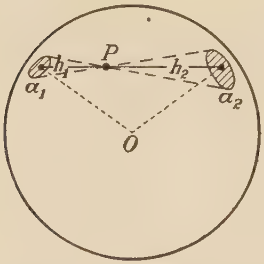


FIG. 42.—Balance of the forces inside spherical shells.

$$\frac{a_1}{a_2} = \frac{h_1^2}{h_2^2}.$$

The quantity of attracting matter in each base is thus proportional to the square of the distance of the matter from P, and the two bases will, therefore, exert equal and oppositely directed forces on a particle at P. Since this is also true for each of the other double cones, the resultant force on a particle at P is zero.

Cavendish remarks:

It follows also from Newton's demonstration that if the repulsion is inversely as some higher power of the distance than the square, the particle at P will be impelled **toward** the center; and if the repulsion is inversely as some lower power than the square it will be impelled **from** the center.

After further analysis, he draws the following conclusions:

a. If the force is inversely as the square of the distance, it is likely that almost all of the charge on a body is lodged close to the surface.

b. If the force lies between the inverse square and the inverse cube, it is likely that the charge will be distributed throughout the body.

c. If the force is inversely as some less power than the square, it is likely that all parts of the body except near the surface will acquire a charge opposite in sign to that imparted to the body.

Figure 43 illustrates the distribution of the charge throughout the volume of negatively charged spherical metal shells for three

different laws of force, namely, the inverse square, the inverse cube, and the inverse 1.4 power. The electric intensities and the values of the potential at points along any radial line, as *OP*, are also represented (not to scale) on these diagrams by the ordinates erected from this radial line to the curves marked *F* and *E*, respectively.

Two years later Cavendish concluded an admirably conceived and painstaking set of measurements to determine to what extent a conducting sphere enclosed within a metal globe and connected to it becomes charged when the outer globe is charged. He could find no evidence that the inner sphere acquired any charge whatsoever. After a study of the sensitiveness of his method he drew the conclusion "that the electric attraction and repulsion must be inversely as some power of the distance between 2.02 and 1.98, and there is no reason to think that it differs at all from the inverse square ratio."[3] Cavendish did not disclose this experimental work and it remained unknown until his manuscripts were edited by Maxwell about 100 years later.

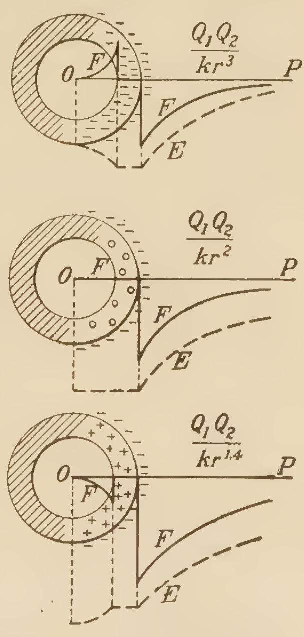


FIG. 43.—Distribution of charge.

From this it appears that before Coulomb, by actual measurement, had arrived at the inverse square law for the force between small charged bodies, it had been deduced by other workers as the law for the force between the postulated particles of the postulated electric fluid.

We have presented the arguments of Priestley and of Cavendish because they show the great influence which was exerted

[3] See *The Electrical Researches of the Honorable Henry Cavendish*, p. 104, edited by Clerk Maxwell and published in 1879. Maxwell, in 1871, repeated the Cavendish experiment with more sensitive electroscopes and drew the conclusion that the exponent in the force formula cannot differ from 2 by more than ± 0.0005.

on the development of electrical theory by the **action-between-particles-at-a-distance** type of explanation which Newton had so successfully applied to gravitational phenomena a century before,[4] and they indicate the antecedents of the postulates proposed below as an answer to questions 1 and 2.

83. Postulates Relating to Fields Containing Conducting Bodies and Dielectrics.—The principles and the postulates by which it is proposed to account for, and to predict, the electric intensities and the distribution of the charge in an electrostatic field containing large conductors are as follows:

a. All electrostatic interactions between bodies are to be accounted for in terms of the postulated forces between the (negative) electrons and the (positive) protons of which all bodies are conceived to be composed.

b. It is **postulated** that each electron and each proton acts upon every other electron and proton, irrespective of whether the electrons may form part of the free electric charge or may be attached to one of the atomic systems, with a force whose average value is expressed by the formula

$$f = \frac{Q_1 Q_2}{4\pi p r^2}, \tag{9}$$

in which, the constant p has a fixed value which is determined by the units in which f, r, and Q are measured, and **which is independent of the material occupying the space between or around the two attracting or repelling elements under consideration.**[5]

c. It is postulated that the resultant force on any elementary charge is to be obtained by the linear addition or superposition

[4] Newton's *Principia* appeared in 1686.

[5] Equation (9) should not be regarded as the expression of the complete law of force but as an approximation which is a very precise expression for the **average value** of the force when the distance between the elementary charges is large compared with their radii. The electrons and protons composing bodies have high vibratory velocities of thermal agitation, and still higher velocities in their orbits of revolution within the molecular structures. Subsequent experiments will show that, because of these velocities, the electrons must be conceived to be subject to rapidly varying forces of a magnetic nature. Therefore, Eq. (9) is not to be regarded as expressing the instantaneous value of the force on an electron, but as expressing the average value, or the net resultant averaged over a long interval of time.

(as outlined in Sec. 30) of the forces due to all the elementary charges.

d. A system of forces which hinders electrons and molecules from leaving the parent substance is postulated to act at the surfaces separating conductors from insulators. (The nature of these forces is discussed in Chap. VII.)

e. The electrostatic distribution of the free electrons of the conductors in the field will be such that:

1. The resultant force on any electron in the surface film of the conducting bodies has no component parallel to the surface of the conductor at the point.

2. The resultant force on any electron beneath the surface films of the conductors is zero.

f. If the field contains, not only conducting masses, but also non-conducting masses, or dielectrics, such as glass, paper, or oil, it is postulated that the effect of the dielectric on the electric intensities and on the distribution of charges on the conductors is to be accounted for in terms of the postulated shift in opposite directions of the electrons and nuclei of the molecules of the dielectric. In other words, it is postulated that the effect of the dielectric is to be explained by taking into account the forces (computed by Eq. (9)) arising from the layers of **concealed** charge in the dielectric which result from the polarization of the molecular structures of the dielectric.

Before applying these **action-of-element-upon-element** postulates to the calculation of the properties of systems of conductors, such as condensers, electrostatic voltmeters, and buried and overhead conductors, we proceed to show that the inverse square law may be expressed in a form involving the surface-integral of the electric intensity, and also by differential equations involving either the electric intensity or the potential. We will then contrast the application of these alternative expressions of the law by using them to compute the electric intensities at points in the field resulting from the charges distributed over parallel plates, concentric spheres, and coaxial cylinders.

84. The Surface-integral of a Distributed Vector. Flux (DEFINITION).—A **surface-integral** pertains to a given surface in a vector field. The surface must be definitely laid out, or

specified, and, in addition, one direction across (through) the surface must be arbitrarily specified. The specified direction may be indicated by an arrow, and may be referred to either as the **arrow direction** or as the **specified direction.**

The sum which results from the following mathematical operations is called the (value of the) **surface-integral of the vector V in the arrow direction across the specified surface,** and the operation is called the operation of finding the **surface-integral** of the vector V.

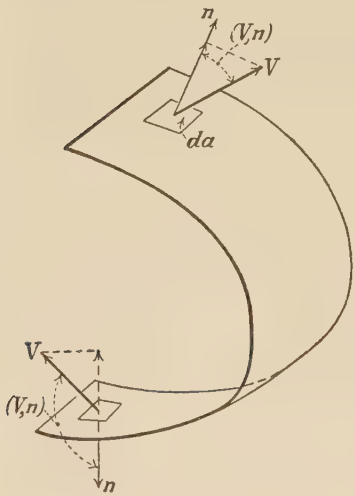


FIG. 46.—Surface integrals.

Operation 1.—Divide the surface into patches in such a manner that the component of the vector V which is normal to the surface, namely, $V \cos (V, n)$, has substantially the same value at all points of any one patch. Let da represent the area of such an elementary patch (Fig. 46).

Operation 2.—Compute the value of the product $V \cos (V, n) da$ for each patch. In this product
V represents the value of the vector at the patch, and
(V, n) represents the angle between the direction of the vector V and the arrow direction along a normal erected to the patch.

It is to be noted that when the vector V points through the surface in the arrow direction, the factor $\cos (V, n)$ is positive, and when it points through in the opposite direction the factor $\cos (V, n)$ is negative (see Fig. 46).

Operation 3.—Take the algebraic sum S_2 of all the products thus formed. This sum is the value of the **surface-integral** of the vector V in the arrow direction across the given surface.

$$S_2 = \int^{\text{over the surface}} V \cos (V, n) da. \tag{64}$$

The expression $V \cos (V, n)\, da$ is the product of three factors, an area, a directed quantity, and the cosine of an angle between the directed quantity and the normal to the area. Products made up of three factors of this type occur in many branches of physics. In some physical problems, such a product represents a familiar physical entity; in other cases it does not. As an instance of the familiar type, suppose that the problem were to compute the quantity of water per second flowing across a specified surface in a stream of running water. If the velocity V with which the water is flowing can be measured from point to point in the fluid, the quantity of water crossing the surface will be given by the following formulas.

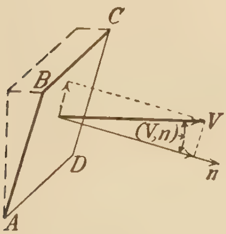


FIG. 47.—Flow across a surface.

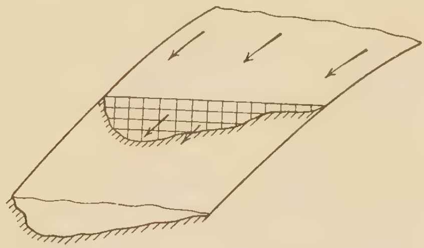


FIG. 48.—Surface integration.

If the surface area is small, and (V, n) is the angle between the normal to the area and the velocity vector (Fig. 47), the velocity V may be resolved into a component $V \sin (V, n)$ parallel to the plane area $ABCD$, and a component $V \cos (V, n)$ perpendicular to the area. Clearly, the quantity of water crossing the area a per second is $Va \cos (V, n)$.

If the area swept over is large and if the velocity of the water is not uniform at all points of this area, the quantity crossing per second will be found by dividing the large area into small patches of area da, as in Fig. 48, measuring the velocity V at the center of each area, and then taking the sum of all the products $V \cos (V, n) da$.

$$\text{Flow (in cu. cm. per sec.)} = \Sigma V \cos (V, n)\ (da) = \int V \cos (V, n) da. \tag{65}$$

In other vector fields the surface-integral does not represent a quantity with such a familiar physical meaning. For example,

in the next section it is shown that the surface-integral of the electric intensity over a closed surface (bounding a volume) is related in a simple way to the charge contained within the surface. The surface-integral of this vector quantity thus appears in an important law of electrostatics, but the integral does not call up such a definite physical picture as it does in the case of the flowing water.

As the general name for the product of an area by the component of a distributed vector normal to the area, the term **flux** is widely used. This term comes from the Latin *flux* (flow) and is suggested by the fact that the most commonplace example of this product is that involved in finding the flow of a stream (in cubic feet per second) across a surface.

The term FLUX in mathematical and physical science always designates either the product of an area by the component of a distributed vector perpendicular to the area, or the summation of many such products.

That is, the term "flux" designates the surface-integral of a distributed vector, an appropriate adjective being used to indicate the nature of the vector quantity involved. Thus, in electrostatic fields, we have **electrostatic flux**; in the field of a small hot body which is radiating energy, we have **radiant flux**; in the field of a source of light, **luminous flux**; and in the magnetic field, **magnetic flux.**

85. The Expression of the Inverse Square Laws by Means of Surface-integrals. Gauss's Theorem (DEDUCTION).—The following mathematical theorem, known as Gauss's theorem, since it was given by Gauss in 1813 in a treatment of gravitational fields, characterizes all centrally directed vector fields to which the inverse square law applies.

GAUSS'S THEOREM (1813—DEDUCTION).—**If any CLOSED surface is taken in an electrostatic field, the surface-integral (or the flux) of the electric intensity in the outward direction over the entire surface is directly proportional to the net quantity of electricity enclosed by the surface.**

To prove this, let S in Fig. 49 represent any closed surface in the fields of the charges Q_1, Q_2, Q_3, etc., some of which are within and others without the surface.

Let us consider the surface-integral resulting from an elementary charge of Q coulombs, located at the point P within the surface. Let APB represent any elementary cone with its vertex at P, enclosing an infinitesimal angle of $d\omega$ steradians. This cone intercepts on the surface S an elementary portion GHK of area da, which may be treated as a small plane surface, the normal to which makes some angle (F, n) with the axis of the cone.

If the distance PC is represented by r, the electric intensity at all points of GHK may be taken to be parallel to the axis of the

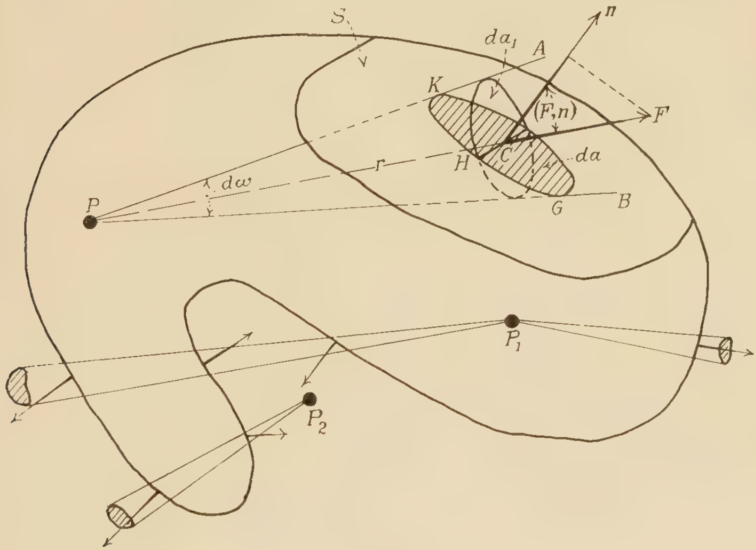


FIG. 49.

cone in the outward direction and of the value $Q/4\pi pr^2$. Then the value of the flux of the electric intensity in the outward direction over the elementary area GHK is

$$F \cos (F, n)\, da = \frac{Q}{4\pi pr^2} \cos (F, n)\, da.$$

But an examination of the figure will show that $\cos (F, n)da$ is equal to the projection da_1 of the area da on a sphere passing through C; and in the limit, da_1/r^2 is the value in steradians of the solid angle of the cone. Hence.

$$F \cos (F, n)\, da = \frac{Q}{4\pi p} d\omega. \tag{66}$$

Now if P lies inside the closed surface S, every elementary cone which can be drawn with P as a vertex must cut through the surface an odd number of times, since the cone must eventually extend outside of any surface of finite extent. For those areas at which such a cone emerges from the surface, (F, n) is less than 90 degrees, and for those areas at which the cone may cut into the surface again, (F, n) is greater than 90 degrees. Therefore, the values of the surface-integrals of the electric intensity in the **outward** direction over the areas cut out by such a cone which re-enters the closed surface one or more times are alternately

$+\frac{Qd\omega}{4\pi p}$ and $-\frac{Qd\omega}{4\pi p}$, and the net sum for the cone is $+\frac{Qd\omega}{4\pi p}$.

Since the total solid angle about a point is 4π steradians, it follows that the surface-integral of the electric intensity in the outward direction over the entire closed surface enclosing P is

$$\int^{\text{closed surface}} F \cos (F, n)\, da = \frac{Q}{4\pi p} \int^{\text{around a point}} d\omega = \frac{Q}{p}. \tag{67}$$

Every cone which can be drawn with its vertex at a charge Q_2 at a point P_2 which lies outside the closed surface must cut through the surface an even number of times. Therefore, the net contribution of any charge which lies outside the surface to the integral over the entire surface will be zero.

From the principle of the linear superposition of the forces of elementary charges, the surface-integral of the electric intensity in the outward direction over the surface due to all the charges in the field will be the sum of the integrals resulting from the individual charges, namely:

$$\int^{\text{closed surface}} F \cos (F, n)\, da = \frac{1}{p}(Q_1 + Q_2 + Q_3 + \ . \ . \ .)$$

$$\int^{\text{closed surface}} F \cos (F, n)\, da = \frac{1}{p}\Sigma Q, \tag{68}$$

in which Q_1, Q_2, Q_3, etc. represent the **algebraic** values of the charges which lie within the surface.

If ρ represents the volume density of charge at any element of volume dv within the surface, then Eq. (68) may be written in the following equivalent form

$$\int^{\text{over closed surface}} F \cos (F, n)\, da = \frac{1}{p} \int^{\text{throughout enclosed volume}} \rho dv \tag{69}$$

in which the surface-integral is to be taken over a given closed surface and the volume-integral is to be taken throughout the space enclosed by the surface.

86. Expression of the Inverse Square Law by Differential Equations. Poisson's and Laplace's Equations (DEDUCTIONS).—From Gauss's theorem, the surface-integral or flux of the electric

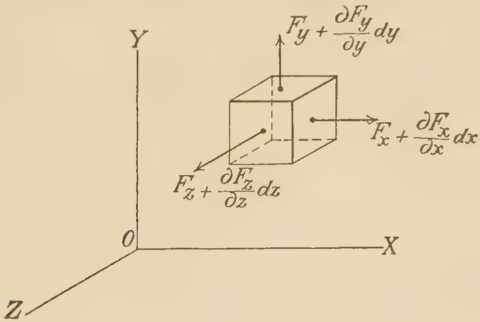


FIG. 50.

intensity in the outward direction over the surface of the infinitesimal cubical volume shown in Fig. 50 is equal to $1/p$ times the quantity of electricity within the cube.

Whence,

$$-F_x\, dy\, dz + \left(F_x + \frac{\partial F_x}{\partial x} dx\right) dy\, dz - F_y\, dx\, dz + \left(F_y + \frac{\partial F_y}{\partial y} dy\right) dx\, dz$$

$$-F_z\, dx\, dy + \left(F_z + \frac{\partial F_z}{\partial z} dz\right) dx\, dy = \frac{1}{p}(\rho\, dx\, dy\, dz)$$

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\, dy\, dz = \frac{\rho}{p}\, dx\, dy\, dz. \tag{70}$$

Dividing both members of the equation by the volume of the cube, it becomes

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = \frac{\rho}{p}. \tag{71}$$

We have seen (Sec. 52) that F_x, the X component of the electric intensity, is equal to the negative of the potential gradient in the X direction.

$$F_x = -\frac{\partial E}{\partial x}. \tag{30}$$

Accordingly, Eq. (71) may be written in the form

$$\left(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}\right) = -\frac{\rho}{p} \tag{72}$$

For regions which are free of charge, Eq. (72) becomes

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = 0. \tag{73}$$

Laplace in 1782 pointed out that the **potential function** in the gravitational field satisfies Eq. (73) in the regions which are free of matter, and Eq. (73) is known as Laplace's equation.

In 1813 Poisson showed that within the substance of the attracting body the potential function satisfies Eq. (72). This equation is frequently called Poisson's equation.

87. Expression of the Inverse Square Law in Vector Notation.—In vector analysis, the **divergence** of a vector is defined as follows:

87a. Divergence of a Vector (Definition).—By the **divergence of a distributed vector** F **at a point** P is meant the value of the quotient obtained by dividing the outward surface-integral of the vector taken over an infinitesimal closed surface around P by the volume enclosed by the surface. The divergence of F is represented by the symbol div F.

$$\text{div } \mathsf{F} = \frac{\int F \cos (F, n) da}{v} \text{ (defining div } \mathsf{F}) \tag{74}$$

Now the left member of Eq. (71) was obtained by dividing the surface-integral of the intensity over an infinitesimal cube by the volume of the cube, and it is, therefore, the divergence of the electric intensity.

Whence, in vector notation Eq. (71) is written in the form

$$\text{div } \mathsf{F} = \frac{\rho}{p}. \tag{75}$$

88. The Field between Two Parallel Metal Plates.—Figure 51 may represent the two parallel metal plates of an absolute disk electrometer or of a condenser. Let us suppose that the distance between the plates is quite small in comparison with their diameters. Under these conditions, if the plates are charged positively and negatively, respectively, by transferring electrons from one plate to the other, the positive and negative charges will be found mainly upon the adjacent surfaces of the plates. The surface density of the charge will be quite uniform over all of the adjacent surfaces

except for a narrow strip of the surface bordering on the edge of each plate. At the rounded edge of each plate the surface density of the charge may be from 50 to 250 per cent (depending upon the radius of curvature of the edge) greater than the surface density over the large central portion of the surface.

88a. Calculation by the Inverse Square Law.—Let us use the inverse square law to compute the electric intensity which the charge on one plate alone causes at a point P which lies in the space between the plates near their center.

Let σ represent the surface density of the charge (in coulombs per square centimeter) upon the plate, and h represent the perpendicular distance PO from the point P to the charged surface.

Consider the intensity at P due to the charge on a circular strip of surface of radius x and of width dx described with the foot of the perpendicular PO as a center.

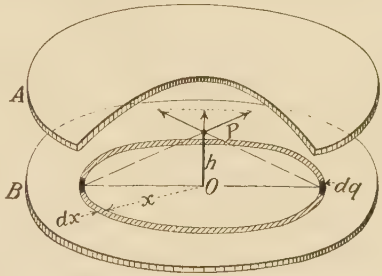


FIG. 51.—The field between parallel plates.

Any small element of charge dq on this strip gives rise to an intensity dF at P equal to

$$dF = \frac{dq}{4\pi p(x^2 + h^2)}.$$

But that component of this intensity which is parallel to the surface will be neutralized by the like component of an equal element of charge which lies diametrically across from the first element. Therefore, the only effective component of the intensity at P due to each element of charge on the strip will be the component perpendicular to the surface, namely,

$$dF_1 = \frac{dq}{4\pi p(x^2 + h^2)} \frac{h}{\sqrt{x^2 + h^2}}.$$

Therefore, the entire charge on the strip, namely, $2\pi x(dx)\sigma$, causes at P an electric intensity in the direction OP equal to

$$dF_2 = \frac{2\pi\sigma h\ xdx}{4\pi p(x^2 + h^2)^{3/2}}. \qquad (76)$$

The electric intensity at P due to all the charge lying on a circular area of radius r described about O as a center is the sum of the intensities due to charge on all the circular strips contained in the area, namely,

$$F = \int_0^r \frac{2\pi\sigma h}{4\pi p} \frac{xdx}{(x^2 + h^2)^{3/2}}$$

$$= \frac{\sigma h}{2p}\left[-\frac{1}{\sqrt{x^2 + h^2}}\right]_0^r = \frac{\sigma h}{2p}\left[\frac{1}{h} - \frac{1}{\sqrt{r^2 + h^2}}\right]$$

$$F = \frac{\sigma}{2p}\left(1 - \frac{h}{\sqrt{r^2 + h^2}}\right). \qquad (77)$$

If the radius of the charged area is 1000 or more times as great as the distance of the point P from the charged surface, this expression takes the simple form

$$F \text{ (volts per cm.)} = \frac{\sigma}{2p} \text{ (coulombs per sq. cm.)}. \tag{78}$$

Since the charge of opposite sign on the surface of the other plate causes at P an equal intensity, it follows that the resultant electric intensity at any point lying between two extended parallel plates near their center will be

$$F \text{ (volts per cm.)} = \frac{\sigma}{p} \text{ (coulombs per sq. cm.)} \tag{79}$$

and the potential difference between the plates will be

$$E \text{ (volts)} = bF = \frac{b\sigma}{p}, \tag{80}$$

in which, b represents the distance between the plates.

Since the electric intensity at the surface of the plate A due to the charge on plate B is $\sigma/2p$, and since the quantity of electricity on a square centimeter of the surface of A is σ, it follows that the charge on each square centimeter of the charged surface A will be attracted toward B with a force of $\sigma^2/2p$ dyne-sevens; that is, each square centimeter of the charged surface of the conducting plates will be subject to a force whose value is expressed by the formula

$$f \text{ (dyne-sevens per sq. cm.)} = \frac{\sigma^2}{2p} \text{ (coulombs per sq. cm.)}. \tag{81}$$

88b. Calculation by Gauss's Theorem.—Let us now use Gauss's theorem to calculate the intensities. The equipotential surfaces between the plates, except near their edges, will be plane surfaces parallel to the surfaces of the plates. Consider a right cylinder $ABCD$ of Fig. 52 of cross-sectional area a, with its elements perpendicular to the equipotential surfaces. Suppose one base lies in the positively charged plate and the other coincides with any of the equipotential surfaces. The quantity of electricity within this cylinder is $a\sigma$. Therefore, the flux of the electric intensity in the outward direction over the surface of this cylinder must be $a\sigma/p$. But the electric intensity over the base within the conducting plates is zero; and along the sides (elements) the intensity is parallel to the surface. Therefore, the surface-integral over the base of area a lying in the space between the plates must be $a\sigma/p$ and since the intensity is perpendicular to the base, the intensity must be σ/p at every point of the base.

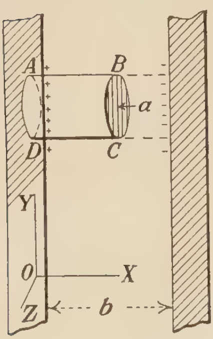


FIG. 52.—Flux between parallel plates.

88c. Calculation by Poisson's Equation.—Let us now apply Poisson's equation to calculate the intensities. Let the axes of Y and Z be taken

parallel to the faces of the plates. Then $\frac{\partial^2 E}{\partial y^2}$ and $\frac{\partial^2 E}{\partial z^2}$ are each zero and the equation reduces to

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\rho}{p}.$$

Whence
$$\frac{\partial E}{\partial x} = -\int \frac{\rho}{p}\,dx.$$

Let x be measured from a point O which lies within the body of the positive plate. Then, in taking the above integral from the point O to any point P which lies in the space between the two plates, the only portion of the path which contributes any value to the integral is the elementary length passing through the layer of charge on the surface of the positive plate. (At all other portions of the path ρ is zero.) The integral of ρdx through this layer of charge is the surface density of charge, σ.

Therefore, the value of the electric intensity at any point P which lies between the plates is

$$\frac{\partial E}{\partial x} = -\frac{\sigma}{p} + \left[\frac{\partial E}{\partial x}\right] \text{at } O.$$

But since the point O lies within a conductor, the intensity at O must be zero. Therefore, the intensity at P is directed from the positive to the negative plate, and has the value σ/p.

The potential E, therefore, at any point between the plates, will be given by the equation

$$E = -\frac{\sigma}{p}\int dx.$$

Hence, the increase in potential from the positive plate to the negative is

$$\triangle E = -\frac{\sigma b}{p},$$

in which b is the distance between the faces of the plates.

89. The Electric Intensity Near the Surface of a Charged Conductor and the Force upon Its Surface.—We have seen that both the electric intensity in the region between two parallel plates and the force exerted upon each square centimeter of charged surface of the plates depend upon the surface density of the charge in the simple manner expressed in Eqs. (79) and (81). We now propose to show, by a demonstration suggested to Poisson by Laplace, that the electric intensity at points immediately outside a charged conductor of any shape whatsoever, and the force upon its charged surface, are expressed by these same formulas.

Let P in Fig. 53 represent a point immediately outside the charged conductor at a place where the surface density of charge is σ. Let a perpendicular be dropped from P to the surface and let P' represent a point on this perpendicular but immediately under the surface of the conductor.

Now the electric intensity at P and at P' may be considered to be made up of two parts: a part F_1 due to the charge Q_1 on a small circular patch of radius r of the surface centered about PP', and a part F_2 due to the remainder of the charge Q_2 on this and other conductors in the field.

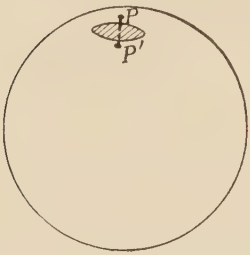


FIG. 53.—Force on the surface of a conductor.

From the previous section, the value of the electric intensity F_1 is given by the expression

$$F_1 = \frac{\sigma}{2p}\left(1 - \frac{h}{\sqrt{r^2 + h^2}}\right). \tag{77}$$

If the points P and P' are at an infinitesimal distance from the surface, h is an infinitesimal and the charge Q_1 on the small patch of surface gives rise to an intensity at P and P' whose value is

$$F_1 = \frac{\sigma}{2p}. \tag{78}$$

At P the intensity F_1 is perpendicular to the surface and is directed outward; at P' the intensity F_1 is directed inward, since P and P' lie on opposite sides of the charge Q_1 on the small patch. Now since P' lies within the conductor, the intensity at P' is zero. All the remainder of the charge Q_2, therefore, must give rise to an intensity F_2 at P' such as to neutralize exactly the intensity F_1 at P'. That is to say, F_2 must be equal to $\sigma/2p$ and must be perpendicular to the surface in the outward direction.

Now, since the points P and P' are an infinitesimal distance apart and since they are similarly located with reference to the charge Q_2, the intensity F_2 at P must be identical in value and direction with the intensity F_2 at P'. These conclusions are expressed in the following statement:

At a point P immediately outside a conductor at a place where the surface density of charge is σ, the charge Q_1 on a small por-

tion of the surface area near P and the charge Q_2 on all the remainder of the conducting surfaces each give rise to an electric intensity of the value $\sigma/2p$, which is perpendicular to the surface in the outward direction. The resultant intensity F outside the surface is, therefore, σ/p.

$$F \text{ (volts per cm.)} = \frac{\sigma}{p} \text{ (coulombs per sq. cm.)}. \tag{79}$$

Consider any very small patch of area a on the surface of a charged conductor. The quantity of electricity on this patch is $a\sigma$. The remaining charges in the field cause at points on this patch an electric intensity of the value $\sigma/2p$. Therefore, the outward force f_1 exerted on the charge $a\sigma$ will be

$$f_1 \text{ (dyne-sevens)} = (a\sigma)\frac{\sigma}{2p}$$

and the outward force upon the charged surface per square centimeter of area will be

$$f \text{ (dyne-sevens per sq. cm.)} = \frac{\sigma^2}{2p} \text{ (coulombs per sq. cm.)}. \tag{81}$$

The expression for the force on the surface of a charged conductor may also be written in terms of the electric intensity F which exists immediately outside of the charged surface. Thus, from Eq. (79), $\sigma = pF$. Upon substituting this value of σ in Eq. (81) we obtain

$$f \text{ (dyne-sevens per sq. cm.)} = \frac{pF^2}{2} \text{ (volts per cm.)}. \tag{82}$$

While the electrostatic force on the surface of a charged conductor is large enough to be utilized in electrostatic voltmeters, the following calculation shows it to be a comparatively feeble force. The air in the vicinity of large conductors breaks down if the electric intensity is raised to 30,000 volts per centimeter. Therefore, the greatest electrostatic force which can be exerted upon a conductor immersed in air will not exceed the following:

$$f_m = \frac{pF^2}{2} = \frac{8.85\ (30{,}000)^2}{2 \times 10^{14}} = 3.98 \times 10^{-5}$$
$$= 398 \text{ dynes per sq. cm.}$$

This force would just be sufficient to hold a disk of aluminum 0.15 centimeter thick against the force of gravity.

90. Field between Concentric Spheres.—In Fig. 54, portions of two concentric spherical metal shells of radii r_1 and r_2 are represented. Let Q coulombs of electrons be transferred from the inner to the outer shell. From the spherical symmetry, the charges will be uniformly distributed over the inner surface of the outer shell and over the outer surface of the inner shell. The surface densities of charge will be

$$\sigma \text{ (inner)} = \frac{Q}{4\pi r_1^2} \text{ (coulombs per sq. cm.).}$$

$$\sigma_1 \text{ (outer)} = \frac{Q}{4\pi r_2^2} \text{ (coulombs per sq. cm.).}$$

The demonstration given in Sec. 82 shows that the charge on the outer shell will give rise to zero intensity at all points enclosed by the shell, and that the charge on the inner shell will give rise to zero intensity at all points enclosed by the inner shell.

Let us compute the electric intensity which the charge on the inner shell gives rise to at any point P which lies outside of the inner shell.

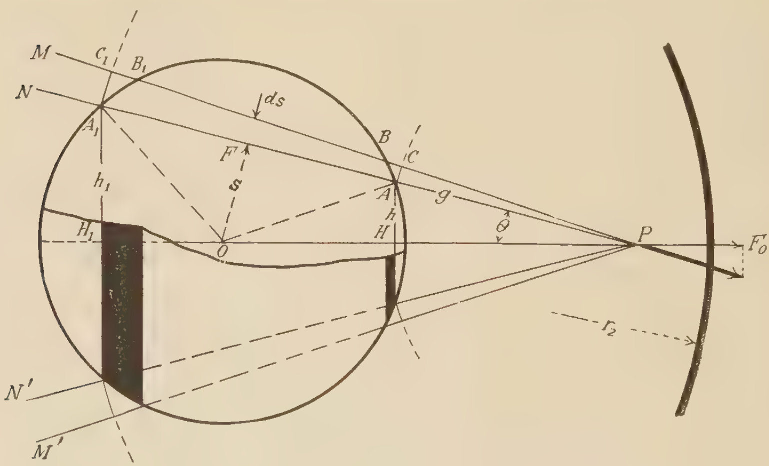


Fig. 54.—Field between concentric spheres.

90a. Computation by the Inverse Square Law.—Consider the two cones MPM' and NPN' having the common axis OP. Let us compute the electric intensity at P due to the charge on the two elementary zones which lie between these cones.

Let b represent the distance OP from the center of the spheres to the point P.

s represent the perpendicular distance OF from the center O to the elements of the inner cone.

g and g_1 represent the distances PA and PA_1 to the nearer and more remote zones, respectively.

k and k_1 the lengths AB and A_1B_1 of the arcs of the elementary zones.

h and h_1 represent the distances AH and A_1H_1, respectively.

v and v_1 represent the lengths of the elementary arcs AC and A_1C_1 described with P as a center.

The charge q on the zone AB which is the nearer to P is, $q = 2\pi hk\sigma$. In this expression

$$\frac{h}{g} = \frac{s}{b} \quad \text{or} \quad h = \frac{gs}{b}.$$

Also,
$$k = \frac{v}{\sin \angle ABC}.$$

But
$$\angle ABC = \angle AOF.$$

Therefore,
$$\sin \angle ABC = \sin \angle AOF = \frac{\sqrt{r_1^2 - s^2}}{r_1}.$$

Also,
$$\frac{v}{ds} = \frac{g}{\sqrt{b^2 - s^2}}.$$

Whence,
$$k = \frac{r_1 g ds}{\sqrt{r_1^2 - s^2}\sqrt{b^2 - s^2}}.$$

Therefore,
$$q = \frac{2\pi\sigma r_1 g^2 s ds}{b\sqrt{r_1^2 - s^2}\sqrt{b^2 - s^2}}.$$

The intensity F_o at P due to the charge on the nearer zone is

$$F_o = \frac{q}{4\pi p g^2}\cos\theta = \frac{q}{4\pi p g^2}\,\frac{\sqrt{b^2 - s^2}}{b}$$

$$F_o = \frac{2\pi\sigma r_1}{4\pi p b^2}\,\frac{s ds}{\sqrt{r_1^2 - s^2}}.$$

In like manner it may be seen that the charge q_1 on the more remote zone is

$$q_1 = \frac{2\pi\sigma r_1 g_1^2 s(ds)}{b\sqrt{r_1^2 - s^2}\sqrt{b^2 - s^2}}$$

and that the intensity F_1 at P due to this charge works out to be equal to F_o. In other words, both elementary charges give rise to the same intensity at P, or the resultant intensity F_2 due to the charges on both zones is

$$F_2 = F_o + F_1 = \frac{4\pi\sigma r_1 s ds}{4\pi p b^2\sqrt{r_1^2 - s^2}}. \tag{83}$$

The intensity F at P due to the charge on the entire inner shell may now be found by integrating the above expression.

$$F = \Sigma F_2 = \int_0^{r_1} \frac{4\pi\sigma r_1}{4\pi p b^2}\,\frac{s ds}{\sqrt{r_1^2 - s^2}} = \frac{4\pi\sigma r_1}{4\pi p b^2}\Big[-\sqrt{r_1^2 - s^2}\Big]_0^{r_1}$$

$$F = \frac{4\pi r_1^2 \sigma}{4\pi p\, b^2}. \tag{84}$$

But $4\pi r_1^2\sigma$ is the charge Q on the inner shell. Therefore, the expression for the electric intensity at P may be written in the form

$$F = \frac{Q}{4\pi p b^2}. \tag{85}$$

Since b is the distance of the point P from the center of the sphere, we see that the following simple rule may be formulated for computing the electric intensities due to the charge Q.

The electric intensity due to a charge Q which is uniformly distributed over a spherical surface is zero for all points within the spherical surface; for all points exterior to the surface the intensity is the same as if the charge Q were all concentrated at the center of the spherical surface.

90b. Computation by Gauss's Theorem.—This same conclusion may be arrived at with much less work by applying Gauss's theorem in the following manner. Imagine a spherical surface of radius b concentric with the two shells and lying between them. If the charge on the inner shell is Q, the surface-integral of the intensity over the spherical surface must be Q/p. But from the spherical symmetry, it follows that the intensity vectors will be perpendicular to this surface at all points. Therefore, the electric intensity at any point P on the spherical surface of radius b is

$$F = \frac{Q}{p} \div 4\pi b^2 = \frac{Q}{4\pi pb^2}.$$

The electric intensities, F_1 and F_2, immediately outside the inner sphere and inside the outer sphere are seen to be $Q/(4\pi pr_1^2)$ and $Q/(4\pi pr_2^2)$ respectively.

But the surface densities of charge, σ_1 and σ_2, on these spheres are $Q/(4\pi r_1^2)$ and $Q/(4\pi r_2^2)$. These calculations, therefore, show that the electric intensity (in space) at the charged surfaces of the spheres is equal to the surface density of charge σ, divided by the permittivity p, thus confirming Eq. (79) of the previous section.

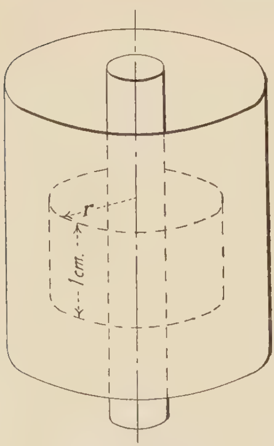


Fig. 55.—Flux between coaxial cylinders.

91. The Field between Coaxial Cylinders.—Let us use Gauss's theorem to compute the electric intensities between two coaxial metal cylindrical shells whose length is very great in comparison with the radius of the larger. If a charge of q coulombs of electrons per centimeter of axial length is transferred from the inner shell to the outer, the charges will be uniformly distributed over the outer surface of the inner and the inner surface of the outer shell, except that the surface density of the charge on each shell will be somewhat greater than the average for the shell for a short distance in from each end of the shell.

Consider the portion of a cylinder of radius r which lies between the two shells (Fig. 55) and is included between parallel planes 1 centimeter apart. From the cylindrical symmetry, the intensity vectors are radial and therefore perpendicular to this cylindrical surface and parallel to its plane ends.

Since the quantity of electricity within this cylinder is q, the surface-integral of the intensity over its surface is q/p, and the value of the electric intensity at points on its cylindrical surface is

$$F = \frac{q}{p} \div 2\pi r = \frac{q}{2\pi pr}. \tag{86}$$

Now it may readily be seen that, if the inner shell were removed, a charge of q coulombs per centimeter of length along the axis of the outer cylinder would give rise to this same intensity. From this, the following conclusion may be drawn.

The electric intensity at points exterior to, and not too far from, the midpoint of a long cylinder which is uniformly charged with q coulombs per centimeter of length is the same as if the charge were uniformly distributed along the axis of the cylinder.

92. Calculation of Capacitance.—From the data of the preceding sections the capacitance of condensers made up of parallel plates, concentric spheres, or coaxial cylinders may be readily calculated.

92a. Two Parallel Plates.—If a charge of q coulombs per square centimeter of surface has been transferred from one of two extended parallel metal plates to a second plate which is separated from it by a layer of insulating material of thickness b, the electric intensity at all points between the plates is q/p and the potential difference E between the plates is qb/p. Therefore, the capacitance C of one plate to the other, which is the ratio of the total transferred charge to the resulting potential difference, is

$$C \text{ (farads)} = \frac{pa}{b} \text{ (cm.)}, \tag{87}$$

in which, b is the thickness of the insulating material (in the direction of the lines of intensity).
a is the cross-sectional area of one face of one plate, or it is the cross-sectional area of the insulating material (perpendicular to the lines of intensity).
p is the permittivity of the insulating material.

It will be noted that the capacitance depends entirely upon the dimensions and the kind of insulating material between the plates, and not at all upon the material of which the plates are composed, and not upon the dimensions of the plates, except as they serve as boundaries for the dielectric.

The above formula takes no account of the edge effect. The actual capacitance is somewhat greater because of the increased surface density of charge on and near the edges of the plates.

92b. Two Concentric Spheres.—If a charge of Q coulombs of electrons is transferred from the inner sphere to the outer, the electric intensity in the space between the two spheres at a distance x from the center is directed

radially outward and has the value $Q/4\pi p x^2$. Therefore, the drop in potential E, from the inner sphere to the outer, is

$$E = \int_{r_1}^{r_2} \frac{Q}{4\pi p x^2}\, dx = -\left[\frac{Q}{4\pi p x}\right]_{r_1}^{r_2} = \frac{Q}{4\pi p}\,\frac{r_2 - r_1}{r_1 r_2}. \tag{88}$$

The capacitance of one sphere to the other is

$$C \text{ (farads)} = 4\pi p\,\frac{r_1 r_2}{r_2 - r_1} \text{ (cm.)}, \tag{89}$$

in which r_1 and r_2 are the radii of the smaller and larger spheres, respectively.

For the limiting case, in which r_2 becomes very large in comparison with r_1, this simplifies to

$$C \text{ (farads)} = 4\pi p r_1 \text{ (cm.)}. \tag{90}$$

For the other limiting case, in which the thickness b of the insulating material between the spheres, namely, $r_2 - r_1$, becomes very small in comparison with the mean radius $r = (r_2 + r_1)/2$, Eq. (89) simplifies to

$$C \text{ (farads)} = p\,\frac{4\pi r^2}{b} = p\,\frac{a}{b} \text{ (cm.)},$$

which is identical with the Eq. (87), which was derived for parallel plates.

92c. Two Coaxial Cylinders.—If a charge of q coulombs of electrons per centimeter of axial length has been transferred from one coaxial cylinder to another, the electric intensity in the space between the cylinders is directed radially outward from the axis, and has the value $q/2\pi p x$ at all points at a distance x from the axis.

Therefore, the rise in potential E from the outer cylinder to the inner is

$$E = -\int_{r_2}^{r_1} \frac{q}{2\pi p x}\, dx = \frac{q}{2\pi p}\Big[\log x\Big]_{r_1}^{r_2} = \frac{q}{2\pi p}\log\frac{r_2}{r_1}. \tag{91}$$

If the cylinders are of equal length l the total quantity of electricity transferred is ql, and the capacitance of one cylinder to the other (neglecting the end effect) is

$$C \text{ (farads)} = \frac{2\pi p l}{\log \dfrac{r_2}{r_1}} \text{ (cm.)}, \tag{92}$$

in which, l is the length of one cylinder,
r_2 is the radius of the larger cylinder,
r_1 is the radius of the smaller cylinder.

For the limiting case in which the thickness b of the insulating material between the cylinders becomes very small in comparison with the mean radius $r = (r_1 + r_2)/2$, we may write with no appreciable error

$$\log\frac{r_2}{r_1} = \log\frac{r_1 + b}{r_1} = \log\left(1 + \frac{b}{r}\right) = \frac{b}{r}.$$

Substituting this in Eq. (92), it simplifies to the parallel-plate formula, thus

$$C = \frac{2\pi p l r}{b} = \frac{pa}{b}.$$

93. Theorems Used in Mapping Fields by Equipotential Surfaces and Tubes of Electric Intensity.—The possibility of mapping electrostatic fields by a system of tubes of electric intensity and a system of equipotential surfaces has been referred to in Secs. 39 and 57. In these earlier sections the method was neither discussed nor illustrated, since Gauss's theorem was not available at that stage for the demonstration of the theorems relating to the tubes and to the distribution of the charges, which we now proceed to give.

The following corollaries are simple deductions from the definitions of equipotential surfaces and of lines of electric intensity.

93a. Corollary.—No work will be done by or against the forces of the field in moving a charge around upon an equipotential surface.

93b. Corollary.—The surface of any homogeneous conductor, or system of connected conductors, in an electrostatic field is an equipotential surface.

For if points of the same conductor were at different potentials, this would imply the existence of forces on the electrons in the body of the conductor. Under these forces a drift of the free electrons of the conductor to the points of high potential would occur until the forces within the conductor vanished.

93c. Corollary.—The electric intensity at any point whatsoever of an equipotential surface is at right angles to the surface. The lines of electric intensity therefore cut through any equipotential surface at right angles to the surface.

93d. Corollary.—If the field is mapped out by surfaces at equal intervals of potential, say 1 or 10 volts apart, then the intensities at different points of the field are inversely proportional (approximately) to the perpendicular distances between the nearest surfaces.

The following theorems, which are readily demonstrated by means of Gauss's theorem, express properties which are peculiar to inverse square systems.

93e. Theorem.—If a closed surface is so taken that every point of the surface lies in a homogeneous conducting material, the net charge inside the surface is zero.

Since the intensity is zero at all points of the surface, the surface-integral of the intensity over the surface is zero, and from Gauss's theorem the total charge enclosed by the surface must be zero.

93f. Theorem.—The charge imparted to a conductor resides on its surface and not in its interior.

Any point or portion of the interior can be surrounded by a closed surface lying entirely in the conductor. At all points of this surface the intensity must be zero, and therefore the charge enclosed must be zero.

93g. Theorem.—If any number of charged bodies are placed inside a hollow closed conducting vessel, the charge on its inner surface will be equal in magnitude but opposite in sign to the net charge on the bodies inside.

A closed surface can be taken in the conducting material immediately surrounding the inner surface of the vessel. The intensity at all points of this surface is zero, and therefore the net charge enclosed by it must be zero.

93h. Theorem.—If a hollow, closed, conducting vessel is itself uncharged but contains charged bodies whose complimentary charges are on objects exterior to the vessel, then the charge on the exterior surface of the vessel is equal in magnitude and in sign to the net charge on the charged bodies enclosed.

If the charge on the conducting vessel is zero, then, since the charge on the inner surface is equal and of opposite sign to the net charge on the bodies, the charge on the outer surface must also be equal but of the same sign to the net charge on the bodies enclosed.

93i. Theorem.—The potential cannot have a maximum or a minimum value at any point in the field which is not occupied by an electric charge.

If the potential has a maximum value at any point P, then at all points on a small closed surface surrounding P the potential must be less, and at all points of a second surface just outside the first the potential must be still less. Hence P must be surrounded by a closed surface at all points of which the electric intensity is directed outward. Therefore the surface-integral of the electric

intensity over this surface cannot be zero, but must be positive. Hence, from Gauss's theorem the surface must contain a positive charge, or there must be a positive charge at P. In like manner, it follows that if the potential at P is a minimum there must be a negative charge at P.

93j. Theorem.—If the potential is a maximum at any point, the point must be occupied by a positive charge; if the potential is a minimum at any point, that point must be occupied by a negative charge. (See the demonstration of 93*i*.)

If a line of electric intensity is traversed in the positive direction, it passes through surfaces at lower and lower potentials. Hence lines of intensity can begin only at points where the potential is a maximum and end only at points where it is a minimum. Combining this with Sec. 93*j* it follows that:

93k. Theorem.—Lines of electric intensity can begin only on positive charges, and can end only on negative charges, but a line of electric intensity cannot spring from and end upon the same (equipotential) conductor.

93l. Theorem.—If the space enclosed by a hollow conducting shell contains no charged bodies, the intensity at all points in the space enclosed by the shell is zero, and the potential at all these points is that of the shell.

Since there are no charges in the space on which lines of intensity can begin or end, and a line cannot begin and end on the interior conducting surface, there can be no lines of intensity.

93m. Theorem.—The negative charge on which a tube of electric intensity terminates is numerically equal to the positive charge on which it starts.

Consider the tube originating on the conductor A of Fig. 56 and terminating on the conductor B. The intensity is parallel to the walls and is zero at each of the two terminal surfaces, since they lie inside the conductor. The surface integral of the intensity over the tubular surface and its ends is, therefore, zero, and from Gauss's theorem the net charge within the tube must be zero.

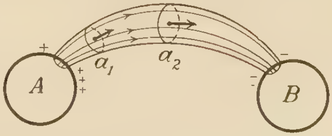


FIG. 56.—Tubes of intensity.

93n. Theorem.—The product of the electric intensity times the normal cross-sectional area of any given tube which contains no charge except at its terminal surfaces is a constant for all cross-sections, and is equal to q/p, in which q represents the algebraic value of either terminal charge.

$$F_1a_1 = F_2a_2 = \frac{q}{p}. \tag{93}$$

In other words, the flux (surface-integral) of the electric intensity over all diaphragms of a given tube in a region free of charge is a constant, or the tubes are tubes of **constant flux of electric intensity.**

Consider the closed surface consisting of that portion of the tube of intensity of Fig. 56 which is included between the terminal surface in the conductor and either of the normal cross-sectional diaphragms, a_1 or a_2. The charge within this surface is q, and from Gauss's theorem it follows that the surface-integral of the intensity over the surface must be q/p. But the only portion of the surface contributing anything to the integral is the diaphragm, and a_1 represents the area of this diaphragm and F_1 the intensity normal to the area. Therefore F_1a_1 must equal q/p.

93o. Theorem.—If a tube of intensity passes through a layer of electric charge containing q coulombs (within the tube), the flux changes by q/p.

Figure 65 represents such a situation in which the layer of charge lies on the surface of an insulating material.

93p. Definition.—A tube of unit flux of electric intensity, or a unit tube, is defined to be a tube of such cross-sectional area that the flux (surface-integral) of the electric intensity over any diaphragm is unity.

The following corollaries are true of fields which are diagrammed by means of unit tubes.

93q. Corollary.—The value of the electric intensity at any point P will be equal to the number of these **unit tubes** which cut across a square centimeter of the equipotential surface at P.

93r. Corollary.—Each coulomb of electricity will be the origin, if positive, or terminus, if negative, of $1/p$ unit tubes.

In drawing the diagrams to represent a given field by means of these unit tubes, the outlines of the tubes themselves are not drawn, but each tube is represented by a line of electric intensity drawn along its axis. This practice leads to the substitution of **line** for unit **tube** in the above corollaries, resulting in such expressions as:

93q′. Corollary.—The value of the electric intensity at any point P is equal to the number of lines of electric intensity which cut across a square centimeter of the equipotential surface at P.

93r′. Corollary.—Each coulomb of electricity will be the origin, if positive, or terminus, if negative, of $1/p$ lines of electric intensity.

Of course, if the intensities are very high at some points in the field which is to be represented (say 20,000 volts per centimeter), it is impracticable to draw one line per unit tube. In this case the diagram of the field would be constructed to the scale of one line for 1000 or 5000 or 10,000 unit tubes.

94. Examples of Electric Fields. In the following maps of the fields about particular conductors, the heavier lines represent the center lines of tubes of equal flux of electric intensity, and the lighter lines represent the intersections of equipotential surfaces with the plane of the paper. The surfaces have been taken at equal intervals of potential.

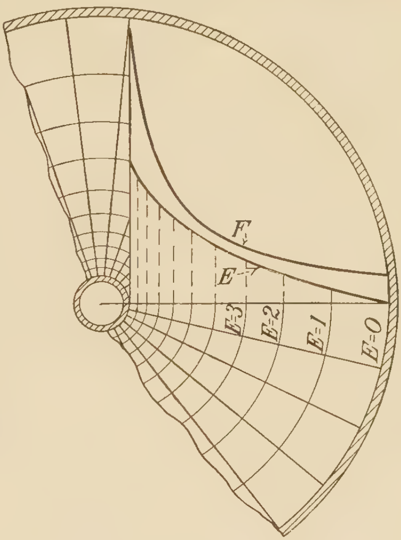


FIG. 58.—Map of the field of coaxial cylinders.

94a. Two Coaxial Cylinders.—The field of two long coaxial circular cylinders charged, respectively, with the quantities of electricity $+q$ and $-q$ per unit length is mapped in Fig. 58. The axes of the cylinders are perpendicular to the plane of the paper.

We have seen that the electric intensity at any point between the two charged cylinders and at a distance x from the axis is directed radially outward and has the value

$$F \text{ (volts per cm.)} = \frac{q}{2\pi p x}. \tag{86}$$

The values of the intensity at points on any radial line, as OR, are shown by the lengths of the ordinates from the points on the line OR to the curve marked F on the figure.

The potential E at a point on the line OR and at the distance r from the axis is obtained from Eq. (26), thus

$$E = -\int_{r_2}^{r} F \cos (F, l)\, dl = -\int_{r_2}^{r} \frac{q dx}{2\pi p x}$$

$$E \text{ (volts)} = \frac{q}{2\pi p} \log \frac{r_2}{r}. \qquad (94)$$

The values of the potential at points on the line OR are shown by the lengths of the ordinates from the points on the line OR to the curve marked E on the figure.

94b. Two Parallel Wires.—The field of two long parallel straight wires of circular cross-section charged, respectively, with the quantities of electricity

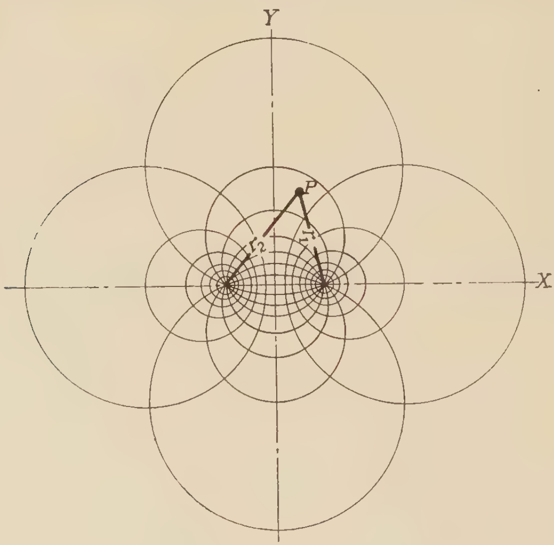


FIG. 59.—Map of field of parallel wires.

$+q$ and $-q$ per unit length is shown in Fig. 59. The axes of the wires are perpendicular to the plane of the paper. The interaxial distance of the wires is assumed to be so great in comparison with the diameter of the wires that the surface density of the charge may without appreciable error be assumed to be uniform at all points of the surface of the wire (except near the ends).

Under the above conditions the potential of a point P which lies at the distances r_1 and r_2 from the axes of the positive and negative wires is

$$E = \frac{q}{2\pi p} \log \frac{r_2}{r_1}. \qquad (95)$$

To obtain the equations in rectangular coordinates of the curves representing the intersections of the equipotential surfaces with the plane of the paper, let us pass the X axis through the centers of the two wires and then take the origin of coordinates at a point midway between the two wires. Then, if s represents the distance between the centers of the wires, the above equation may be written

$$E = \frac{q}{2\pi p} \log \frac{\sqrt{y^2 + (x + s/2)^2}}{\sqrt{y^2 + (x - s/2)^2}}.$$

Solving, $\left(y^2 + x^2 - sx + \frac{s^2}{4}\right) \epsilon^{\frac{4\pi pE}{q}} = y^2 + x^2 + sx + \frac{s^2}{4}$

or

$$x^2 + y^2 - xs \frac{\epsilon^{\frac{4\pi pE}{q}} + 1}{\epsilon^{\frac{4\pi pE}{q}} - 1} + \frac{s^2}{4} = 0.$$

By assigning to the potential E in the above equation a numerical value, positive or negative, we obtain the equation of the intersection of the equipotential surface of that potential with the plane of the paper.

Putting $E = 0$, we obtain $x = 0$. That is, the equipotential surface of zero potential is a plane surface of infinite extent midway between the two wires.

Starting with the value zero and assigning to E larger and larger positive numerical values, we obtain the equations of successive circles whose diameters become smaller and smaller, and whose centers shift along the X axis from points infinitely remote and approach the center of the positive wire as a limit. The positive equipotential surfaces of these two linear distributions of charge are thus cylindrical surfaces which lie entirely on the same side of the plane of zero potential as the positive wire. By assigning negative numerical values to E, a similar set of cylindrical equipotential surfaces is found around the wire with the negative charge. The largest value, E_m, which it is physically permissible to assign to E is found by taking the point P so that it lies on the surface of one of the conductors. It is

$$E_m = \frac{q}{2\pi p} \log \frac{s}{r}, \tag{96}$$

in which r is the radius of the wire, and s is the distance between the centers of the wires.

94c. Bodies of Irregular Shape.—The features of the lines of intensity and equipotential surfaces in the field of bodies of irregular shape are illustrated in Fig. 60, which is for the case in which electrons have been transferred from an extended plane conductor A to an irregular shaped conductor B above it. The other conductors C and D are charged by inductive action only. The curves cannot be calculated, and can only be sketched in to conform in a general way with the theorems of the previous section. The lines of intensity start from the surfaces of the conductors at right angles to the surfaces and cut all equipotential surfaces at right angles. They converge upon protruding portions, particularly edges and corners, of the

conductors, and are widely separated at the re-entrant surfaces. The equipotential surfaces near the surface of the conductors conform somewhat closely to the shape of the surface, crowding more closely to the protuberances and receding from the surface at the re-entrant portions. Thus the irregularities of the surface become gradually smoothed out in successive equipotential surfaces which differ more and more from the potential of the conductor.

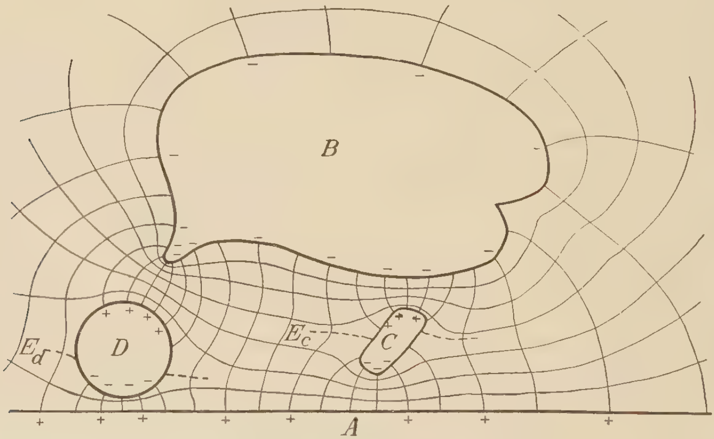


Fig. 60.—Map of field about irregular conductors.

95. Faraday's Conception of Tubes of Electric Intensity.—The invention of tubes of electric intensity as a means of accounting for and picturing the features of the electric field around charged conductors is due to Faraday. In the foregoing paragraphs we have developed the concept of the tubes and of their properties as a most effective scheme for investigating and picturing the properties of an inverse square field of force. This, however, was not the origin of the concept with Faraday. Mathematical methods and theorems, other than graphical processes and the theorems of arithmetic and Euclidean geometry, played a very small part in Faraday's mental processes, and the tubes were Faraday's only device for correlating in a quantitative way the diverse phenomena of the electrostatic field. They apparently originated in his mind as a scheme for portraying peculiar states in a hypothetical **medium** (an **ether**), surrounding the charged bodies. Faraday conceived that the action of one charged body upon another remote body depended "on induction being an action of the contiguous particles of the dielectric, which, being thrown into a state of polarity and tension, are in mutual relation by their forces in all directions." From a study of the fields of a number of cases he was led to conclude that "the direct inductive force, which may be conceived to be exerted in (these) lines between the two limiting and charged conducting surfaces, is accompanied by a lateral

or traverse force equivalent to a dilation or repulsion of these lines; or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction."[7]

From the results obtained in Sec. 89 we may by the following argument derive expressions for the magnitude of these hypothetical stresses. Since the electric intensity F immediately outside any charged conducting surface is

$$F \text{ (volts per cm.)} = \frac{\sigma}{p} \tag{79}$$

and since the force f on the surface is

$$f \text{ (dyne-sevens per sq. cm.)} = \frac{\sigma^2}{2p}, \tag{81}$$

it follows that the distribution of the forces on the charged conductors in any field is the same as the distribution which would be computed if the hypothetical medium is imagined to be in a state of tension along the tubes of intensity, such that the value of the tensile stress f_t across any normal cross-section of a tube is expressed by the formula

$$f_t \text{ (dyne-sevens per sq. cm.)} = \frac{1}{2}pF^2. \tag{82}$$

If we now **postulate** that the sum of the forces acting on any elementary portion of the hypothetical medium must be zero, the following examination of the case of the conical tubes between two concentric spherical conductors shows that we must imagine a system of compressive stresses of the same value to be exerted across the tube walls. Let the charge on the inner sphere be Q.

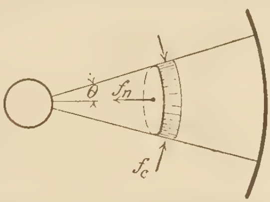


FIG. 61.—The transverse stress.

Consider the hypothetical forces acting on the small dished wafer of the medium lying within the conical tube of force and between the two spherical surfaces of radius r and $r + dr$ as shown in Fig. 61. Let the tube be taken as the frustrum of a circular cone whose surface elements make the infinitesimal angle θ with its axis. Then if f_c represents the value of a possible compressive stress exerted across the walls of the tube, the net force f_n acting on the small wafer of the medium, tending to move it radially inward along the axis of the cone, is

$$f_n = \frac{p}{2}\left[\frac{Q}{4\pi p r^2}\right]^2 \pi\left[r \sin\theta\right]^2 - \frac{p}{2}\left[\frac{Q}{4\pi p(r + dr)^2}\right]^2 \pi\left[(r + dr)\sin\theta\right]^2$$
$$-f_c \sin\theta\,(2\pi r \sin\theta\, dr).$$

Equating this to zero and solving for f_c, we find

$$f_c = \frac{p}{2}\left[\frac{Q}{4\pi p\, r^2}\right]^2$$

or

$$f_c \text{ (dyne-sevens per sq. cm.)} = \frac{p}{2}F^2. \tag{97}$$

[7] *Experimental Researches*, Vol. I, Secs. 1164, 1165, 1221, 1224, 1231, 1297, 1298, 1304, 1613 to 1616.

96. Fields with Two or More Dielectrics.—In deriving the important theorems of electrostatics, such as Gauss's theorem and the theorems about tubes of electric intensity, we have hitherto dealt with the fields around conductors surrounded by a single **homogeneous** medium, such as free space, or air at atmospheric pressure. If a field, containing two or more dielectrics, such as air and slabs of paraffin, or plates of glass, or beakers of oil, is studied, the relations are at first sight more involved. For example, if a slab of paraffin is close to the balls of the Coulomb torsion balance, the force of attraction between the balls may not vary inversely as the square of the distance between the balls. Of course, the same thing is true if a mass of metal is placed near the balls of the balance. In the latter case, we do not say that the inverse square law is not valid when the field contains large conducting masses, but the apparent departure from the law is precisely accounted for by the taking into account the effect (computed by the inverse square law) of the induced charges on the disturbing masses of metal.

In the same way, it is now postulated (as stated in 83f) that the effect of a second dielectric, such as a plate of paraffin, on the electric intensities and on the distribution of the charges on the conductors is to be explained and accounted for by taking into account the forces (computed by the inverse square law) arising from the layers of **concealed** *charge in the dielectric. These layers of concealed charge may be conceived to be the net statistical result of a postulated shift in opposite directions along the lines of electric intensity of the electrons and nuclei* **of the molecules of the dielectric.**

For our purposes, we may picture the molecules of a plate of paraffin under normal conditions either as unpolarized structures, as in (*a*) Fig. 62, or as polarized structures with their axes distributed indiscriminately in all directions, as in (*b*) Fig. 62. We may conceive that the effect of placing the plate of paraffin between the charged plates of a parallel-plate condenser is either to cause an elastic shift of the electrons and protons within the molecular structures, thus **polarizing** the previously unpolarized structures, or it is to twist around and partially align the previously polarized structures of (*b*). In either case, the result

is the state of affairs pictured at (*c*) in Fig. 62. The result of such a process is that the surface layer of paraffin facing the positively charged plate contains an excess of electrons and the surface layer facing the negatively charged plate has a deficit of electrons.

We will find it convenient to refer to these **induced** layers of charge on the face of a dielectric as the **concealed** charges of the field, and to call those charges which accumulate on or in dielectrics by rubbing contact and by conduction from other bodies, together with the charges which are either placed upon or induced **upon conductors,** the **obvious** charges of the field. The **concealed** charges are those resulting from a relative shift of electrons and protons **within** the molecular structures. The

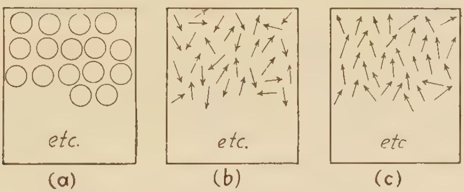


FIG. 62.—Structure of dielectrics.

obvious charges are those resulting from a shift of the free electrons, that is, from electric conduction. The obvious charge on a portion of a conductor can be taken out of the field by the simple expedient of cutting that portion of the conductor free from the remaining portion and then removing it with its charge to another region. The layer of concealed charge induced on the surface of a dielectric cannot be removed from the field by cutting, because the plane of cutting passes **between** the molecular structures and never separates the positively polarized end of a molecule from its negative end.

97. Experimental Data on Dielectrics in Condensers.—Let a given kind of wax or of oil be substituted for the air between the electrodes of a plate, a cylindrical or a spherical condenser, and let the quantity of electricity which must be transferred from one electrode of the condenser to the other to cause any specified difference of potential between the electrodes be measured, first,

with the condenser evacuated of all air, and, second, with the evacuated spaces completely filled with the given oil. It will be found that the quantity of electricity for the oil-insulated condenser is always a definite multiple (say p_r times as great as) of the quantity for the evacuated condenser. The value of the factor p_r is always the same for a given sample of oil, regardless of the proportions of the condenser in which it is tested. No substance has been found for which the value of this factor p_r, which is called the **relative permittivity** of the dielectric, is less than unity. Table 76*b* gives the relative permittivities of a few dielectrics.

98. Dielectric Properties Accounted for in Terms of Concealed Charge.—How may the above experimental relations be interpreted and accounted for? The two condensers were charged to the same specified potential difference. This means that the line-integral of the electric intensity over any and all paths leading from electrode to electrode **through the evacuated space or through the oil** has the same given value, although the obvious charge on the conducting electrodes of the oil condenser is p_r times that on the electrodes of the evacuated condenser. Now the **fact that the line-integral of the intensity has the same value for all paths regardless of the configuration of the test condenser** would most clearly and most simply be accounted for by the following assumptions.

1. Assume the existence of a layer of concealed charge on the boundary surfaces of the dielectric which are in contact with the conducting electrodes, the concealed charge of the dielectric being of the opposite kind to the adjacent layer of charge on the conductor.

2. Assume that the surface density σ_c of the layer of concealed charge is everywhere directly proportional to the surface density σ_o of the obvious charge on the adjacent surface of the conductors, bearing to it the following relation:

$$\sigma_c \text{ (coulombs per sq. cm.)} = -\left(1 - \frac{1}{p_r}\right)\sigma_o. \qquad (98)$$

For, clearly, the resultant or net interface density of charge, namely, $\sigma_o + \sigma_c$, will equal σ_o/p_r, or will everywhere be equal to the surface density of the charge on the conductors of the evacu-

ated condenser. Therefore, by the use of the same inverse square law, intensities will be computed to be identical in value at corresponding points in the oil and in the evacuated space.

The assumption expressed by Eq. (98) leads to the following deductions.

a. The net interface density of charge σ_n is

$$\sigma_n = \sigma_o + \sigma_c = \frac{\sigma_o}{p_r}. \tag{99}$$

b. The electric intensity in the dielectric near its surface but directly below the layer of concealed charge is

$$F_d = \frac{\sigma_n}{p_o} = \frac{\sigma_o}{p_r p_o}. \tag{100}$$

c. The surface density of the concealed charge **at the surface of a dielectric bordering on a charged conductor** is related to the electric intensity F_d in the dielectric by the relation

$$\sigma_c \text{ (coulombs per sq. cm.)} = p_o(p_r - 1)\, F_d \text{ (volts per cm.)}. \tag{101}$$

Let us now postulate that the shift expressed by Eq. (101) holds at all points in the dielectric except for points in a surface film several molecules thick. This postulate may be thus expressed.

98d. Fundamental Postulate Concerning Dielectrics.—**An electric intensity F_d at any point in a dielectric of relative permitivity p_r is accompanied by the shift across any small plane area which is taken in the dielectric perpendicular to the electric intensity, of a quantity of electricity which is directly proportional to the electric intensity at the area—namely, the equivalent shift of $p_o(p_r - 1)F_d$ coulombs of positive electricity per square centimeter in the direction of F_d.**

This postulate will now be regarded as the general postulate underlying the more special relation expressed by Eq. (98). We proceed to apply it in conjunction with the postulate of Sec. 83*b*, namely, that the constant p appearing in the denominator of the inverse square law of force has a fixed value which is **independent of the material occupying the space between or around the attracting or repelling elements under consideration.**

99. The Possibility of a Volume Distribution of Charge Throughout the Dielectric.

—It now becomes necessary to determine whether the postulated polarization which will result in a layer of charge on those surfaces of the dielectric in contact with

the conductors having the obvious charge will also result in a distribution of charge throughout the volume of the dielectric.

Let us consider the field in a region made up, as in Fig. 63, of evacuated space (of permittivity p_o) and one or more homogeneous dielectrics of any configuration and having the relative permittivities p_{r1}, p_{r2}, etc.

Let q be any element of charge which is contributing to the forces of the field, and let $MNOP$ be any closed surface which does not enclose the charge q. Then, under the postulate of Sec. 83*b*, Gauss's theorem enables us to state that the surface integral over this surface, of **that component of the electric intensity which is contributed directly by q,** is zero. Therefore, since in any given dielectric the quantity of electricity which shifts across each unit element of this surface is directly proportional to the normal component of the intensity, it follows that, **if the closed surface lies wholly within a single dielectric,** the quantity of electricity which shifts out across the surface under the intensities directly caused by the charge q, is zero. But this surface represents any surface whatsoever that lies wholly within a given dielectric.

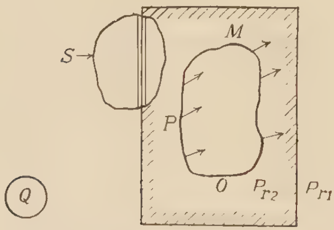


Fig. 63.—Volume distribution of charge in dielectrics.

99a. *Deduction.*—Therefore, the shift postulated in Sec. 98*d* and expressed in Eq. (101) will give rise to no concealed charge within the body of a dielectric, but only at its surface.

100. Interface Conditions.—Let us now consider the conditions at an interface between two dielectrics having different permittivities.

Consider the two points in Fig. 64. A_1 is in the dielectric of lower relative permittivity p_{r1}, and A_2 is in the dielectric of higher relative permittivity p_{r2}. Let these points be adjacent to each other at infinitesimal distances from the interface. Then the portion of the interface near the points may be regarded as a small plane surface, as illustrated in the figure.

Now the two points are identically spaced from all charges causing the field save the layer of concealed charge on the infin-

itesimal patch of the interface which lies between the points. Therefore, the intensities at A_1 and A_2 may each be regarded as the resultant of two components, namely:

a. A component F_c, which is contributed by the layer of concealed charge on the small plane patch of the interface separating the two points. At the two points A_1 and A_2 these components have the same numerical value but they are oppositely directed along the normal from the points to the interface.

b. A component F_s, which has the same value and the same direction at each of the two points. This component is the

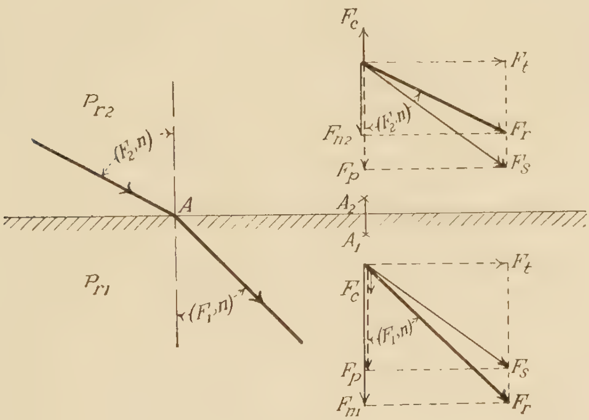


FIG. 64.—Intensities at the interface between dielectrics.

contribution of **all the charges in the field save the concealed charge on the infinitesimal patch of interface,** and it may make any angle whatsoever with the normal to the interface.

Let F_t and F_p represent the components of F_s which are respectively tangential and perpendicular to the interface. By a consideration of the sign and relative value of the shift of electricity in the two dielectrics it may be seen that:

c. If the direction of the perpendicular component F_p is from the dielectric of larger to the dielectric of smaller permittivity, the sign of the resultant or net concealed charge on the patch of interface is positive and the electric intensity F_c contributed by it is directed away from the interface in each dielectric.

d. If the direction of the perpendicular component F_p is reversed, the sign of the concealed charge and the direction of F_c are each reversed.

e. In either case, the intensity F_c due to the concealed layer subtracts from the perpendicular component of F_s in the dielectric of larger permittivity, and adds to it in the dielectric of smaller permittivity.

Therefore, if F_{n1} and F_{n2} represent the components of the resultant intensity which are normal to the interface in the dielectrics of small and of large permittivity, respectively, we have

$$F_{n1} = F_p + F_c. \tag{103}$$

$$F_{n2} = F_p - F_c.$$

$$F_{t1} = F_{t2}. \tag{104}$$

Whence, from Eq. (101), we may write,

$$\sigma_1 = p_o(p_{r1} - 1)(F_p + F_c).$$

$$\sigma_2 = p_o(p_{r2} - 1)(F_p - F_c).$$

Hence the surface density σ_n of the net charge on this interface is

$$\sigma_n = \sigma_2 - \sigma_1 = p_o[(p_{r2} - p_{r1})F_p - (p_{r2} + p_{r1})F_c + 2F_c].$$

But from Eq. (78)

$$F_c = \frac{\sigma_n}{2p_o}.$$

Whence, $F_c = \frac{1}{2}[(p_{r2} - p_{r1})F_p - (p_{r2} + p_{r1})F_c + 2F_c]$.

Solving for F_c,

$$F_c = \frac{p_{r2} - p_{r1}}{p_{r2} + p_{r1}} F_p. \tag{105}$$

Therefore,

$$F_{n1} = (F_p + F_c) = \frac{2p_{r2}}{p_{r2} + p_{r1}} F_p \tag{106}$$

$$F_{n2} = (F_p - F_c) = \frac{2p_{r1}}{p_{r2} + p_{r1}} F_p \tag{107}$$

and

$$\frac{F_{n1}}{F_{n2}} = \frac{p_{r2}}{p_{r1}} \tag{108}$$

or

$$p_{r1}F_{n1} = p_{r2}F_{n2} \tag{109}$$

and

$$\sigma_n = p_o(F_{n1} - F_{n2}). \tag{110}$$

From any point A (Fig. 64), on the interface between two dielectrics, let (F_1, n) and (F_2, n) represent the angles between

the electric intensity vectors and the normals to the surface in the dielectrics of small and of large permittivity, respectively.

$$\tan (F_1, n) = \frac{F_t}{F_{n1}}. \qquad \tan (F_2, n) = \frac{F_t}{F_{n2}}.$$

Therefore,
$$\frac{\tan (F_1, n)}{\tan (F_2, n)} = \frac{F_{n2}}{F_{n1}} = \frac{p_{r1}}{p_{r2}}. \tag{111}$$

The relations expressed by Eqs. 104, 108, and 111 may be stated in the following language:

100f. (DEDUCTION).—*At adjacent points on opposite sides of, but close to, the interface separating two dielectrics, the tangential components of the intensities are equal, but the normal components are inversely proportional to the relative permittivities of the two dielectrics.*

Or, again:

100g. (DEDUCTION).—*In passing from one dielectric to another, a line of electric intensity undergoes an abrupt change in direction, being refracted at the interface in such a way that*

a. The incident and refracted lines lie in the same plane, which is perpendicular to the interface at the point of incidence.

b. The angle (F, n) between the line of intensity and the normal to the surface is always the greater in the dielectric of greater permittivity, the ratio of the tangents of these angles in the two dielectrics being a constant which is equal to the ratio of the two permittivities (see Fig. 64).

101. The Reframing of Gauss's Theorem to Provide for the Effect of Concealed Charge.—Gauss's theorem when stated in the form "the surface-integral of the electric intensity over any closed surface is equal to the quotient obtained by dividing the net quantity of electricity enclosed within the surface by the permittivity p_o of free space" applies to a field containing two or more dielectrics, **provided** we include in the quantity of electricity any concealed charges induced on the interfaces between dielectrics. It cannot be applied in the above form if the concealed charges are ignored. However, the relation expressed in Eq. (109), namely,

$$p_{r1}F_{n1} = p_{r2}F_{n2}, \tag{109}$$

suggests the basis for reframing Gauss's theorem in a form which will automatically provide for the effect of the concealed charges, and will thus leave only the obvious charges to be specifically dealt with.

Consider any tube of electric intensity, as that in Fig. 65, which originates on a positively charged body, passes through several dielectrics, and terminates on a negatively charged body. At any interface, the values of the normal components of the electric intensities jump abruptly from a value F_{n1} on one side to a value

$$F_{n2} = \frac{p_{r1}}{p_{r2}} F_{n1} \tag{112}$$

on the other side. The result is that the surface-integrals of the electric intensity, if taken over the tube diaphragms $ABCD$ close to the interface in one dielectric and $MNOP$ of equal area immediately on the other side of the interface, are in the ratio of p_{r2} to p_{r1}.

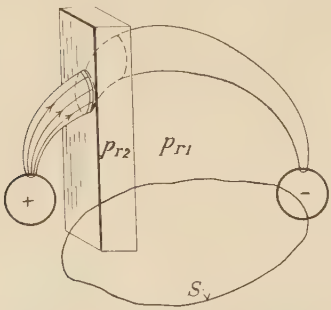


FIG. 65.—Tubes in two dielectrics.

Suppose, however, that we introduce a new distributed vector D which is defined in terms of the electric intensity F by means of the straight-line equation

$$\mathsf{D} = p_r p_o \mathsf{F} \tag{113}$$

or

$$\mathsf{D} = p\mathsf{F}.$$

The definition of this new vector quantity, which we will call the **electrostatic flux density**[8] at a point, is as follows:

101a. (DEFINITION).—**By the ELECTROSTATIC FLUX DENSITY at a point P in an electric field is meant a vector quantity which is obtained by multiplying the electric intensity at the point by the permittivity of the dielectric in which the point P is located.**

Since, by Eq. (109), D_{n1} (or $p_{r1}\ p_o\ F_{n1}$) is equal to D_{n2} (or $p_{r2}\ p_o\ F_{n2}$), it follows that the surface-integral (flux) of the distrib-

[8] This vector was introduced by Maxwell, who called it the **electric displacement.** The name **electrostatic flux density** is recommended in the Standardization Rules of the American Institute of Electrical Engineers.

uted vector D over the diaphragm $ABCD$ is equal to its surface-integral over the diaphragm $MNOP$.

Since (a) the surface-integrals of the D vector over the diaphragms $ABCD$ and $MNOP$ in the dielectrics 1 and 2 are equal, and since

(b) within any given dielectric the surface-integral of the electric intensity has the same value for all diaphragms of any given tube, and since

(c) within any given dielectric the distributed vector D is simply some constant multiple of the intensity F,—it follows that the surface-integrals of the vector D over all diaphragms of a **given** tube of electric intensity will be identical in value.

Now from the theorem of Sec. 93n and from Eq. (100), the flux of the electric intensity over a diaphragm is equal to the obvious charge at the end of the tube divided by $p_r p_o$. It follows, therefore, that the surface-integral of the vector D **over any diaphragm whatsoever** in a given tube of electric intensity is equal to the obvious charge at either terminus of the tube.

Suppose that we now express the properties of an inverse-square field containing several dielectrics in terms of the properties of tubes of electrostatic flux density. We may think of **lines of electrostatic flux density** as lines whose direction at each point indicates the direction of the D vector, and of **a tube of electrostatic flux density** as the tubular surface formed by the lines of electrostatic flux density passing through each point of the boundary of any small area. Except in certain non-isotrophic dielectrics (generally of crystalline structure, for example, quartz); the lines of electrostatic flux density coincide with the lines of electric intensity.

The difference between tubes of electric intensity and tubes of electrostatic flux density is that the values of the surface-integral of the F vector over the diaphragms of a tube suffer a discontinuity at the interface between dielectrics, while the values of the surface-integral of the D vector are continuous at the interface. In other words, new unit tubes of **electric intensity** originate at the interface, but tubes of **electrostatic flux density originate only at the obvious charges** and never at the interface between dielectrics (unless an obvious charge has been imparted to the surface of one of the dielectrics by **conduction** from another region).

Each coulomb of charge, **obvious or concealed,** gives rise to $1/p_o$ unit tubes of electric intensity, no matter what the surrounding dielectric may be. Each coulomb of obvious charge gives rise to one unit tube of electrostatic flux density, and the notion of electrostatic flux density is a scheme for the very purpose of automatically providing for the effect of the concealed charge without specifically bringing it into future calculations.

In crossing the interface separating two dielectrics, the tubes of intensity and of flux density are refracted by the same angle. In the case of the electric intensities, at adjacent points on opposite sides of the interface, the tangential components of the intensities are equal, and the normal components satisfy the relation

$$\frac{F_{n1}}{F_{n2}} = \frac{p_{r2}}{p_{r1}}. \tag{108}$$

In the case of the electrostatic flux densities, it is the normal components of the D vectors which are equal and the tangential components satisfy the relation

$$\frac{D_{t1}}{D_{t2}} = \frac{p_{r1}}{p_{r2}}. \tag{114}$$

Suppose that S in Fig. 65 represents any closed surface in an electric field cutting through two or more dielectrics. Let this surface be divided up into small patches and let tubes of electrostatic flux density be passed through the boundaries of all these patches. Now the surface-integral of the electrostatic flux density in the outward direction over this surface is the net sum of the surface-integrals over the tubes which cut through the surface. But the flux density vector has been so defined that its surface integral over any diaphragm in a tube is constant and is equal to the obvious charge (quantity of electricity) at the end of the tube. With this as a basis, an examination of the figure will justify the following restatement of Gauss's theorem.

101b. GAUSS'S THEOREM (Restated in terms of electrostatic flux density).—If any closed surface is taken in an electrostatic field, the surface-integral of the electrostatic flux density in the outward direction over the entire surface is equal to the algebraic sum of the obvious charges enclosed by the surface.

$$\int^{\text{closed surface}} D \cos (D,n)da = \Sigma q. \tag{115}$$

102. Electrostatic Flux and Units of Flux and of Flux Density. A special name has been coined for the **surface integral of the** D **vector.** It is called the **electrostatic flux,** and is invariably represented by the symbol ψ. We thus have the following definition.

102a. (DEFINITION).—**The ELECTROSTATIC FLUX, ψ, IN A SPECIFIED DIRECTION across a given surface is defined to be the algebraic value of the surface-integral of the electrostatic flux density taken over the surface, with the specified direction as positive. That is, in the following integrand which defines ψ, (D, n) represents the angle between the flux density vector and the specified direction along the normal to the surface.**

$$\psi \text{ (coulombs)} = \int D \cos (D,n) da \text{ (defining } \psi). \qquad (116)$$

From Gauss's theorem as stated in Sec. 101b, the electrostatic flux in the outward direction over any closed surface is equal to the obvious charge enclosed by the surface.

$$\psi \text{ (over a closed surface)} = \Sigma q \text{ (coulombs).} \qquad (117)$$

Since the unit of electricity is called the **coulomb,** we may use this same name for the unit of electrostatic flux. Thus:

102b. (DERIVED NAME AND SIGNIFICANCE).—*The unit of electrostatic flux, the "coulomb," is the flux over a surface enclosing a net obvious charge of* 1 *coulomb of electricity.*

Since electrostatic flux is the product of flux-density times area, or, vice versa, electrostatic flux density is a flux divided by an area, we may call the unit of electrostatic flux density the **coulomb per square centimeter.** Thus:

102c. (DERIVED NAME AND SIGNIFICANCE).—*The unit of electrostatic flux density, the "coulomb per square centimeter," is the flux density at the surface of a sphere of* 1 *square centimeter surface area enclosing* 1 *coulomb of electricity concentrically distributed, or it is the electrostatic flux density immediately outside a conducting surface on which the surface density of charge is* 1 *coulomb per square centimeter.*

103. The Elastance and Permittance of Elementary Portions of the Dielectric.—Suppose that Q coulombs of electrons have been transferred from a conductor A to a conductor B and that from considerations of geometrical symmetry the configurations

of the resulting equipotential surfaces are known. Now imagine a number of very thin films of metal in the field, each one coinciding with one of the above-mentioned equipotential surfaces. Such films would not alter the intensities in the field. We may now look upon the condenser constituted by A and B and their surrounding dielectric as the equivalent of a great number of elementary condensers in series, each elementary condenser consisting of two adjacent metal films and the intervening dielectric. It is evident that, if we can calculate the **elastances** of these elementary condensers, then the elastance of the conductor A relative to B will be the sum of the elastances of the elementary condensers. This leads us to the practice of thinking of the **elastance** or of the **permittance** of the elementary portions into which the dielectric surrounding two charged conductors may be divided by means of the equipotential surfaces and the tubes of electrostatic flux.

103a. (Definition).—**The ratio of the electrostatic flux ψ over a given tube of flux to the potential difference E between any two equipotential surfaces is called the PERMITTANCE or CAPACITANCE of the portion of the dielectric lying between the specified end surfaces.**

$$C \text{ (farads)} = \frac{\psi \text{ (coulombs)}}{E \quad \text{(volts)}}. \tag{118}$$

103b. (Definition).—**The ratio of the potential difference E between any two equipotential surfaces to the electrostatic flux ψ over a given tube of flux is called the ELASTANCE between the specified end surfaces of the portion of the dielectric enclosed.**

$$S \text{ (darafs)} = \frac{E \quad \text{(volts)}}{\psi \text{ (coulombs)}}. \tag{119}$$

103c. Elastance and Permittance of Right Cylinders.—If the portion of a given dielectric lying within a tube of electrostatic flux and two equipotential surfaces is a right cylinder, or is for all practical purposes a right cylinder, the elastance and the permittance of this right cylinder may be readily stated in terms of its dimensions. Thus, if the electrostatic flux over a right cylinder of cross-sectional area a and length l is ψ, the flux density D is ψ/a and the electric intensity F is

$$F = \frac{D}{p} = \frac{\psi}{ap}.$$

Since the F vector is parallel to the walls of the cylinder and is uniform throughout the length l, the potential difference between the plane end surface is

$$E = Fl = \frac{l\psi}{ap}.$$

Therefore, the permittance and the elastance between the parallel bases of a right cylinder of dielectric of permittivity p (or of elastivity s) are

$$C \text{ (of a cylinder)(farads)} \left(= \frac{\psi}{E} \right) = \frac{ap}{l} \text{ (cm.).} \tag{120}$$

$$S \text{ (of a cylinder)(darafs)} = \frac{l}{ap} = \frac{ls}{a} \text{ (cm.).} \tag{121}$$

In these formulas, p represents the permittivity of the dielectric, namely, $p_r p_o$ or $p_r \times 8.85 \times 10^{-14}$.

s represents the reciprocal of the permittivity. It is termed the **elastivity** of the dielectric.

$$s = \frac{1}{p}. \tag{122}$$

104. Energy Stored per Unit Volume of Dielectric.—When the electrostatic field surrounding the electrodes of a condenser has in imagination been divided into elementary portions by means of many equipotential surfaces, and when the condenser is then regarded as made up of a great many condensers in series, it is natural, and in some respects helpful, to associate the energy stored in the condenser with the elementary portions of the dielectric constituting the condenser.

The energy stored in a charged condenser has been shown to be

$$W = \tfrac{1}{2}(QE) = \tfrac{1}{2}(\psi E). \tag{58}$$

If the energy associated with the elementary portion of the dielectric included between two equipotential surfaces is taken to be $\frac{1}{2}(\psi E_1)$, in which E_1 is the potential difference between the surfaces, it is evident that the energy stored in all the elementary portions—or $\Sigma\frac{1}{2}(\psi E_1)$ or $\frac{1}{2}(\psi \Sigma E_1)$—will sum up to $\frac{1}{2}(\psi E)$, since $\Sigma E_1 = E$.

A similar argument enables us to say that the energy associated with each elementary tube of constant flux ψ_1 may be taken to be $\frac{1}{2}(\psi_1 E)$, and that the energy associated with the elementary

length of a tube included between two adjacent equipotential surfaces may be taken as

$$w_1 = \tfrac{1}{2}\psi_1 E_1. \tag{123}$$

Now such an elementary portion of a tube is, for all practical purpose, a right cylinder having the volume $a_1 \times l_1$. Therefore the expression for the energy per unit volume of dielectric becomes

$$w \text{ (joules per cubic centimeter)} = \frac{w_1}{a_1 l_1} = \frac{1}{2}\frac{\psi_1}{a_1}\frac{E_1}{l_1}.$$

But ψ_1/a_1 and E_1/l_1 represent the values of the electric displacement and the electric intensity in the elementary volume. Therefore the expression for the energy per cubic centimeter of dielectric may be written in the forms:

$$w \text{ (joules per cubic cm.)} = \tfrac{1}{2}FD \text{ (volts, coulombs, cm.)} \tag{123a}$$

$$= \tfrac{1}{2}pF^2 \text{ (volts per cm.)} \tag{123b}$$

$$= \frac{D^2}{2p} \text{ (coulombs per sq. cm.)} \tag{123c}$$

105. Electric Intensities in Graded Cable Insulation.—The following is an illustration of the use of elastances in the calculation of the distribution of the electric intensities in insulating materials.

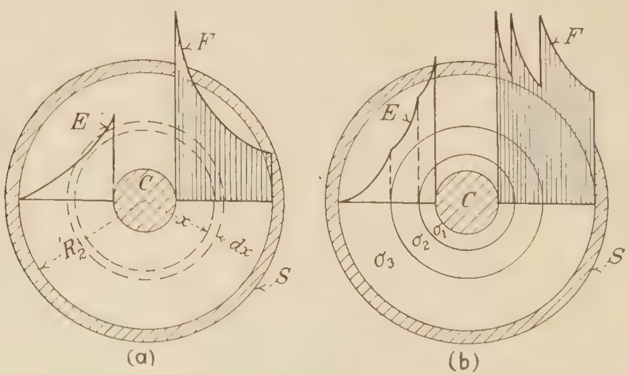


FIG. 66.—Electric intensities in cable insulation.

Figure 66 is a cross-sectional view of an underground cable. A solid copper conductor C of circular cross-section is insulated with two or more dielectrics which have been applied to it in concentric cylindrical layers. Suppose that in the operation of the cable the copper conductor is maintained at a potential E volts higher than the lead sheath S which surrounds and protects the insulation. The problem is: Given the potential difference E, and the dimensions and the elastivities of the materials, to derive expressions and to plot curves for the intensities.

The equipotential surfaces are cylindrical surfaces concentric with the wire, and we may regard the dielectric between C and S as the equivalent of a number of condensers in series.

Consider two equipotential cylindrical surfaces of length l and of radii x and $x + dx$. The dielectric between these has the properties of a right cylinder of length dx and of cross-sectional area $2\pi xl$. Therefore, the elastance of this elementary layer is

$$dS = \frac{sdx}{2\pi lx}$$

and the elastance of a tubular layer of homogeneous dielectric of elastivity s and having internal and external radii of r_1 and r_2, respectively, is

$$S = \int_{r_1}^{r_2} \frac{sdx}{2\pi lx} = \frac{s}{2\pi l} \log \frac{r_2}{r_1}. \tag{124}$$

The elastance, therefore, from the conductor to the sheath is

$$\begin{aligned} S_t &= (S_1 + S_2 + S_3 + \ . \ . \ .) \\ &= \frac{1}{2\pi l}\left[s_1 \log \frac{r_2}{r_1} + s_2 \log \frac{r_3}{r_2} + s_3 \log \frac{r_4}{r_3} + \ . \ . \ . \right] \end{aligned} \tag{124a}$$

$$\psi = \frac{E}{S_t} = \frac{E}{(S_1 + S_2 + S_3 +)}.$$

The potential across any layer, as the layer of elastance S_n, is

$$E_n = S_n\psi = \frac{S_nE}{S_t}. \tag{125}$$

The potential increase across the elementary layer from x to $x + dx$ is

$$de = \psi dS = -\frac{E}{S_t}\frac{sdx}{2\pi lx}$$

Therefore the intensity at points a distance r from the center is

$$F = -\frac{de}{dx} = \frac{E}{2\pi lS_t}\frac{s}{r}. \tag{126}$$

The potential at any layer at distance r from the center is

$$E_r = -\int_{r_2}^{r} \frac{E\,s}{2\pi lS_t}\frac{dx}{x}. \tag{127}$$

Since the elastivity has a different value in each substance, this integral must be evaluated layer by layer. This may readily be done. For layers which are completely traversed in taking the integral, the expression for the potential E_n across the nth layer is already given in Eq. (125).

If the conductor is insulated with a single homogeneous dielectric, the expressions for the electric intensity F and for the potential E_r at a point at a distance r from the center of the wire become

$$F \text{ (volts per cm.)} = \frac{E}{r \log \frac{r_2}{r_1}}. \tag{128}$$

$$E_r \text{ (volts)} = E\,\frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}. \tag{129}$$

The intensity is a maximum at the surface of the conductor and has the value

$$F_m(\text{at conductor.}) = \frac{E}{r_1 \log \frac{r_2}{r_1}}. \tag{130}$$

The curves F and E in Fig. 66*a* show, respectively, the values of the intensities and the potentials at points along any radial line in a single layer of homogeneous insulator 1.5 centimeters deep insulating a wire of circular cross-section 1 centimeter in diameter. The curves have been drawn for the case of an assumed allowable maximum intensity for the particular dielectric of 60,000 volts per centimeter. Substituting the above values in Eq. (130) and solving for E, the allowable potential difference between wire and sheath is found to be 41,500 volts. The electric intensity in the dielectric next to the sheath is only one-fourth as great as it is next to the core. The outer portion of the dielectric, therefore, is not effectively used, since the intensities in it are so far below the safe working value.

By the expedient of **grading the dielectric,** that is, of using the materials of the higher permittivity next to the core and of lower and lower permittivity in successive layers, the intensities in the outer layers can be made to equal or even to exceed the intensity next to the core. This is illustrated by the curves of Fig. 66*b* which have been plotted for the same conductor and sheath, insulated, however, by three grades of rubber insulation applied in concentric layers of the following dimensions:

Layer	Thickness, centimeters	Relative permittivity
First (innermost)	0.25	6
Second	0.45	4
Third	0.80	2.5

These curves have been plotted for the same maximum intensity of 60,000 volts per centimeter. The allowable potential difference (which is proportional to the vertically hatched area under the intensity curve) is now 70,000 volts.

106. Forces on Polarized Dielectrics.

—The following experiments demonstrate some of the features of the forces which act upon polarized dielectrics in an electrostatic field.

106a. Experiment 1.—If a small glass rod is suspended in an electrostatic field so that it is free to turn in any direction about its center of gravity, it tends to set itself along the line of electric intensity passing through its center. If the rod, or a dielectric of any shape, as a glass or paraffin sphere, is inserted in a non-uniform field, the body tends to move to regions of greater electric intensity.

106b. Experiment 2.—The existence of forces in liquid dielectrics along the lines of electric intensity may be demonstrated by the apparatus shown in Fig. 67. It consists of a vessel containing two liquid dielectrics having different permittivities and different densities, the lighter floating on the denser, or of a gaseous over a liquid dielectric. Two extended metal plates, one in each dielectric, are mounted with their plane faces parallel to the common interface.

If a potential difference is maintained between the plates, that portion of the interface of the two dielectrics which lies between the two charged plates shifts to the new position indicated by the dotted line. That is to say, in the space between the plates the material of higher displaces the material of lower permittivity until the hydrostatic pressures resulting from the gravitational pulls on the liquids of different densities just balance the electrostatic forces on the concealed charge on the common interface.[9]

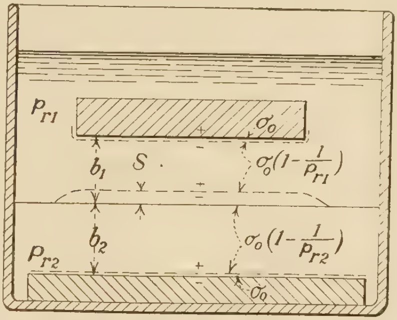


FIG. 67.—Force on dielectric interface.

106c. Experiment 3.—Upon immersing the disks of the absolute electrometer of Fig. 22 in a dielectric, such as oil, having a relative permittivity represented by p_r, Quincke[10] experimentally found the pull on the suspended disk to be expressed by the formula

$$f\,(\text{dyne-sevens per sq. cm.}) = \frac{\sigma_o^2}{2p_r p_o}\ (\text{coulombs per sq. cm.}), \qquad (131)$$

in which σ_o represents the surface density of charge on the disk.

Since the same disk in evacuated space would experience a pull of $\sigma_o^2/2p_o$ dyne-sevens per square centimeter, it follows that the oil facing the disks is exerting a pressure P_d outward from the oil to the metal equal to the difference between these or

$$P_d\,(\text{dyne-sevens per sq. cm.}) = \frac{\sigma_o^2}{2p_o} - \frac{\sigma_o^2}{2p_r p_o}$$
$$= \frac{\sigma_o^2}{2p_o}\left(1 - \frac{1}{p_r}\right)\ (\text{coulombs per sq. cm.}). \qquad (132)$$

[9] This statement is true only when the potential difference is an alternating potential difference of a frequency high enough to make the leakage or conduction currents negligibly small in comparison with the displacement current. When a continuous potential difference is maintained between the plates, the relative conductivities, and not the relative permittivities, of the two dielectrics determine direction of the force on the separating surface.

[10] Phil. Mag., 1883, Vol. XVI, p. 1; Nature, 1887, Vol. XXXV, p. 334.

106d. Experiment 4.—The existence of forces in liquid dielectrics transverse to the lines of electric intensity was demonstrated and measured by Quincke[10] with the apparatus illustrated in Fig. 68. It consists of two parallel metal plates mounted close together in a liquid dielectric such as oil. By means of a tube tapped into the upper plate at its center, air is blown into the space between the plates until it has displaced the liquid from a

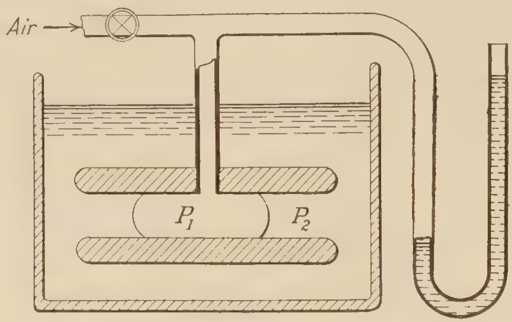


FIG. 68.—Force on dielectric interface.

large central space between the plates. A stopcock in the air supply is then closed and the air pressure within the bubble is then read by means of the manometer shown. Upon applying a potential difference between the plates, the manometer at once shows an increase in the pressure, indicating the existence of forces at the interface traverse to the lines of intensity and tending to force the oil into the space occupied by the air.

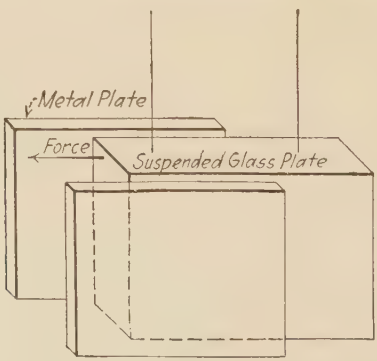


FIG. 68*a*.—Force on a dielectric.

A similar experiment with a solid dielectric is illustrated in Fig. 68*a*. A glass plate G is so suspended between the two metal plates of a condenser that it projects beyond the right edge of the plates and does not extend to the left edge. Upon applying a potential difference between the metal plates, the glass plate is drawn into the space between the plates.

107. Analysis of Forces on Dielectrics.—The fundamental postulate concerning dielectrics, namely, "an electric intensity F_d in a dielectric of relative permittivity p_r is accompanied by a shift across any (and each) plane area which is taken perpendicular to the electric intensity, of a quantity of electricity equivalent to $p_o(p_r - 1)F_d$ coulombs of positive electricity per square centimeter in the direction of F_d" (Sec. 98*d*), puts before us an arrangement of electric charge which is definite in the **smoothed-over** sense. It should enable us to calculate the electric force on any portion of the dielectric which contains so many molecules that smoothed-over, or average, values apply to it.

The polarized molecules along a tube of electric intensity are crudely pictured for a given instant of time in Fig. 69. In such a medium the electric intensity varies enormously from point to point and from instant to instant. Since we are interested in deriving expressions for the forces on large areas or bodies, our calculations will deal with the **smoothed-over** value of the intensity and its **systematic variation** in space, and not with these erratic or haphazard values.

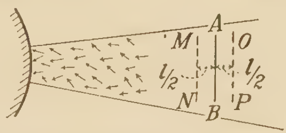


FIG. 69.—Polarized molecules in a tube of electric intensity.

We will discuss in the order listed:

a. The oppositely directed electric forces tending to separate the + and − electricity of the individual molecules of the dielectric, and the opposing force, which we propose to call the **inner molecular force of polarization.**

b. The net or unbalanced electric force on a small interior group of polarized molecules or the **volume element force,** and the resulting **hydrostatic pressure** which must exist in fluid dielectrics to maintain equilibrium.

c. The net or unbalanced electric force on the boundary film of polarized molecules, or the **boundary film force,** and the **boundary pressure** necessary to maintain equilibrium in fluid dielectrics.

107a. The Forces on Individual Molecules.—In an electric field, a shift of electricity may be conceived to occur within each molecular structure of the dielectric, so that for all points at a distance from any polarized molecule it acts as an electric doublet;

that is, the molecule is pictured as having the properties of a system of two equal + and − charges q separated by a distance l along an axis which coincides with the direction of the electric intensity. Since the charges do not separate completely and pass to the surface as in a conductor, we picture equal and opposite inner **molecular forces** as acting on each charge and holding it within the molecule.

If the molecules lie in a region in which the field is uniform, that is, a region in which the smoothed-over intensity F_d does not vary from point to point (as between the parallel plates of a condenser), then, on the average, the positive and negative charges of the doublets are acted upon by equal and opposite forces, or the net force tending to cause a movement of translation of any considerable group of molecules is zero.

But if the field is non-uniform, that is, if along a tube of intensity there is a systematic variation in the value and direction of the smoothed-over intensity, then one charge of the doublet representing a molecule is acted on by a force which is different in magnitude and direction from the force which acts on its complement. We may, therefore, consider these forces acting on the two charges to consist of a balanced pair tending to stretch the doublet, and an unbalanced force which then requires that an external force shall act on the molecule to hold it in equilibrium. This unbalanced force is, obviously, the vector sum of the forces on the two complementary charges (see Fig. 69b, which is drawn to illustrate the fact that the unbalanced resultant may be due as much to a change in direction of the intensity as to a change in magnitude). The unbalanced forces are comparatively small except when summed up for a great many molecules. In the following section they will be considered as the source of a hydrostatic pressure in fluid dieletrcics.

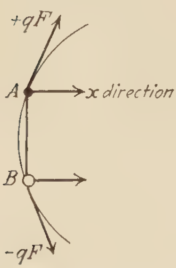


Fig. 69b.—Forces on a doublet in a non-uniform field (greatly exaggerated).

107b. The Force on Interior Volume Elements and the Resulting Hydrostatic Pressure.—We have seen that polarized molecules lying in a non-uniform field, such as shown in Fig. 69b,

are on the average acted on by a small unbalanced force which may, in general, have any direction relative to the lines of intensity. Let us deal first with the resultant force on a single doublet and then extend our argument to arrive at the force per unit volume of the dielectric.

Continue to let a polarized molecule be represented by a doublet, $+q$ at A and $-q$ at B, separated a distance l along a line of intensity, as in Fig. 69b. Since, in general, we do not know the direction of the resultant force, let us deal with its component in an arbitrarily chosen direction x. The electric force on the charge $+q$ at A in the direction x is the product of q and the component of the intensity in this direction. This component is given by $\frac{-\partial E_A}{\partial x}$ (if we think of the point A as being movable in the direction x). Therefore

$$f_x' \text{ (on } +q) = -q\frac{\partial E_A}{\partial x}.$$

Similarly, the force in this direction on charge $-q$ at B is

$$f_x'' \text{ (on } -q) = q\frac{\partial E_B}{\partial x}$$

and the resultant force on the doublet in this direction x is

$$f_x''' \text{ (on doublet)} = q\frac{\partial}{\partial x}(E_B - E_A) = ql\frac{\partial}{\partial x}F_d. \tag{133}$$

If there are N polarized molecules per cubic centimeter of dielectric, the force per unit volume may be written

$$\frac{f_x}{\text{vol.}} = Nql\frac{\partial}{\partial x}F_d. \tag{134a}$$

But every molecular doublet whose center lies within the distance $l/2$ of an equipotential surface contributes a shift of q coulombs to the postulated shift of electricity across this surface. The number of molecules whose centers lie within this cylindrical volume of length l and of unit cross-sectional area is N times the volume or Nl. Consequently, the expression for the quantity of electricity which shifts across unit area of the equipotential surface is Nlq. This is an expression for the shift σ_c, whose value was postulated in Sec. 98d to be $p_o(p_r - 1)F_d$. Whence,

$$Nlq = \sigma_c = p_o(p_r - 1)F_d.$$

Substituting this value of Nlq in Eq. (134a), we get

$$\frac{f_x \text{ (dyne-sevens)}}{\text{vol. (cu. cm.)}} = p_o(p_r - 1)F_d \frac{\partial F_d}{\partial x}. \tag{134}$$

We may note that the argument up to this point applies to a dielectric, whether it is homogeneous or not. Let us now specify that p_r shall be constant throughout the region considered, thereby limiting all conclusions drawn on this basis to homogeneous dielectrics. If p_r is constant, Eq. (134) may be written

$$\frac{f_x}{\text{vol.}} = \frac{\partial}{\partial x}\left[\frac{p_o(p_r - 1)F^2{}_d}{2}\right]. \tag{135}$$

Now this electric force which is exerted on an interior volume of dielectric must be balanced by some other force. If the dielectric is a solid, this force can be balanced only by forces which are associated with deformations or strains in the dielectric. If the dielectric is a fluid (liquid or gaseous), this force sets up hydrostatic pressures in the fluid. If the fluid is not thereby set in circulation but remains in equilibrium, this driving force must be everywhere balanced by the variations in the hydrostatic pressure. By the argument below it is demonstrated that the system of forces specified by Eq. (135) is one that may be balanced by the variations of a pressure.

Now the net mechanical force exerted (by reason of variations of hydrostatic pressure) in any given direction x on a small volume of fluid by the bounding fluid is

$$f_x = -(\text{vol.})\frac{\partial}{\partial x}(P)$$

$$\frac{f_x}{\text{vol.}} = -\frac{\partial}{\partial x}(P), \tag{136}$$

in which P represents the hydrostatic pressure.

A comparison of this equation for the hydrostatic force exerted per unit volume with Eq. (135) for the force exerted electrically per unit volume shows that the two forces will be equal and opposite, provided only that the two derivatives are equal. Whence

$$\frac{\partial}{\partial x}(P_v) = \frac{\partial}{\partial x}\left[\frac{p_o(p_r - 1)F_d{}^2}{2}\right], \tag{137}$$

in which P_v represents the pressure caused by the volume element force.

Since the direction x is any direction arbitrarily chosen, this differential equation requires that

$$P_v \text{ (dyne-sevens per sq. cm.)} = \frac{p_o(p_r - 1)F_d^2}{2} + K, \quad (138a)$$

in which the integration constant K represents any pressure which is constant throughout the fluid and therefore constant at the boundary. If there is any point on the boundary surface of the fluid where F_d is zero, and where, by leaving out of consideration pressures due to other causes (such as gravitational attraction, centrifugal effects, surface tension, atmospheric pressure, etc.), the pressure may be taken as zero, then in such a system K must have the value zero, and the pressure at any point in the fluid caused by the electrical forces on interior volume elements is

$$P_v \text{ (dyne-sevens per sq. cm.)} = \frac{p_o(p_r - 1)F_d^2}{2}. \quad (138)$$

The forces on the boundary film of polarized molecules are considered in the next section.

107c. The Force on the Boundary Film of a Dielectric.—By the boundary film of a dielectric we mean the molecules of the dielectric included between two surfaces, one lying within the dielectric sufficiently deep so that upon it the density of the molecules is the same as in the interior, and the other lying outside the dielectric sufficiently far so that upon it the density of the molecules of the dielectric is zero. It has been shown (Sec. 100) that within any given small patch of the boundary film, the tangential component F_t of the smoothed-over intensity is for present purposes constant, but the normal component of the intensity increases from the value F_n at the inner surface to the value p_rF_n in free space outside the dielectric.

The surface, when thought of in terms of molecules, is very irregular, but to calculate the force on the molecules of this boundary film we may use smoothed-over values by imagining the molecules in the boundary film to be arranged in orderly layers in some such fashion as that illustrated in Fig. 70. The molecular density is represented as becoming less and less as the surface is approached. Since the component of the intensity tangential to the surface is constant throughout any small

patch, the electric force on the molecules in the boundary film has no component tangential to the surface.

In Fig. 70, the ordinates of the curve labeled F show the manner in which the normal component of the intensity decreases as we pass through the boundary film from the outer to the inner surface. The ordinates of the curve labeled q represent two things: (a) the concealed charge (in coulombs per square centimeter) lying between the outer surface and the plane to which the ordinate applies; and (b) the quantity of electricity (in coulombs per square centimeter) at either end of the layer of polarized molecules through which the plane cuts.

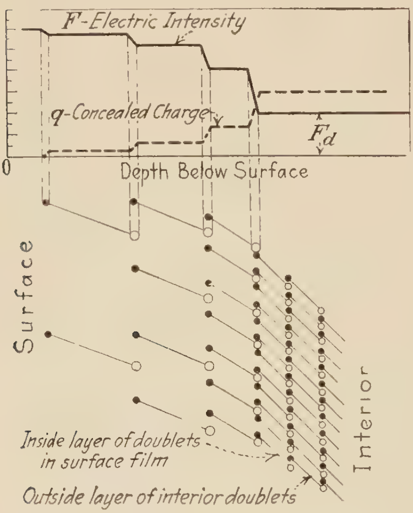


FIG. 70.—Polarized layers in surface film.

In each region in which the concealed charge q increases by the amount dq, the electric intensity decreases by the amount dq/p_o

Consider the force on any layer of polarized molecules containing q coulombs per square centimeter at each end. The outer charges of the layer lie in a region in which the electric intensity is dq/p_o volts per centimeter higher than at the inner. Therefore the net unbalanced force per square centimeter on this layer of polarized molecules will be along the normal to the surface in the outward direction and will have the value

$$df = \frac{q\,dq}{p_o}$$

and the force normal to the surface in the outward direction exerted on all layers of polarized molecules in a square centimeter of the surface film will be

$$f_s = \int^{\text{through film}} df = \int_0^{\sigma_c} \frac{q\,dq}{p_o} = \left[\frac{q^2}{2p_o}\right]_0^{\sigma_c}$$

$$f_s \text{ (dyne-sevens per sq. cm.)} = \frac{\sigma_c^{\,2}}{2p_o} \text{ (coulombs per sq. cm.)} \quad (139)$$

Substituting for σ_c its value in terms of the intensity as expressed in Eq. (101), this expression becomes

$$f_s \text{ (dyne-sevens per sq. cm.)} = \frac{p_o(p_r - 1)^2 F_n^2}{2} \text{ (volts per cm.)} \quad (140)$$

in which F_n represents the value of the normal component of the electric intensity in the dielectric just beneath the boundary layer.

This force acts at any existing bounding surface of the dielectric and would come into existence should a cleavage develop in the body of the dielectric.

108. Pressures Exerted by Fluid Dielectrics.—We have seen that the electric force exerted on volume elements in the interior of a polarized fluid dielectric gives rise to the following hydrostatic pressure at points within the dielectric

$$P_v = \frac{p_o(p_r - 1)F_d^2}{2} + K, \quad (138a)$$

and that the force on the boundary film gives rise to the following outwardly directed pressure normal to the surface

$$P_s = \frac{p_o(p_r - 1)^2 F_n^2}{2}. \quad (140)$$

The total pressure exerted at its surface by a fluid dielectric in the outward direction along a normal to its surface is the sum of these, or

$$P \text{ (dyne-sevens per sq. cm.)} = \frac{p_o(p_r - 1)}{2}[F_d^2 + (p_r - 1)F_n^2] + K. \quad (141)$$

Since $F_d^2 = F_n^2 + F_t^2$, in which F_t is the tangential component of F_d, Eq. (141) may be written in the alternative form.

$$P \text{ (dyne-sevens per sq. cm.)} = \frac{p_o(p_r - 1)}{2}\left[F_t^2 + p_r F_n^2\right] + K. \quad (142)$$

In these formulas the value of K is zero if at any point on the boundary of the dielectric F_d is zero.

108a. Pressure upon an Immersed Conductor.—At any point of the interface between a dielectric and a conducting body, the electric intensity is normal to the surface of the conductor

(electrostatic conditions being assumed). It follows from Eq. (142) that a fluid dielectric exerts a pressure P_c upon the conductor of the value

$$P_c \text{ (dyne-sevens per sq. cm.)} = \frac{p_o p_r (p_r - 1) F_n^2}{2} + K. \tag{143}$$

Since the surface density σ_o of the obvious charge on the surface of the conductor is equal to $p_o p_r F_n$, the expression for the pressure due to the electrical forces on a fluid dielectric which extends to a region in which the intensity is zero, may be written

$$P_c \text{ (dyne-sevens per sq. cm.)} = \frac{\sigma_o^2}{2p_o}\left(1 - \frac{1}{p_r}\right). \tag{144}$$

Now the arguments of Sec. 89 have shown that the outward force per square centimeter upon the charge on the surface of a charged conductor is

$$f = \frac{\sigma_o^2}{2p_o}. \tag{81}$$

From Eqs. (81) and (144) it follows that the net force per square centimeter applied to the surface of a conductor which is immersed in a fluid dielectric of permittivity p_r is

$$f_n = f - P_c = \frac{\sigma_o^2}{2p_r p_o} \tag{145}$$

$$f_n = \frac{p_r p_o F_d^2}{2}. \tag{145a}$$

That is, the pressure of a fluid dielectric upon any conducting surfaces having a given surface density of charge σ_o reduces the net force on the conductor to $1/p_r$ of the force which the conductor would experience if it were charged to the same surface density in free space. These Eqs. (144) and (145) are in agreement with the experimental results obtained by Quincke as recited in Sec. 106e.

From this discussion the following important conclusion may be drawn:

The force between charged conductors immersed in a single infinitely extended homogeneous fluid dielectric may be computed by dealing only with the obvious charges on the conductors, provided the law of force between the elementary portions of the obvious charge is written as in Eq. 9a, that is, provided the value appropriate to the dielectric is assigned to the factor p, which appears in the

denominator of the inverse square law as it was originally written in Eq. (9).

$$f \text{ (dyne-sevens)} = \frac{q_1 q_2}{4\pi p r^2} \quad \frac{\text{(coulombs)}}{\text{(cm.)}} \tag{9}$$

$$f \text{ (dyne-sevens)} = \frac{q_1 q_2}{4\pi p_r p_o r^2} \quad \frac{\text{(coulombs)}}{\text{(cm.)}} \tag{9a}$$

108b. Net Force on the Common Interface between Two Dielectrics.—If two fluid dielectrics have a common interface, each dielectric presses outward upon the common interface with a pressure expressed by Eq. (142). The resultant pressure of the electrical forces is always from the material of higher (p_{r2}) to the material of lower (p_{r1}) permittivity. The former dielectric tends to displace the latter. If it cannot displace it, it compresses it until the ordinary hydrostatic pressure in the dielectric of lower permittivity exceeds that in the higher by the difference between the values of the two electrically produced pressures. The expression for this difference, simplified by recalling that F_t has the same value on opposite sides of the interface and that $p_{r1}F_{n1} = p_{r2}F_{n2}$, takes the form

$$P_{2 \text{ to } 1} = \frac{p_o}{2}[(p_{r2} - p_{r1})F^2{}_t + p_{r1}F^2{}_{n1} - p_{r2}F^2{}_{n2}] \tag{146}$$

or the form

$$P_{2 \text{ to } 1} = \frac{p_o(p_{r2} - p_{r1})(F_t{}^2 + F_{n1}F_{n2})}{2}. \tag{146a}$$

If the interface is normal to the lines of intensity, as in Fig. 67, this simplifies to

$$P_{2 \text{ to } 1} = \frac{p_o(p_{r1}F^2{}_{n1} - p_{r2}F^2{}_{n2})}{2} \tag{147}$$

and if the interface coincides with the walls of tubes of intensity, as in Fig. 68, the pressure is

$$P_{2 \text{ to } 1} = \frac{p_o(p_{r2} - p_{r1})F_t{}^2}{2}. \tag{148}$$

109. Mutual Elastance between Two Condensers.—Let us consider four insulated conductors A, B, C, and D definitely spaced in specified surroundings, as illustrated in Figs. 30 and 71 and described in Sec. 72 (Secs. 72 to 75 should be reviewed). If conductors A and B are so used that electricity is transferred

from one to the other, then these two conductors are regarded as constituting a condenser—call it condenser 1. Likewise, if C and D are so used that electricity may be transferred from one to the other, they may be regarded as constituting condenser 2. We propose to consider the relations between the potential differences of the condensers and the charges they contain when the electrodes A, B, C, and D of the condensers are so located with reference to each other that any transfer of charge in one condenser causes a difference in potential between the conductors constituting the other condenser.

Let e_1 represent the potential difference between the conductors A and B which results from the charge q_2 in condenser 2—condenser 1 being uncharged.[11]

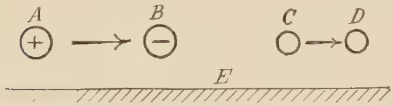


FIG. 71.—Condensers having mutual elastance.

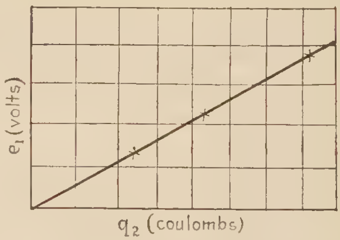


FIG. 72.—Relation between potential difference and charge transferred.

Let e_2 represent the potential difference between the electrodes of condenser 2 which results from the charge q_1 in condenser 1 —condenser 2 being uncharged.

If the potential differences e_1 corresponding to various values of the charge q_2 are plotted, the straight-line relation illustrated in Fig. 72 is obtained. This straight-line relation[12] between the two variables is expressed by the equation

$$e_1 \text{ (volts)} = S_{1,2}q_2 \text{ (darafs, coulombs)}. \qquad (151)$$

[11] The statement "condenser 1 being uncharged" means that no charge has been transferred from the electrode A to electrode B of condenser 1. It does not mean that no transfer of electricity has taken place on conductor A itself, or on conductor B. In general, a charge cannot be transferred from C to D without inducing a redistribution of the free electrons on any neighboring conductor (such as A).

[12] This straight-line relation may be regarded either as an experimentally determined relation or as a consequence of the principle of the linear superposition of electric fields (see the footnote to Sec. 72).

It follows that a similar straight-line equation will express the relation between the potential difference across condenser 2 and the charge in condenser 1, as

$$e_2 \text{ (volts)} = S_{2,1}\, q_1 \text{ (darafs, coulombs).} \tag{151}$$

The proportionality constants $S_{1,2}$ and $S_{2,1}$ appearing in these equations are called the **mutual elastances** between condensers 1 and 2. They are explicitly defined by rewriting the equations in the form

$$S_{1,2} = \frac{e_1}{q_2}$$

$$S_{2,1} = \frac{e_2}{q_1}. \tag{152}$$

In the next section, it will be demonstrated that the ratio of e_1 to q_2 is always equal to the ratio of e_2 to q_1. Consequently, $S_{1,2} = S_{2,1}$, and when no more than two condensers are to be dealt with, the subscripts $_{1,2}$ and $_{2,1}$ may be dropped and we may write

$$S_m = S_{1,2} = S_{2,1} = \frac{e_1}{q_2} = \frac{e_2}{q_1}. \tag{153}$$

If there are more than two condensers, the subscripts $_{1,2}$, $_{2,3}$, $_{3,1}$ etc. must be retained, but the order in which the subnumbers are written is immaterial.

To specify completely the conditions in a field containing several condensers, it is necessary to specify the directions of the transfers, and of the resulting potential differences; that is, to specify whether the transfer of electrons has been from A to B or from B to A, etc. If this complete information is to be embodied in equations such as Eq. (151), the quantities q, e, and S must be treated as algebraic quantities, in which the algebraic signs of q and e specify the directions of transfer, etc., by definitely recognized conventions.

109a. In this text, the conventions by which the algebraic signs of the quantities q, e, and S are related to directions in space are as follows:

1. *For convenience in specifying directions, an arrow will be arbitrarily drawn on (the charging circuit of) each condenser, pointing from one electrode to the other. The electrodes may then be called the head and the tail electrodes.*

2. *The algebraic value of the charge in a condenser (the charge transferred) will be represented by the symbol $+q$. A positive numerical value will be assigned to q if the transfer results in a positive charge on the head electrode.*

3. *The algebraic value of the potential difference between the electrodes of a condenser will be represented by the symbol $+e$. A positive numerical value will be assigned to e if the potential of the head electrode is higher than that of the tail electrode.*

Using potential difference and charge in the algebraic sense fixed by the above conventions, the following definition may now be formulated for the algebraic value of the mutual elastance.

109b. MUTUAL ELASTANCE (DEFINITION).—**By the MUTUAL ELASTANCE S_m between two condensers is meant the ratio of the potential difference between the electrodes of an uncharged condenser to the charge in a second condenser, which charge is the cause of the potential difference.**

$$S_m \text{ (darafs)} = \frac{e_1}{q_2} = \frac{e_2}{q_1} \frac{\text{(volts)}}{\text{(coulombs)}}. \tag{153}$$

As an example of the use of the conventions let the arrow directions be taken as shown for the two condensers shown in Fig. 71. Imagine that condenser 1 is charged as shown. Then q_1 is a negative quantity. The head electrode of condenser 2 is at a higher potential than the tail; therefore, e_2 is a positive quantity. Consequently, the mutual elastance between the two condensers will have a negative value.

110. Energy Stored in Two Condensers Having Mutual Elastance (DEDUCTION).—Let two condensers having the elastances S_1 and S_2 have mutual elastance, so that the relations between the potential differences and the charges are expressed by the following equations:

$$e_1 = S_1Q_1 + S_{1,2}Q_2. \tag{154}$$
$$e_2 = S_2Q_2 + S_{2,1}Q_1.$$

By an argument in which the two condensers are carried through a complete cycle of changes, the two constants $S_{2,1}$ and $S_{1,2}$ may be shown to be equal. The steps in the argument are as follows:

Initial Condition.—Let both condensers be initially uncharged.

Step 1.—Let the quantity of electricity Q_1 be transferred in the arrow direction (in infinitesimal amounts) from one plate to the other of condenser 1. The work done, and the energy thereby stored, is $\frac{1}{2}S_1Q_1^2$.

Step 2.—Let the quantity Q_2 be now transferred in the arrow direction in infinitesimal amounts from one plate to the other of condenser 2. The work done in transferring the charge, and the energy thereby stored, is

$$W = \int_O^{Q_2}(S_{2,1}Q_1 + S_2Q)dQ = S_{2,1}Q_1Q_2 + \frac{1}{2}S_2Q_2^2.$$

Step 3.—Let condenser 1 be now discharged by transferring the quantity Q_1 in the opposite direction. The work done by the forces of the field, or the energy thereby returned, is

$$W = \int_O^{Q_1}(S_{1,2}Q_2 + S_1Q)dQ = S_{1,2}Q_1Q_2 + \frac{1}{2}S_1Q_1^2.$$

Step 4.—Let condenser 2 be now discharged. The energy thereby returned is $\frac{1}{2}S_2Q_2^2$.

Conclusion.—The condensers are now in their initial condition. The energy delivered to the condensers during the cycle is

$$\frac{1}{2}S_1Q_1^2 + S_{2,1}Q_1Q_2 + \frac{1}{2}S_2Q_2^2.$$

The energy returned by the field during the cycle is

$$\frac{1}{2}S_1Q_1^2 + S_{1,2}Q_1Q_2 + \frac{1}{2}S_2Q_2^2.$$

If these two quantities are not equal, this cycle, or the reverse of this cycle, will violate the principle of the conservation of energy. Therefore

$$S_{2,1} \text{ must equal } S_{1,2} \tag{155}$$

and if there are only two condensers to be dealt with, the subscripts $_{2,1}$ and $_{1,2}$ may be discarded for the common symbol S_m.

The expression for the energy stored in the two condensers having the charges Q_1 and Q_2 becomes

$$W \text{ (joules)} = \frac{1}{2}S_1Q_1^2 + S_mQ_1Q_2 + \frac{1}{2}S_2Q_2^2. \tag{156}$$

Further study will show that if there are more than two condensers having mutual elastance, say three condensers, the expression for the stored energy may be written

$$W = \frac{1}{2}[S_1Q_1^2 + S_2Q_2^2 + S_3Q_3^2] + S_{1,2}Q_1Q_2 + S_{1,3}Q_1Q_3 + S_{2,3}Q_2Q_3. \quad (156a)$$

Equations (156) and (156*a*) may be written

$$W = \frac{1}{2}[S_1Q_1 + S_{1,2}Q_2]Q_1 + \frac{1}{2}[S_2Q_2 + S_{1,2}Q_1]Q_2$$

and

$$W = \frac{1}{2}[S_1Q_1 + S_{1,2}Q_2 + S_{1,3}Q_3]Q_1 + \frac{1}{2}[S_2Q_2 + S_{1,2}Q_1 + S_{2,3}Q_3]Q_2 + \frac{1}{2}[S_3Q_3 + S_{1,3}Q_1 + S_{2,3}Q_2]Q_3$$

or

$$W \text{ (joules)} = \frac{1}{2}E_1Q_1 + \frac{1}{2}E_2Q_2 + \frac{1}{2}E_3Q_3 \quad (157)$$

or

$$W \text{ (joules)} = \Sigma\frac{1}{2}EQ. \quad (158)$$

That is to say:

110a. The energy stored in a system of condensers having mutual elastance is equal to the sum of the products obtained by multiplying the quantity of electricity transferred in each condenser by the average of the potential differences existing across the condenser at the beginning and the end of the transfer.

111. Systems of Conductors. Coefficients of Elastance and Capacitance.—The discussion of mutual elastance in the sections immediately above has been illustrated by the case of a field containing two or three condensers, and these condensers have each had two distinct electrodes. But the discussion applies also to the general case in which there may be any number of conductors between which a transfer of electricity may take place and in which any one conductor may serve as an electrode which is common to a number of condensers.

Figure 73 is an example of a simple system of conductors encountered in engineering practice. It consists of a three-phase power-transmission line comprising three wires A, B, and C and two telephone wires N and D strung above the earth's surface E. In some cases the power conductors are so connected to the generating apparatus that charge is transferred back and forth between A and B, between B and C, and between C and A; each

wire thus serves as an electrode common to two condensers. In other cases, the connections are such that the transfer of charge is back and forth between E and A, E and B, and E and C. The earth thus serves as an electrode common to all three condensers. For certain purposes it is desirable to be able to predict the induced potential difference across the condenser ND, and for other purposes across the condensers NE or DE.

In many cases it is very convenient to select some one conductor (generally the earth) and to regard it as the electrode common to all condensers, and then in the discussion to write of the potentials of the various conductors (relative to the reference conductor), rather than of the potential differences across the condensers. This practice frequently contributes to hazy attempts (on the part of beginners) to deal with one charge without any knowledge of, or interest in, the location of its complementary charge. On this account, it cannot be too strongly emphasized that in the treatment from the engineering point of view, of the electrostatic relations in a system of conductors we are generally interested in transfers of electricity from one specified conductor to another and with potential differences between specified conductors.

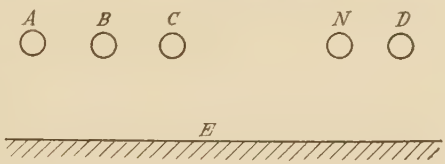


FIG. 73.

111a. Elastances.—Let the system contain n condensers numbered 1, 2, 3 . . . n. Let Q_1, Q_2, Q_n represent the algebraic values of the quantities of electricity transferred in the arrow direction in the respective condensers.

Let E_1 E_2 . . . E_n represent the algebraic value of the potential differences across the respective condensers.

Then the relation between the potential differences and the quantities of electricity in the condensers will be expressed by the following equations:

$$\begin{aligned} E_1 &= S_{1,1}Q_1 + S_{1,2}Q_2 + S_{1,3}Q_3 + \;.\;.\; + S_{1,n}Q_n \\ E_2 &= S_{2,1}Q_1 + S_{2,2}Q_2 + S_{2,3}Q_3 \;.\;.\;.\; + S_{2,n}Q_n \\ E_3 &= \\ E_n &= S_{n,1}Q_1 + S_{n,2}Q_2 + S_{n,3}Q_3 \;.\;.\;.\; + S_{n,n}Q_n \end{aligned} \tag{159}$$

The factors $S_{1,1}$, $S_{1,2}$, $S_{k,k}$, $S_{k,p}$, etc. are called the **elastances,** or the **coefficients of elastance,**[13] of the system. $S_{k,p}$ is the factor by which the charge in the pth condenser must be multiplied to obtain the potential difference which this charge causes between the electrodes of the kth condenser, all the other condensers being uncharged. Its value may be experimentally measured by the leaving all condensers uncharged except the pth, transferring a known positive charge Q_p in the arrow direction in the pth condenser, and measuring (by an electrostatic voltmeter of negligible capacitance) the potential difference which is set up between the electrodes of the kth condenser.

$$S_{k,p} = \frac{E_k}{Q_p}. \text{ (every } Q \text{ except } Q_p \text{ being zero)} \qquad (152a)$$

$S_{k,p}$ is the **mutual elastance** between the kth and the pth condenser. From Eq. (152), it follows that the factors with numbers $S_{1,1}$ $S_{2,2}$, $S_{n,n}$ are the ordinary (self) elastances of the respective condensers.

111b. Capacitances.—The above system of n linear equations explicitly expresses the value of the potential difference of each of the n condensers in terms of the n quantities of electricity in the condensers. By solving these simultaneous equations for Q_1, Q_2, etc., n linear equations may be obtained, each equation explicitly expressing the value of the charge in one of the condensers in terms of the n potential differences. These equations will be of the form:

$$\begin{aligned} Q_1 &= C_{1,1}E_1 + C_{1,2}E_2 + C_{1,3}E_3 + \ . \ . \ . \ C_{1,n}E_n. \\ Q_2 &= C_{2,1}E_1 + C_{2,2}E_2 + C_{2,3}E_3 + \ . \ . \ . \ C_{2,n}E_n. \qquad (160) \\ Q_n &= C_{n,1}E_1 + C_{n,2}E_2 + C_{n,3}E_3 + \ . \ . \ . \ C_{n,n}E_n. \end{aligned}$$

The factors $C_{1,1}$, $C_{1,2}$, $C_{k,p}$, etc. are called the **capacitances** or the **coefficients of capacitance**[14] of the system. $C_{k,p}$ is the factor by which the potential difference across the pth condenser must be multiplied to obtain the charge which the kth condenser

[13] For the case in which all condensers have one common electrode which is at zero potential, these factors are what many writers, notably Maxwell, have called **coefficients of potential.**

[14] For the case in which all condensers have one common electrode which is at zero potential, many writers, notably Maxwell, have called the $C_{p,k}$ factors **coefficients of induction,** and the $C_{k,k}$ factors **coefficients of capacity.**

will contain when the potential differences across the kth condensor and across all other condensors save the pth are zero.

Its value may be experimentally measured by connecting the head electrode of each condensor to its tail electrode save for the pth condensor, and measuring the charge Q_k which passes through the jumper from the tail to the head electrode of the kth condensor while the head electrode of the pth condensor is raised from zero to a value E_p volts higher than that of its tail electrode. The value of $C_{p,k}$ is given by

$$C_{k,p} = \frac{Q_k}{E_p}. \quad \text{(every } E \text{ except } E_p \text{ being zero)} \tag{161}$$

111c. Relation between Capacitances and Elastances.—Although from the formulas, the capacitance $C_{k,p}$ has the appearance of being the reciprocal of the elastance $S_{k,p}$, this is not the case, since the connections of the condensers are quite different when the potential differences E_p are being measured in the two cases. When measuring E_p for use in the capacitance formula, each condenser save the pth condenser must have its electrodes connected together by a connecting jumper.

Formulas expressing the capacitances in terms of the elastances may be obtained by using determinants to obtain equations for the Q's from Eq. (159).

Thus for a system containing four condensers we obtain the following equation for Q_2:

$$Q_2 \begin{vmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} \end{vmatrix} = \begin{vmatrix} S_{1,1} & E_1 & S_{1,3} & S_{1,4} \\ S_{2,1} & E_2 & S_{2,3} & S_{2,4} \\ S_{3,1} & E_3 & S_{3,3} & S_{3,4} \\ S_{4,1} & E_4 & S_{4,3} & S_{4,4} \end{vmatrix} \tag{162}$$

Using the symbols,

$$D = \begin{vmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} \end{vmatrix} \tag{163}$$

and

$$M_{3,2} = \begin{vmatrix} S_{1,1} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,3} & S_{2,4} \\ S_{4,1} & S_{3,4} & S_{4,4} \end{vmatrix} = \text{minor of 3rd row and 2nd column} \tag{164}$$

we may write

$$Q_2 = -\frac{M_{1,2}}{D}E_1 + \frac{M_{2,2}}{D}E_2 - \frac{M_{3,2}}{D}E_3 + \frac{M_{4,2}}{D}E_4. \quad (165)$$

From this it follows that

$$C_{2,1} = -\frac{M_{1,2}}{D}, \qquad C_{2,3} = -\frac{M_{3,2}}{D}$$

$$C_{2,2} = \frac{M_{2,2}}{D}, \qquad C_{2,4} = \frac{M_{4,2}}{D}$$

or, in general,

$$C_{k,p} = (-1)^{p+k}\frac{M_{p,k}}{D}. \quad (166)$$

Now the minor $M_{p,k}$ is the same as $M_{k,p}$ with its rows and columns interchanged. Since the interchanging of the rows with the columns does not alter the value of a determinant, $M_{p,k} = M_{k,p}$ and, consequently,

$$C_{k,p} = C_{p,k}. \quad (167)$$

111d. Voltage Ratios.—In engineering practice the data and the problem relating to systems and conductors most frequently take the following form: A number of condensers have mutual elastance (for example, several power-transmission lines and a telephone line strung with definite spacings). The potential differences across the condensers of the power system are known. Compute the induced potential difference between the two wires of the telephone circuit.

In the solution of such problems, neither the elastances nor the capacitances of the system can be used strictly. An examination of Eqs. (159) and (160) indicates that if there are n condensers across which the potential differences are $E_1, E_2 \ldots E_n$, then the potential difference thereby induced between the electrodes of a condenser which is itself uncharged, and which we will call the ath condenser, may be expressed by a linear equation of the form

$$E_a = P_{a,1}E_1 + P_{a,2}E_2 + P_{a,3}E_3 + \ldots P_{a,n}E_n \quad (168)$$

In recent work, the factors $P_{a,1}$ $P_{a,2}$ etc. have been called the **coefficients of induction** of the system, but since this name has been rather extensively applied to the capacitances $C_{1,2}$ $C_{1,3}$, etc., it may lessen the confusion to call these P factors the **voltage ratios.**

$P_{a,k}$ is the factor by which the potential difference across the kth condenser must be multiplied to obtain the potential difference induced between the electrodes of the ath condenser when the potential difference across all the other condensers is zero. Its value may be experimentally measured by connecting the head electrode to the tail electrode of each condenser save the ath and the kth, impressing a known potential difference E_k across the kth condenser and measuring the induced potential difference across the ath

$$P_{a,k} = \frac{E_a}{E_k}\cdot \text{ (every } E \text{ except } E_a \text{ and } E_k \text{ being zero)} \qquad (169)$$

Now in the system consisting of only the condensers 1 to n,—(the a condenser being left without a jumper from head to tail),

$$\left.\begin{aligned} Q_1 &= C_{1,k}\,E_k \\ Q_2 &= C_{2,k}\,E_k \\ Q_k &= C_{k,k}\,E_k \\ Q_n &= C_{n,k}\,E_k \end{aligned}\right\}\text{ every } E \text{ except } E_k \text{ being zero}$$

and since in the system consisting of the a and the n condensers, the voltage of the a condenser is given by

$$E_a = S_{a,a}Q_a + S_{a,1}Q_1 + S_{a,2}Q_2 + \ .\ .\ + S_{a,k}Q_k + \ .\ .\ + S_{a,n}Q_n$$

we obtain the following expression for the potential across the a condenser in terms of E_k (all other potentials save E_a and E_k being zero).

$$E_a = E_k[S_{a,1}C_{1,k} + S_{a,2}C_{2,k} + \ .\ .\ + S_{a,k}C_{k,k} + \ .\ .\ + S_{a,n}C_{n,k}]$$

Upon substituting this value of E_a in Eq. (169), the following expression is obtained for calculating the value of $P_{a,k}$

$$P_{a,k} = S_{a,1}C_{1,k} + S_{a,2}C_{2,k}\ .\ + S_{a,k}C_{k,k}\ .\ + S_{a,n}C_{n,k} \qquad (170)$$

Examples of the calculation of the values of the elastances, the capacitances, and the voltage ratios of a system will be given in Sec. 113.

112. Kelvin's Method of Electric Images.—A problem of frequent occurrence is that of finding the electric intensities and the distribution of the charges in a field containing both a given distribution of charges on certain conductors and an unknown distribution of charges upon the surface either of other conductors or of dielectrics—the unknown distribution being that which is

induced by the given distribution. For example—the problem may be to compute the force by which the concentrated charge Q at a distance h from the extended flat conducting surface S of Fig. 74 is attracted toward the surface by the charge which Q itself induces thereon. A second problem is that of computing the capacitance of the telephone wire A of Fig. 41 to the second parallel wire B in the presence of the charges which are induced on the (conducting) surface of the earth S by the given charges on the wires.

Problems of this kind, in which the conductors are of the simpler geometrical shapes, are most readily solved by the aid of a concept which was evolved by Kelvin in 1848—the concept of **electric images.** We proceed to develop the concept.

Electrostatic Boundary Relations to Be Satisfied.—The following statements can be made relative to the distribution of the induced charge.

112a. The distribution of the induced charge on the surface of any conducting body will be such that under the action of all charges in the field (given and induced) the electric intensities at points immediately outside the surface are normal to the surface; or, in other words, the surface of any conducting body is an equipotential surface.

112b. The distribution of the (induced) concealed charges on the surface of any dielectric will be such that (under the action of all charges) at adjacent points on opposite sides of the interface between two dielectrics, the tangential components of the electric intensities are equal, and the normal components are inversely proportional to the values of the permittivities of the two dielectrics.

The Image Method of Seeking the Solution.—Let us first consider the case in which the field due to a given distribution of charges exists in a greatly extended homogeneous medium, such as air, which contains other conducting bodies with their unknown induced charges but no non-conductors, such as glass or oil. Let us imagine all conducting bodies in the field to be coated with a non-conducting film of stiff collodion-like material having the same permittivity as the homogeneous medium by

which the conductors are surrounded. Each conductor, or conducting system, is now completely enclosed by a surface film. Now imagine the charges and the conducting material to be taken from each enclosure and to be replaced by air. We now have a homogeneous medium of infinite extent, free of electric charge, with the location of the surfaces of the original conductors mapped out by the stiff non-conducting films.

Let us now search for an **auxiliary** system of charges in this medium, which, together with the given distribution of charges, will make each closed surface (indicated by the films) an equipotential surface having the same potential as the known potential of the original conductors. If such an auxiliary system can be found (and it can only be found for a few simple geometrical configurations), then the electric intensity at any point in the field due to the combined effects of the auxiliary system and the given distribution of charge can be computed. That is to say, the electric intensities at points immediately outside the films may be computed. These intensities will, of course, be normal to the collodion film. Therefore it is evident that in such a field the non-conducting equipotential films may be replaced by metal films without causing any alteration in the field, since no movement of charge will take place **along** the film from one region to another, and only an infinitesimal separation of a finite charge will take place **across** the metal film.

Now by making use of the relation which always exists between the surface density of charge and the electric intensity immediately outside a conducting surface, namely,

$$\sigma \text{ (coulombs per sq. cm.)} = pF,$$

the surface density of the charge which will be found on any small patch of the outside surface of the metal film may be readily obtained from the previously computed value of the intensity at a point immediately outside the patch.

Having replaced the collodion by the metal film, any part or all of the auxiliary charge inside any enclosure (space formerly occupied by conducting matter) may be moved about without affecting the field outside the enclosure (Faraday's ice-pail experiment). The system of auxiliary inside charges may even be imparted to the film (thereby reducing the surface density of

charge at all points on the inside surface to zero) without affecting the intensity on the outside surface. Finally, the space enclosed by the metal films may be filled in with conducting material without in any way affecting the intensities.

By these steps, the system has in imagination been transformed from two sets of charges in free space (the given distribution and the auxiliary set), for which it is easy to compute the intensities, to a system consisting of the given distribution and the surface distribution on the actual configuration of conductors.

Now in the optical case of a candle in front of a mirror, the illumination of any small surface in the region in front of the mirror is produced partly by light directly from the candle, and partly by the light reflected from the mirror. The method of optical images enables us to replace the effect of the light reflected from the surface of the mirror by the light from an imaginary candle back of the mirror, which is called the optical image of the actual candle in the surface. By analogy the set of auxiliary charges which has the same effect as the surface distribution is called an **electric image.** Some writers call it the electric image of the surface distribution of electricity on the conductor, and others call it the image of the charges outside the surface which induce the surface distribution.

112c. (Definition).—**By an ELECTRIC IMAGE with respect to a closed conducting surface** S **(the surface of a conducting body or system of bodies) is meant that auxiliary point charge or distribution of charge inside the surface which would produce in the space outside the surface the same electric intensities which the actual electrification of that surface really does produce.**

The following examples illustrate the application of the method of images.

112d. (Problem).—Let the problem be to determine the surface density of the charge induced by the point charge Q of Fig. 74 on the surface S of the infinitely extended conducting plate, and to determine the force with which Q is attracted toward the plate.

Since the plate extends to regions infinitely remote, its surface S is at zero potential. Now if the location of the surface S is marked (as by the non-conducting collodion film) and the plate is removed, it is evident that a charge $-Q$ placed on the normal from Q to the surface S and at a distance h behind the surface will, together with the charge Q, make the potential of all points of the surface zero. Therefore, the image of the surface distribu-

tion is a charge $-Q$ located at a distance h behind the surface S as shown in the figure.

At any point P which lies in the space in front of the plate and at the distances r_1 and r_2 from the charge Q and its image, the values of the potential and the intensity will be

$$E = \frac{Q}{4\pi p r_1} - \frac{Q}{4\pi p r_2}$$

and

$$F = \frac{Q}{4\pi p r_1^2} + \frac{-Q}{4\pi p r_2^2} \text{ (a vector addition).}$$

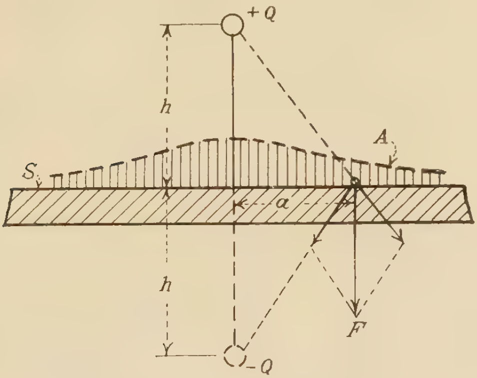


FIG. 74.—Image of the surface distribution on a plane.

The electric intensity at Q due to the image will have the value $Q/(16\pi ph^2)$, and therefore the charge Q will be attracted toward the plate with the force

$$\text{f (dyne-sevens)} = \frac{Q^2}{16\pi ph^2}. \tag{171}$$

The intensity at a point A immediately outside the surface S and at a distance a from the foot of the perpendicular has the value

$$F = \frac{Qh}{2\pi p(h^2 + a^2)^{3/2}}.$$

Therefore the surface density of the charge on the plate at a distance a from the foot of the perpendicular has the value

$$\sigma(\text{coulombs per sq. cm.}) = pF = \frac{Qh}{2\pi(h^2 + a^2)^{3/2}}. \tag{172}$$

The variation of the surface density is represented to scale in Fig. 74 by the depth of the cross-hatched portion.

112e. (PROBLEM).—Find the image of a point charge of Q coulombs which lies between the two infinitely extended metal plates illustrated in Fig. 75.

The surfaces of the plates are surfaces at zero potential. A study of the figure will show that these surfaces will be at zero potential under the action of the charge Q and the infinite series of point charges having the signs and locations tabulated below.

This series of charges is, therefore, the image of surface distribution of the charge on the two plates. Since the more remote charges contribute very little to the intensities, the intensities may be obtained with reasonable accuracy by taking into account the effect of the nearer charges only.

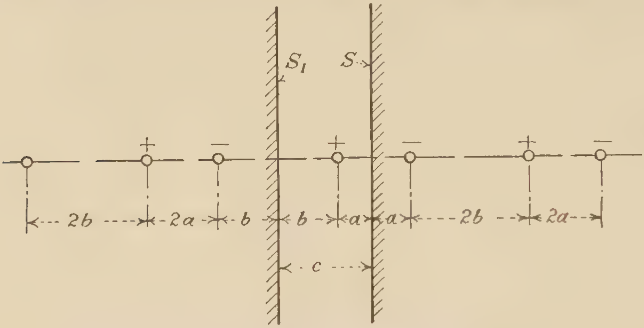


Fig. 75.—Image of a point charge in two planes.

To the right of the surface S	To the left of the surface S_1
$-Q$ at distance a	$-Q$ at distance b
$+Q$ at distance $c + b$	$+Q$ at distance $c + a$
$-Q$ at distance $2c + a$	$-Q$ at distance $2c + b$
$+Q$ at distance $3c + b$	$+Q$ at distance $3c + a$
$-Q$ at distance $4c + a$	$-Q$ at distance $4c + b$
etc.	etc.

112f. (Problem).—To find the surface density at any point on a conducting sphere of radius R, under the action of a charge of Q_1 coulombs concentrated at a point at a distance b from the center.

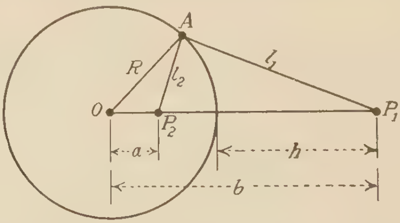


Fig. 76.—Image of the surface distribution on a sphere.

Let the charge of Q_1 coulombs be at the point P_1 of Fig. 76. Suppose that a charge Q_2 is located on the radial line OP_1 at a point P_2 whose distance a from the center of the sphere is such that

$$a = \frac{R^2}{b}. \tag{173}$$

Let A represent any point on the surface of the sphere of radius R. Then the value of the potential at A is

$$E = \frac{1}{4\pi p}\left(\frac{Q_1}{l_1} + \frac{Q_2}{l_2}\right).$$

Now the triangles OP_2A and OAP_1 are similar, since the angle AOP_2 is common and P_2 has been so taken that

$$\frac{OP_2}{OA} = \frac{OA}{OP_1}.$$

Therefore $\frac{l_2}{l_1} = \frac{R}{b}$ and the potential of A may be written

$$E = \frac{1}{4\pi p}\left(\frac{Q_1}{l_1} + \frac{Q_2}{l_1}\frac{b}{R}\right).$$

From this it will be seen that the potential at A (which represents any point on the sphere) will be zero, provided Q_2 is related to Q_1 by the relation

$$Q_2 = -\frac{Q_1R}{b}. \tag{174}$$

Therefore the charge $Q_2 = -Q_1R/b$ located inside the sphere at a distance $a = R^2/b$ from the center of the sphere is the image of the surface distribution of electricity which exists on the sphere when the sphere is at zero potential under the influence of the charge Q_1 which lies outside the sphere on the same radial line as Q_2 and at the distance b from the center.

The electric intensity at any point immediately outside the sphere due to the charge Q_1 and its image Q_2 may be calculated, and from these calculations the surface density of the charge on the sphere may be readily obtained by making use of the relation $\sigma = pF$.

It should be noted that the net quantity of electricity on the surface of the sphere must be equal to the quantity in the image, namely, $-Q_1R/b$. This quantity lying at the distance R from the center together with the quantity $+Q_1$ lying at the distance b from the center may be seen by inspection to give zero for the potential of the center.

112g. If a charge of Q coulombs uniformly distributed over the surface of the sphere is superposed on the surface distribution of the previous case, the surface will still be an equipotential surface but the potential of the sphere will be raised from zero to the value

$$E = \frac{Q}{4\pi pR}.$$

Now the image of the charge Q uniformly distributed over the sphere is a charge Q at the center of the sphere. Therefore it follows that the image of the surface distribution of charge on a sphere at the potential E and under the inductive influence of a charge Q_1 at a distance b from its center consists of a charge

$$Q = 4\pi pRE \tag{175}$$

at the center of the sphere, and a charge of $-Q_1R/b$ at the distance R^2/b from its center.

If the sphere is uncharged, *i.e.*, the net charge is zero, Q must equal $-Q_1R/b$ and the potential of the sphere will be $Q_1/(4\pi pb)$. This same result may be arrived at directly by noting that the center of the sphere lies at a distance b from the charge Q_1 and at the same distance R from all surface charge, of which the net sum is zero.

112h. To find the distribution of the charge on a conducting sphere of radius R placed in an extended uniform field of intensity F, we need but to place the charge Q_1 at a distance b which is very great in comparison with R and give to it the value

$$Q_1 = 4\pi pb^2F$$

The image of the resulting surface distribution will consist of a charge having the value

$$Q_2 = -\frac{Q_1R}{b} = -4\pi pbRF$$

at the infinitesimal distance R^2/b from the center, and an equal charge of opposite sign at the center of the sphere.

An arrangement of two point charges of equal magnitude and of opposite sign with a very small distance l between centers is termed an **electric doublet.** *The straight line drawn through the centers is termed the* **axis** *of the doublet, and the product of the distance of separation times the magnitude of either charge is termed the electric* **moment** *of the doublet.*

$$M(\text{coulomb, cm.}) = ql. \tag{176}$$

The electric image of the charge on the surface of a conducting sphere placed in an extended uniform field of intensity F (before the introduction of the sphere) is thus an electric doublet located at the center of the sphere, with its axis parallel to the F vectors, and having an electric moment of the value

$$M = 4\pi pR^3F. \tag{177}$$

112i. (PROBLEM).—Let two metal spheres 1 and 2 of Fig. 77 having the radii R_1 and R_2 and the distance S between centers be maintained at the potentials E_1 and E_2, respectively. To find the images of the surface distributions of electricity on the spheres.

Let us first find the images when the sphere of radius R_1 is at the potential E_1 and the other sphere at zero potential.

A point charge Q_1 having the value of $4\pi pR_1E_1$ if placed at the center of sphere 1 will make the potential of sphere 1 equal to E_1, but it would raise the potential of the sphere 2 to the value $Q_1/4\pi pS$.

To keep sphere 2 at zero potential, a point charge Q_2 having the value $-Q_1R_2/S$ must be placed at the distance $a_2 = R_2^2/S$ from the center of sphere 2.

This charge Q_2 will alter the potential of sphere 1. To keep the potential of sphere 1 unaltered in the presence of Q_2, a charge $Q_3 = -Q_2R_1/(S - a_2)$ must be placed at the distance $a_3 = R_1^2/(S - a_2)$ from the center of sphere 1.

The charge Q_3 will, in turn, alter the potential of sphere 2 and to keep this unaltered, a charge $Q_4 = -Q_3R_2/(S - a_3)$ must be placed at a distance $a_4 = R_2^2/(S - a_3)$ from the center of sphere 2, and so on indefinitely.

In like manner, the images for the distributions when sphere 2 is at the potential E_2 and sphere 1 at zero potential may be formed. By superimposing these two cases, we have the case of sphere 1 at potential E_1 and sphere 2 at a potential E_2.

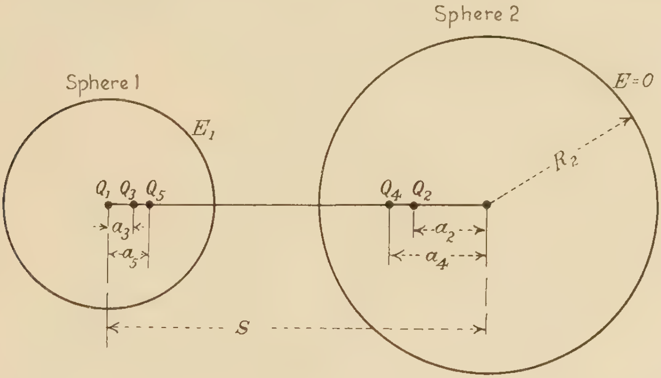


FIG. 77.—Image of the surface distribution.

It is to be noted that if the distance S is large in comparison with R the successive point charges decrease rapidly in magnitude and only a few need be considered. Thus if the spheres are equal in diameter, and if the gap between them is equal to the diameter, or $S = 4R$, and if the spheres are at equal potentials above and below earth potential, the magnitudes of the successive charges are as follows:

IMAGE SYSTEM: $R_1 = R_2 = 0.25S$

Sphere 1 at potential E_1	Sphere 2 at zero potential
	$Q_1 = 4\pi pRE_1$ at center of sphere 1
$Q_2 = -0.25Q_1$ at $a_2 = 0.25R$	$Q_3 = +0.0667Q_1$ at $a_3 = 0.266R$
$Q_4 = -0.0178Q_1$ at $a_4 = 0.2676R$	$Q_5 = +0.0048Q_1$ at $a_5 = 0.268R$
$Q_6 = -0.0013Q_1$ at $a_4 = 0.268R$	$Q_7 = +0.0003Q_1$ at $a_7 = 0.268R$

A study of this table will show that the intensities can be calculated to an accuracy of one-tenth of 1 per cent for this case by taking the image system to consist of point positive charges of the magnitudes Q_1 and 0.341 Q_1 at the center and at the distance $0.255R$ from the center in sphere 1, and point negative charges of equal magnitude similarly located in sphere 2.

113. Calculation of Elastances and Capacitances.—As an illustration of the calculation of the elastances, capacitances, and voltage ratios of a system

of condensers, let the system shown in Fig. 78 be considered. It consists of the two very long wires of a power circuit, 1 and 2, strung parallel to each other and to the surface of the earth and of two long telephone wires, 3 and 4, strung parallel to the power wires. The discussion will pertain to the case in which the transfer of charge takes place between wires 1 and 2 or between the wires 1 and 2 and the earth, but in which the telephone wires remain uncharged.

In setting up the equations for the system, the surface of the earth will be taken as an electrode common to all condensers, and the arrow directions will be taken as indicated.

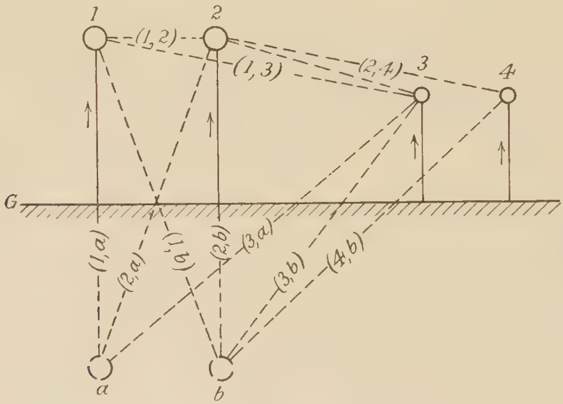


FIG. 78.—Image of power wires.

Let E_1, E_2, E_3, etc. represent the algebraic value of the potential differences of the condensers $G1$, $G2$, $G3$, etc. (that is, the potentials of the wires 1, 2, 3, etc. relative to earth).

Let q_1 and q_2 represent the algebraic value of the charges transferred in the arrow directions per unit length of wire in condensers 1 and 2, and let Q_1 and Q_2 represent the corresponding total charges transferred.

The equations for the potential differences in terms of the elastances and charges are

$$E_1 = S_{1,1}Q_1 + S_{1,2}Q_2.$$
$$E_2 = S_{2,1}Q_1 + S_{2,2}Q_2.$$
$$E_3 = S_{3,1}Q_1 + S_{3,2}Q_2.$$
$$E_4 = S_{4,1}Q_1 + S_{4,2}Q_2.$$

The first problem is to compute the values of the elastances. The wires are assumed to be so long that the increase in the charge per unit length near the ends of the wires may be ignored and each wire may be assumed to be charged with the same quantity of electricity per unit length throughout its entire length.

For all practical purposes, the surface of the earth remains at zero potential, no matter what transfers of electricity may be made between the wires.

Clearly then, if charges of q_1 and q_2 per unit length of wire have been transferred from the earth to wires 1 and 2, the image of the distribution of charge which is thereby induced on the surface of the earth in a broad strip paralleling the power line will be two similar wires a and b at distances below the surface equal to the heights of the power wires above the surface, and containing the charges $-q_1$ and $-q_2$ per unit length.

$S_{1,2}$, the mutual elastance between the first and second condensers, is the factor by which the charge on wire 2 must be multiplied to obtain the potential to which this charge raises wire 1, all other condensers being uncharged.

$$S_{1,2} = \frac{E_1}{q_2}. \tag{152}$$

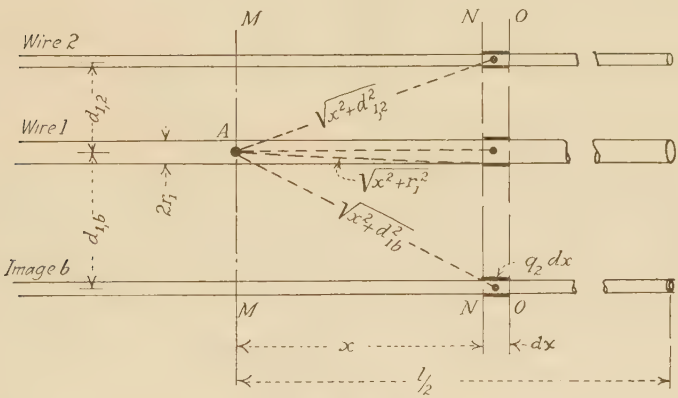


FIG. 79.

Let us calculate the value of $S_{1,2}$ by calculating the potential to which wire 1 is raised by the charge q_2 per unit length on wire 2.

Figure 79 is drawn to facilitate this calculation. In this diagram, MM represents the intercept on the paper of a plane taken perpendicular to the wires at the midpoint of the line.

Let the potential of wire 1 be found by calculating the potential at A, the midpoint on its axis.

Let $d_{1,2}$, $d_{1,b}$, etc. represent the distances between centers of the wires designated by the subscripts.

h_1, h_2, etc. represent the heights above the ground of the wires.

r_1, r_2, etc. represent the radii of the wires.

Let two planes NN and OO be drawn parallel to MM at the distances x and $x + dx$ from it. The elements of charge on wire 2 and its image b included between these planes have the values q_2dx and $-q_2dx$. For the case in which the distance between the wires is many times the radius of wire 2, the distances from these charges to the point A on the axis of wire 1 may, without appreciable error, be taken as $\sqrt{x^2 + d^2_{1,2}}$ and $\sqrt{x^2 + d^2_{1,b}}$.

Therefore potential at A due to these elements of charge is

$$de = \frac{q_2 dx}{4\pi p\sqrt{x^2 + d^2_{1,2}}} + \frac{-q_2 dx}{4\pi p\sqrt{x^2 + d^2_{1,b}}}.$$

The potential at A due to the charge on the entire length of wire 2 and its image is given by the integral

$$E_1 = 2\int_0^{\frac{l}{2}} \frac{q_2}{4\pi p}\left[\frac{dx}{\sqrt{x^2 + d^2_{1,2}}} - \frac{dx}{\sqrt{x^2 + d^2_{1,b}}}\right] \tag{178}$$

$$E_1 = \frac{q_2}{2\pi p}\left[\log\left(x + \sqrt{x^2 + d^2_{1,2}}\right) - \log\left(x + \sqrt{x^2 + d^2_{1,b}}\right)\right]_0^{\frac{l}{2}}$$

$$= \frac{q_2}{2\pi p}\left[\log\frac{x + \sqrt{x^2 + d^2_{1,2}}}{x + \sqrt{x^2 + d^2_{1,b}}}\right]_0^{\frac{l}{2}}$$

$$E_1 = \frac{q_2}{2\pi p}\left[\log\frac{l/2 + \sqrt{(l/2)^2 + d^2_{1,2}}}{l/2 + \sqrt{(l/2)^2 + d^2_{1,b}}} - \log\frac{d_{1,2}}{d_{1,b}}\right]. \tag{179}$$

If $l/2$ (the half length of the line) is greater than 10 times the distance $d_{1,2}$ between wires, the first term in the bracket differs so little from $\log l/l$, or zero, that it may be neglected in comparison with the second term, and without appreciable error the equation may be written

$$E_1 \text{ (volts)} = \frac{q_2}{2\pi p}\log\frac{d_{1,b}}{d_{1,2}}. \tag{180}$$

Accordingly, $$S_{1,2} \text{ (darafs)} \left(= \frac{E_1}{Q_2}\right) = \frac{1}{2\pi pl}\log\frac{d_{1,b}}{d_{1,2}}. \tag{181}$$

In like manner $$S_{2,1}\left(= \frac{E_2}{Q_1}\right) = \frac{1}{2\pi pl}\log\frac{d_{2,a}}{d_{1,2}}.$$

Since $d_{1,b} = d_{2,a}$, it follows that $S_{1,2} = S_{2,1}$, thus confirming by direct calculation for a special case the equality of the mutual elastances between two condensers which was deduced in Sec. 110 by a general argument.

In like manner the value of $S_{1,1}$ may be found by calculating the potential to which the point A is raised by charges of q_1 and $-q_1$ coulombs per unit of length on wire 1 and its image a. By noting that the distance of the point A from the elements of charge on wire 1 and its image are $\sqrt{x^2 + r_1^2}$ and $\sqrt{x^2 + d^2_{1,a}}$, we may write the formula for $S_{1,1}$ by substituting $d_{1,a}$ for $d_{1,b}$ and r_1 for $d_{1,2}$. Thus

$$S_{1,1} = \frac{1}{2\pi pl}\log\frac{d_{1,a}}{r_1}. \tag{182}$$

Therefore, the equations for the potentials may be written

$$\begin{aligned} E_1 &= \frac{1}{2\pi p}\left[q_1 \log\frac{d_{1,a}}{r_1} + q_2 \log\frac{d_{1,b}}{d_{1,2}}\right] \\ E_2 &= \frac{1}{2\pi p}\left[q_1 \log\frac{d_{2,a}}{d_{1,2}} + q_2 \log\frac{d_{2,b}}{r_2,}\right] \\ E_3 &= \frac{1}{2\pi p}\left[q_1 \log\frac{d_{3,a}}{d_{1,3}} + q_2 \log\frac{d_{3,b}}{d_{2,3}}\right]. \end{aligned} \tag{183}$$

To derive the formula for the capacitance of wire 1 to wire 2 in the presence of the earth, the above equations may be used by putting $q_2 = -q_1$, which means that the net transfer of electricity is from wire to wire, the net charge on the earth being zero.

If l is the length of one wire of the pair, the total quantity transferred is $Q = ql$.

The difference in the potential of the wires is

$$E = E_1 - E_2 = \frac{q_1}{2\pi p}\left[\log\frac{d_{1,a}}{r_1} - \log\frac{d_{1,b}}{d_{1,2}} - \log\frac{d_{2,a}}{d_{1,2}} + \log\frac{d_{2,b}}{r_2}\right].$$

If the wires have the same radius r, are at the same height h above the ground, and are spaced the distance d center to center, this equation becomes

$$E = \frac{q_1}{\pi p}\log\frac{d}{r}\frac{2h}{\sqrt{(2h)^2 + d^2}}.$$

Consequently, the capacitance of one wire to another when both wires are at the height h above the ground is

$$C \text{ (farads)} \left(= \frac{Q}{E}\right) = \frac{\pi p l}{\log\dfrac{d}{r}\dfrac{2h}{\sqrt{(2h)^2 + d^2}}}. \tag{184}$$

The factor $\dfrac{2h}{\sqrt{(2h)^2 + d^2}}$ takes into account the effect of the induced charge on the surface of the earth, since if h (the height of the wires) is made infinitely great this factor reduces to unity. For the usual spacings of overhead power or telephone circuits this factor differs from unity by less than 0.001. Therefore, for such circuits, this factor may be taken to be unity and the formula may be written

$$C \text{ (farads)} = \frac{\pi p l}{\log\dfrac{d}{r}}. \tag{185}$$

To find the capacitance to earth of a long wire of radius r at the height h above the earth, it is necessary to use only the first term of the voltage Eq. (183). Thus the quantity on the wire is ql, and the difference of potential is

$$E_1 = \frac{q_1}{2\pi p}\log\frac{d_{1,a}}{r} = \frac{q_1}{2\pi p}\log\frac{2h}{r}.$$

Whence,

$$C \text{ (farads)} \left(= \frac{Q}{E_1}\right) = \frac{2\pi p l}{\log\dfrac{2h}{r}}. \tag{186}$$

113a. Exercises.

1. Let a spherical surface S contain a single charge Q located at its center. Calculate the flux of the electric intensity vector in the outward direction across the surface S.

2. Let a charge Q be placed at a point P on the axis of a circular disk whose radius is r and let h represent the distance from P to the surface. Calculate the flux of the electric intensity over the surface of the disk.

What is the significance of the similarity between the expression you obtain and Eq. (77) of Sec. 88? (Make it a question of action and reaction.)

3. Let a charge Q be located at the origin of a rectangular set of axes. For a point P whose coordinates may be represented by x,y,z, derive the expression for the x component (F_x) of the electric intensity, also for the y component (F_y), and the z component (F_z). Do these values when substituted in Eq. (71) of Sec. 86 satisfy it?

4. Compute the rate at which energy is converted into the electrical form when the water dropper generator of Sec. 64 is operating, as follows:

Difference of potential between receptors is 60,000 volts.

Diameter of spherical water drops 3 millimeters.

Electric intensity at the surface of the carriers 20,000 volts per centimeter.

Number of carriers passing through each half of the machine per second = 4.

5. Consider two long coaxial metal cylinders, such as the copper conductor and lead sheath in Fig. 66a, separated by any homogeneous dielectric.

a. Derive an expression for the value of the electric intensity at the surface of the inner cylinder, in terms of the dimensions of the cylinders and the difference in potential, E, between them.

b. Suppose that E is 10,000 volts and that the inner radius, r_2, of the sheath is fixed at 2 centimeters but that the radius of the inner cylinder may have any value from zero to r_2. Plot a curve showing the relation between the intensity at the inner cylinder and the radius of the inner cylinder, the radius being plotted, as abscissa, in per cent of the radius of the outer cylinder.

6. Assume a No. 0000 B. & S. stranded conductor is given a thin lead coating so that the surface of the lead coating is perfectly smooth and round with a diameter of ½ inch.

This wire is to be insulated with impregnated manila paper enclosed in a lead sheath. Manila paper will support for a short time a gradient of 175,000 r.m.s. volts per centimeter, and will operate safely for an hour under a gradient of 100,000 r.m.s. volts per centimeter, but should not be used continuously at a gradient in excess of 40,000 r.m.s. volts per centimeter.

What thickness of insulation would be necessary to limit the gradient at the surface of the inner conductor to 40,000 r.m.s. volts per centimeter for the following r.m.s. working voltages between conductor and sheath: 25, 50, 100, and 200 kilovolts?

7. Determine the voltage across each layer and plot the gradient curves for a high-tension lead-covered cable insulated with four layers of insulation having the properties given below. The radius of the circular inner conductor is 0.9 centimeter and the voltage from conductor to sheath is 50,000 volts. Ignore the effect of conduction.

	Relative Permittivity
Innermost layer of rubber 0.25 cm. thick	6.1
Second layer of rubber 0.23 cm. thick	4.7
Third layer of rubber 0.45 cm. thick	4.2
Layer of paper 0.52 cm. thick	4.0

8. The chemical energy stored per cubic centimeter of high-grade coal is 40,000 joules. The energy per cubic centimeter of hydrogen at atmospheric pressure (to be burned to H_2O) is 13 joules. Compare these figures with the energy which can be stored per cubic centimeter in a glass plate constituting the dielectric of a parallel-plate condenser. Assume that the relative permittivity of glass is 7, and that the allowable electric intensity in the glass is 150,000 volts per centimeter.

9. In Fig. 67 let the liquid dielectric be an oil of relative permittivity 3, and let the upper dielectric be air, in which the electric intensity is 24,000 volts per centimeter. What is the electric intensity in the oil? What hydrostatic pressure is caused by electric forces in the oil between the plates? What is the upward force per square centimeter on the boundary film of oil? To what height (shown by the dotted line) will the oil rise under these forces (density is 0.85)?

10. Calculate the mutual elastance between condenser 1 consisting of spheres A and B, and condenser 2 consisting of spheres C and D, of exercise 18, Chap. III. Let A and D be taken as the "head" electrodes.

11. Suppose that a large, smooth metal sphere has such a negative charge $(-Q)$ that the value of the electric intensity at its surface is 30,000 volts per centimeter. How far must an electron which has been emitted from the metal get from the surface in order that the repulsive force of the charge $(-Q)$ on the electron shall equal or exceed the attractive force between the electron and the charge it induces on the surface? Express the distance in diameters of the electron.

12. A power line consists of two No. 00 B. & S. gage solid copper wires (diameter 0.93 centimeter), separated 180 centimeters center to center and mounted 9 meters above the earth. It is paralleled by a telephone line consisting of two No. 10 wires (diameter 0.26 centimeter), separated 30 centimeters, mounted 9 meters above the earth with a distance of 9 meters between the adjacent power and telephone wires. Assume the two telephone wires are well insulated from ground and from each other.

At given instant one power wire is 30,000 postive to ground and the other is 30,000 negative to ground.

a. Compute the potential difference between the two telephone wires.

b. What is the "voltage ratio" between the power line and the telephone line?

c. Compute the potential difference between each telephone wire and ground.

d, e, f. Repeat the calculations *a, b,* and *c,* for the case in which the more remote power wire is connected to ground and the nearer wire is 60,000 positive to ground.

CHAPTER VI

ELECTRIC CURRENT

THEME: The detection and measurement of electric current.

114. Movements of Electric Charge.—The preceding chapters have dealt with electrical phenomena which are the result of the separation of negative from positive electricity. The separated charges, considered in bulk, have been at rest relative to each other. The essential rôle played by the motions of individual electrons in such phenomena as contact electrification has been mentioned only in a qualitative way. We now propose to consider the phenomena accompanying the motion of electric charge and the laws pertaining to these phenomena. As in the treatment of electrostatics, the first laws will relate to the motion of electricity in bulk, and not to the motions of the constituent elements—the electrons. The rôle played by the motions of the individual electrons will be discussed only in a qualitative way.

The notion of the flow of electricity was first suggested by such phenomena as the ready disappearance of a charge from the bodies called conductors when they were connected to earth, and the ready division of an electric charge between two such bodies when contact was made between them. After becoming interested in such movements, the early investigators made experimental discoveries which may be grouped as answers to two questions:

a. What phenomena or effects attend the movements of electricity, and how may these effects be used to detect and to measure the movements?

b. How may such movements be caused and controlled?

These two topics are the respective themes of the present and the following chapter.

115. The Electric Current and Its Effects.—If two insulated conductors A and B, originally at different electric potentials,

are connected by a conducting body *C*, such as a wire, there will be a flow of electricity, or an **electric current,** through the body *C*. This flow results in a transfer between *A* and *B* of electricity of a kind to equalize the potentials of the two bodies. If *A* and *B* are bodies having small capacitance relative to each other, the flow or the current may be brought to an end in a millionth of a second or less, by reason of the equalization of the potentials. That is, the current is a momentary or **transient** current. If, however, the conductors *A* and *B* are made the receptors of an electric machine, electricity will ultimately be carried from *B* to *A* through the machine at the same rate that it flows from *A* to *B* through the conducting body. In this case electricity continues indefinitely to flow through the connecting body, or the current is a **continuous** current.

While the electric current, transient or continuous, exists in the body *C*, certain phenomena which are called the manifestations or the effects of the electric current are to be observed in the body *C* and in the space surrounding *C*. These effects, which are all utilized in instruments of different types to detect and to measure electric current, are as follows:

a. **Redistribution of charge effect.**—Every transient or variable current results in a redistribution of the charges giving rise to the electrostatic field.

b. **Heating effect.**—All conductors are heated to a greater or less degree by an electric current through them.

c. **Electrochemical effect.**—If the current is through a liquid or molten solution, one or more of the constituents (salts or acids) is in part decomposed into its component parts (chemical elements or radicals) and these parts are either deposited or liberated at the two electrodes by which the current is conveyed to and from the solution.

d. **Magnetic field effects.**

A current in a conductor exerts a mechanical force upon nearby magnets and upon the ferromagnetic materials, iron, nickel, and cobalt.

Mechanical forces—attractive and repulsive—exist between different portions of a conductor carrying current, and between different conductors carrying different currents.

If the current in a conductor varies, or if the conductor carrying the current moves relative to nearby conductors, transient currents are **induced** in the nearby conductors.

e. **Peltier effect.**—When an electric current exists across the junction of two metals, the junction is heated when the current is in one direction, and cooled when it is in the opposite direction across the junction. This effect was discovered by Peltier in 1834 and is known as the Peltier effect.

116. Classification of Currents Based on Mode of Transport. Before discussing the effects listed above, we may observe that the motion of electric charge across a surface may take one of three forms:

a. The charge may be carried across the surface by **charged moving matter,** for example, by rotating disks, endless belts, water drops, dust or mist particles, molecules, or atoms. The flow of electricity through gases or through liquids is, at least in part, of this type. If a strong electric field exists throughout a region filled with a gas such as air, any atoms, molecules, or particles of dust containing excess electrons move from regions of low to regions of high potential, while atoms or particles which have temporarily lost one or more electrons move in the other direction. A current of this character (charge conveyed by moving matter) is called a **convection current.**

b. The charge may flow across the surface through a solid (not liquid or gaseous) conductor. In this case we conceive that the flow is of electrons alone. The conductor is conceived to contain at any instant an atmosphere of free electrons. When a potential difference exists between two points or regions of the conductor, electrons move from the regions of low potential toward the regions of higher potential because of the electric forces directed along the path. The flow of electrons is called a **conduction current.**

c. The electrons and positive nuclei of any layer of molecules of a dielectric may suffer an elastic displacement in opposite directions across the midsurface of the layer, as a result of changing electric intensities in the dielectric. Such an elastic flow of electricity across a surface is included in the current which

will receive consideration in a later chapter as the **displacement current** across a surface.

117. Redistribution of Charge Effect.—The creation or alteration of an electrostatic field of force is the result of some redistribution of electric charge. Redistribution means that electric charge flows or is transferred from one or more bodies or regions to other regions.[1] Since an account of the quantity of electricity transferred to or from any region may be rendered by stating the rate at which electricity crosses some surface enclosing the region, it follows that a direct quantitative connection between electrostatic theory and electric current theory may be established by defining electric current in terms of the quantity of electricity crossing a surface. This we proceed to do in the following section.

118. Quantitative Definition of Electric Current.—In the discussion of any flow, whether of electricity or of a gas or of a liquid, some definite surface over which the flow is occurring must either be specified or understood. In specifying the flow across any surface, it is convenient to indicate arbitrarily either one of the two directions across the surface by an arrow, and to call this direction the **arrow direction,** or the **specified direction** across the surface.

It has been stated that a convection current in gaseous or in liquid conductors is conceived to consist of the flow of positively charged atoms or particles from regions of high to regions of low potential, and of electrons or negatively charged atoms or particles in the opposite direction, while the conduction current in metals is conceived to involve the flow of electrons alone. Now as regards the transfer of electric charge from one region to another, the flow of negative electricity in the arrow direction across a

[1] While a change in an electrostatic field is always attended by an electric current, an electric current is not always attended by changes in the electrostatic field, since the charges which cause the field may be replaced through some source (as an electric machine or a battery) as fast as they flow together. In other words, there may be a continuous circulation of electricity. Thus electric currents may or may not be attended by changes in the electrostatic field. A transient or a variable current is always attended by such changes, and a continuous current is not so attended.

s the equivalent of the flow of an equal quantity of positive electricity in the opposite direction. We shall find that it is also the equivalent in magnetic effect. We may, therefore, state the effect of either kind of current or define the value of either current in terms of the equivalent flow of positive electricity in the following manner:

118a. ELECTRIC CURRENT (DEFINITION).—**The continued passage of electric charge across any surface is termed an ELECTRIC CURRENT across the surface.**

The algebraic VALUE I **of the current in a specified direction across any given surface is defined to be the EQUIVALENT TIME RATE OF PASSAGE of positive electricity across the surface in the specified direction—the rate being expressed in coulombs per second.**

Thus, if, in the interval of time t, Q_1 represents the absolute value of the quantity of positive electricity which crosses the surface in the arrow direction, and Q_2 represents the quantity of negative electricity which crosses in the opposite direction, this is equivalent (as regards separation of charge) to the passage of $Q = Q_1 + Q_2$ coulombs of positive electricity across the surface in the arrow direction. Accordingly, the average value (for the interval t) of the current in the arrow direction across the surface is, by definition,

$$I \text{ (coulombs per second)} = \frac{Q}{t} \frac{\text{(coulombs)}}{\text{(seconds)}}. \tag{187a}$$

The descriptive name of the unit of current is the **coulomb per second.** As is customary with important units, a short name is substituted for the descriptive name, and for this particular unit the name **ampere** has been chosen.

118b. *Unit of Electric Current* (DEFINITION).—*The current across a surface is said to have a value of one "ampere" when electricity crosses the surface at the net rate of one coulomb per second.*

$$I \text{ (amperes)} = \frac{Q}{t} \frac{\text{(coulombs)}}{\text{(seconds)}}. \tag{187}$$

Since the equivalent quantity Q of positive electricity which crosses a surface in the specified or arrow direction may be a positive or a negative algebraic quantity, the (value of the) current in a specified direction is to be regarded as an algebraic quantity which may have a + or − sign.

If the flow is not uniform, the instantaneous value of the current is defined to be the limit approached by the average rate when taken over shorter and shorter intervals.

$$i \text{ (amperes)} = \frac{Q}{t} \text{ (as } t \text{ approaches 0).} \tag{187}$$

By the "current in a wire" or "through an appliance" is meant the current across some cross-section of the wire or across some surface which separates the two terminals of the appliance.

119. Direction of the Electric Current.—In the rules which will be formulated in later chapters for stating the direction of the mechanical force upon conductors carrying current the expression "the direction of the current" occurs. At a time when it was not known whether a conduction current was a flow of positive electricity from regions of high to regions of low potential, or a flow of negative electricity from low- to high-potential regions, or the combined flow of the two electricities in opposite directions, the following arbitrary convention was adopted.

119a. (CONVENTION).—**The direction across the surface in which positive electricity tends to flow is to be called the DIRECTION OF THE CURRENT.**

120. Current Density.—By the current density across a given plane surface at a given point is meant the number of amperes crossing the plane per unit area at the point. By the current density at any point in a medium is meant the maximum current density at the point; that is, the current density measured in a plane normal to the direction of flow of current at the point.

120a. CURRENT DENSITY (DEFINITION).—**The current density J at a point is a vector quantity pointing in the direction of current flow at the point, and having a magnitude equal to the number of amperes per square centimeter crossing a small plane area which is perpendicular to the vector at the point.**

$$\mathrm{J} \text{ (amperes per square centimeter)} = \frac{I}{a} \frac{\text{(amperes)}}{\text{(centimeters)}}. \tag{188}$$

121. Heating Effect. The Nature of Metallic Conduction and of the Conduction Current.—The heating effect of the spark which occurs when charge passes from one body to another was

early observed. A little later it was discovered that heat is generated also in the wire which is used to discharge a condenser. Later still, when continuous currents could be obtained from batteries, much larger heating effects were observed, so that today we may make the general statement: "All bodies are heated to a greater or less degree when an electric current exists in them."

The quantitative relation between the magnitude of the current and the rate at which energy is expended in the body in the form of heat was discovered by Joule in 1841. Joule found that the power so expended in a conductor varies as the square of the current. This law is presented more fully in Chap. VIII.

The development of heat in a conductor when electricity flows through it is readily accounted for in terms of the mechanical theory of heat and the electron theory of conduction. According to the mechanical theory of heat, the **sensible** heat energy of a substance is the kinetic energy possessed by the molecules of the substance by reason of their velocities of vibration (the velocities of thermal agitation). Any increase in the sensible heat, or in the temperature of a substance, signifies that work has been done on the substance in such a manner as to increase the kinetic energy of molecular motion, or the mean molecular velocity.

Let us consider the manner in which an electric current through a substance brings about an increase in the molecular velocity. In all conductors except the electrolytic conductors discussed in the following section, the electric current is regarded as a flow of the **free** electrons through the interstices between the molecules of the substance. When no electric field exists within the conductor, the free electrons move at random in all directions with a mean velocity which is of the order of 10^7 centimeters per second. When an electric field exists within the body of the conductor each electron experiences an accelerating force in a direction opposite to that of the electric intensity, and therefore has a directed velocity superimposed upon its random velocity. The atmosphere of free electrons as a whole is thus driven through the conductor in the direction of the force exerted upon it, or the field does work upon the atmosphere of free electrons. The electrons do not gain momentum indefinitely, because, by colliding with the molecules, they do work upon the molecules, thereby sharing their increased momentum with the molecules or increas-

ing their velocity of thermal agitation. In this manner we conceive the work done by the forces of the field to be converted into the heat energy of random molecular motion.

Since the electronic charge is 1.59×10^{-19} coulombs, a current density of 1 ampere per square centimeter means the passage across a surface of 6.28×10^{18} electrons per square centimeter per second. The evidence as to the number of free electrons per cubic centimeter in a good conductor like copper is conflicting. Estimates based upon different lines of evidence vary from one electron per atom to one per 3000 atoms. There are 8.4×10^{22} atoms of copper per cubic centimeter. On the assumption of one free electron per 3000 atoms, there would be 2.8×10^{19} free electrons per cubic centimeter. This atmosphere would have to move through the conductor with a velocity of only 0.22 centimeter per second to make the current density 1 ampere per square centimeter. The economic current density for copper conductors will be found to be of the order of 150 amperes per square centimeter. The above calculations indicate that such a current density would require that the electron atmosphere in the copper move forward with an average velocity of only 33 centimeters per second. This average velocity is superimposed on a random velocity of the order of 10^7 centimeters per second. The type of conduction described above is known as **metallic conduction** and the current is called a **conduction current.**

Electric currents are very commonly detected by means of the rise in temperature. The same effect is often used to measure the currents, for even with very small currents the temperature rise may be detected if instruments of sufficient delicacy are devised. The heating effect of electric current has an important commercial application in electric incandescent lamps, and in a wide variety of electric furnaces, ovens, and heating appliances. On the other hand, the current which an electric generator can safely deliver is limited by the temperature rise which may be permitted without destroying the insulating materials in the machine. These facts make the heating effect on electric current a very important one.

122. Electrochemical Effect. Electrolytic Conduction.—Conduction currents of the type just pictured have been passed for

scores of years through metallic conductors and from one metal to another (*e.g.*, from copper clamped to iron) without the slightest evidence of physical or chemical change in the conductor. In striking contrast to this metallic conduction is the type of conduction in most chemical compounds, such as molten salts and liquid solutions of the salts, acids, and bases. The passage of current through conductors of this type is almost invariably attended by the separation of the chemical compound into its two constituent atoms or groups of atoms. These constituents do not appear within the body of the compound, but only at the terminals dipping into the fluid; one constituent appears at the terminal or electrode by which the current enters the fluid, and the other at the electrode by which the current leaves. To describe this type of conduction, Faraday coined the following terms from Greek roots.[4] Substances which are decomposed by the electric current are called **electrolytes.** The process of electroseparation of the constituents by the electric current is called **electrolysis,** and the decomposed substance is said to have been **electrolyzed.** The substances are said to conduct **electrolytically.** The places where the current enters and leaves the solutions are called the **electrodes.** The electrode at which the direction of the current is from the metal to the electrolyte is called the **anode,** and that at which the direction is from electrolyte to metal is called the **cathode.** The two constituents into which the molecule of the electrolyte divides are called the **ions**; that which appears at the anode is called the **anion** or the electronegative ion, and that which appears at the cathode is called the **cation** or the electropositive ion (see Fig. 81).

The following are typical examples of electrolytic conduction:

a. If current be passed through a bath of molten sodium chloride ($NaCl$) having a graphite anode and a cathode of iron, metallic sodium appears at the cathode, and chlorine gas is evolved at the anode.

b. Figure 80 shows a glass cell by which the volume of the gases evolved from liquid electrolytes may be measured. If this eudiometer is filled with a solution of hydrochloric acid (HCl) in water and if current is passed through the cell from the platinum

[4] Faraday: *Experimental Researches*, Vol. I, Series VII.

anode A to the platinum cathode C, hydrogen gas is evolved at the cathode C and collects in the tube above by rising to the top and displacing the liquid. Chlorine gas is evolved at the anode A, and after the solution around and above the anode becomes saturated with the gas, chlorine collects in the tube above A.

c. If the cell of Fig. 81 contains a solution of zinc chloride ($ZnCl_2$) instead of HCl, chlorine appears at the anode, and metallic zinc is deposited upon and adheres to the platinum cathode. The amount of zinc deposited may be determined by weighing the cathode before and after the passage of the current.

FIG. 80.—Electrolytic eudiometer cell.

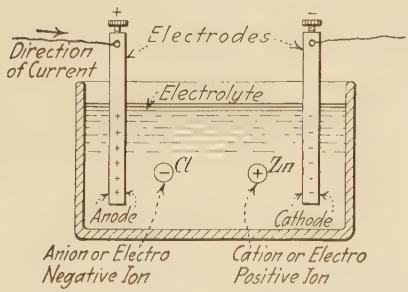


FIG. 81.—Electrolytic cell.

122a. Secondary Changes in Electrolysis.—In many solutions, the ions do not separate out at the electrodes because of secondary chemical reactions which follow the electrolytic separation. In these cases, the ions, upon being set free at the electrodes, enter into chemical combination either with the electrode or with some constituent of the solution. In some cases these secondary reactions make the determination of the actual ions somewhat difficult. The following are examples of electrolysis followed by secondary reactions.

d. If a solution of sulphuric acid in water ($H_2SO_4 + H_2O$) is electrolyzed in the cell of Fig. 80, hydrogen is evolved at the cathode and oxygen at the anode, in the ratio of two volumes of the former to one of the latter. We conceive that in this case

the ions are the hydrogen and the sulphion radical (SO_4) of the acid, and that the (SO_4), instead of being evolved at the anode, enters into combination with the hydrogen atoms of water molecules, thus liberating the oxygen atoms:

$$2H_2O + 2SO_4 = 2H_2SO_4 + O_2.$$

The oxygen given off at the anode is thus the result of a secondary chemical reaction.

e. If the cell of Fig. 81 is provided with an anode of sheet silver, and if a solution of silver nitrate in water is electrolyzed in the cell, metallic silver is deposited on the cathode, and is dissolved from the silver anode at the same rate. In this case, we conceive the ions to be the silver atom and the NO_3 radical of the silver nitrate. The NO_3 ion goes to the anode and there enters into combination with the silver of the anode, forming silver nitrate, which goes into solution.

123. Faraday's Laws of Electrolysis.—Faraday conducted a large number of experiments in which he repeatedly charged given condensers to a given potential difference and then discharged them through electrolytes. By these experiments he discovered that the mass of the ionic substance obtained at the electrodes is always proportional to the quantity of electricity which has passed through the solution, and, within limits, is practically independent of such factors as the size of the cell or of the electrodes, concentration of the solution, rate at which the electricity has passed through, etc. For example, the condensers may be rapidly discharged through the cell through good conducting leads, or slowly through such poorly conducting leads as moistened strings, but the amount of material evolved at the electrodes is the same. Faraday further experimented by connecting several cells containing different electrolytes in series in the discharge path of the condenser, so that the same quantity of electricity passed through each. He then discovered the relative masses of the ions obtained at the electrodes to be proportional to their **chemical equivalents.** These relations, so striking in their simplicity, are embodied in the following laws.

123a. FARADAY'S LAWS OF ELECTROLYSIS (Exp. Det. Rel., 1833).—**The mass** M **of the ionic material set free at either electrode,**

deposited thereon, or dissolved therefrom during the flow of an electric current through an electrolytic conductor is proportional

1. To the quantity of electricity Q which crosses the electrode during the flow.
2. To the ionic weight W of the ion.
3. It is inversely proportional to the valence V of the ion in the compound being electrolyzed.

These laws may be expressed by the formula

$$M \text{ (grams)} = K \frac{W}{V} Q \text{ (coulombs) } (O = 16.) \tag{189}$$

Now in chemical calculations it is found to be convenient to express masses, not in grams, but in **gram-equivalents** of the element or of the ion.

123b. *By a "gram-equivalent" of a specified ion is meant a quantity of the ion equal, in grams, to the number denoting its "chemical equivalent weight," which, in turn, is equal to the ionic weight W divided by the valence of the ion.*

$$1 \text{ gram-equivalent of an ion} = \frac{W}{V} \text{ grams of the ion.} \tag{190}$$

Clearly, then, if in electrochemical calculations we express the masses of the ionic products not in grams but in their **gram-equivalents,** Faraday's laws are expressed by the following formula:

$$M_e \text{ (in gram-equivalents)} = KQ \text{ (gram-equivalents per coulomb, coulombs).} \tag{191}$$

By discharging condensers of known capacity, charged to known potential differences, through electrolytic cells in the manner described in Sec. 137, the value of the proportionality factor K appearing in the above formulas may be determined by absolute methods. By precise measurements, the value of K has been found to be independent of the nature of the compound, nature of the solvent, strength and temperature of the solution, strength of the electrolyzing current, etc. In other words, it has the same value (namely, 1.0359×10^{-5}) for all ions, or it is one of the

general constants of nature.[5] Strange to say, no name has been applied to this general constant. We propose to call it the gram-equivalent of the coulomb.

123c. THE GRAM-EQUIVALENT OF THE COULOMB (DEFINITION). **By the gram-equivalent K of the coulomb is meant the number of gram-equivalents of matter liberated or deposited at either electrode of an electrolytic conductor per coulomb of electricity which passes through the electrolyte. Its value is**

$$K = 1.0359 \times 10^{-5} \text{ gram-equivalents per coulomb } (O = 16). \quad (192)$$

123d. (DEFINITION).—*By the "electrochemical equivalent K_1 of a specified ion" is meant the mass, in grams, of the specified ion which is liberated or deposited per coulomb of electricity which passes through an electrolyte in which the specified ion is one of the carriers.*

The value of the electrochemical equivalent of a specified ion is equal to the gram-equivalent of the coulomb times the ionic weight of the ion divided by its valence.

$$K_1 = \frac{W}{V} K = \frac{W}{V} \frac{1.0359}{10^5} \text{ (grams per coulomb).} \quad (194)$$

124. Nature of Electrolytic Conduction. The Arrhenius Theory of Dissociation.—The simple quantitative relations which Faraday discovered suggested to him the explanation now universally accepted—that the charge is carried through electrolytic conductors by the matter which is deposited or liberated at the electrodes. The first law suggested the idea that each ionic constituent of the decomposed molecules brings or carries the same definite charge to the electrode. Such an assumption clearly accounts for the fact that the amount of material arriving at the electrodes in a given electrolyte is proportional to the quantity of

[5] To deposit one gram-equivalent of any ion, the passage of $1 \div (1.0359 \times 10^{-5}) = 96{,}540$ coulombs of electricity is required. It has been proposed to use this quantity as the unit quantity of electricity in electrochemical calculations, applying to it the name **faraday**.

$$1 \text{ faraday} = 96{,}540 \text{ coulombs.} \quad (193)$$

In terms of these units, Faraday's laws may be written: The mass in gram-equivalents of the ions appearing at either electrode is equal to the number of faradays of electricity which have passed through the electrolyte.

electricity which has passed through the electrolyte. The second law—the fact that the passage of 1 coulomb of electricity through any electrolyte is always attended by the deposition or liberation of the same fractional part of a gram-equivalent of the ions—is fully accounted for by the assumption that every monovalent ion carries the same definite elementary charge, every divalent ion carries two such elementary charges, every trivalent ion carries three such charges, etc. Thus, in the electrolysis of $ZnCl_2$, the current is conceived to consist of zinc cations, each carrying two of the unit positive charges (lacking two electrons) toward the negatively charged cathode, and for every Zn cation two Cl anions, each carrying one of the unit negative charges (one excess electron) toward the positively charged anode. Upon reaching an electrode the ions give up their charges to the electrode and react with one another, or with the solvent, or with the electrode material to form neutral molecules. Electrolytic conduction is thus conceived to be a **convection process,** the electricity being conveyed by streams of positively and negatively charged matter moving in opposite directions.

Since the passage of 1 coulomb through an electrolyte means the deposit or liberation of 1.0359×10^{-5} gram-equivalents of the ions at each electrode, we may compute the charge associated with and conveyed by each monovalent ion, provided we know from other data the number of ions in a gram-equivalent. Now there are 6.062×10^{23} monovalent ions in a gram-equivalent. Therefore, each coulomb of electricity which is transferred to the cathode represents the charge conveyed by

$$(1.0359 \times 10^{-5})\ (6.062 \times 10^{23}) = 6.279 \times 10^{18} \text{ monovalent ions.}$$

Therefore, the charge conveyed by a monovalent ion is 1.591×10^{-19} coulombs. This is the electronic charge. Therefore, each monovalent ion is conceived to have one electron in excess or to lack one electron of the number required to make it electrically neutral.

The forces between the complementary charges upon the anions and cations may be conceived to be the bonds between the ions in the molecules of the compound. Now the feeblest electric intensities in an electrolyte are sufficient to cause current to flow. This suggests that no force is necessary to decompose the mole-

cules of the electrolyte. To account for this and other phenomena, the dissociation theory advanced by Arrhenius is generally accepted. This theory is that the molecules of a molten salt and of the liquid solutions of salts, acids, and bases are always, in greater or less numbers, broken up, or **dissociated,** into their two ionic constituents. The electrolyte is in this partly dissociated or **ionized** condition whether a current is passing or not. Thus a molecule of $AgNO_3$ dissociates, when in water, into a positively charged Ag cation lacking one electron, and a NO_3 anion having one excess electron. A molecule of H_2SO_4 dissociates into two H cations, and an SO_4 anion having two excess electrons. When the electrodes are maintained at different potentials the charges on the electrodes give rise to electric intensities in the electrolyte and the oppositely charged ions are subjected to forces which cause them to move through the solutions in opposite directions toward the two electrodes.

125. The Heating Effect Attending Electrolytic Conduction.—The current through an electrolytic conductor is attended not only by the electrolytic effects at the electrodes, but also by a heating effect within the body of the conductor. This heating effect follows the same law as in metallic conduction; that is, the power so expended in a given conductor is proportional to the square of the current.

We conceive that the development of heat in the electrolyte is to be accounted for in the same general manner as in metallic conduction. In electrolytic conduction, it is the directed velocity acquired under the forces of the field by the ions (rather than by the electrons), which is constantly being converted by collision into the random velocities of increased thermal agitation. This matter will be considered more fully in Chap. VIII.

126. Applications of Electrochemical Effects.—The electrochemical effects of the current are utilized on an enormous scale in such processes as the electrolytic reduction of aluminum, sodium, calcium, and magnesium, the production of caustic soda, bleaching products, and chlorine by the electrolysis of sodium chloride, the production of oxygen and hydrogen by the electrolysis of water, the electrolytic separation and refining of

metals, electroplating, electrotyping, and electrocleaning. The legal unit of current is defined in terms of the rate of deposition of silver as outlined in Sec. 130*b*.

Harmful electrolytic effects are experienced when the return current of direct-current railway systems, instead of flowing from the cars to the generating station entirely by way of the rails and conductors provided for that purpose, leaks from the rails and flows in part along cast-iron water and gas pipes or along the lead sheaths of telephone and power cables. The conduction through the soil from the rails to the buried pipes or lead sheaths is of the electrolytic type. Therefore in those regions in which the iron pipes or the lead sheaths are positive or anodic to the surrounding soil, the electronegative ions, or acid radicals, in the soil are carried to these anodes and attack the iron or the lead.

127. Magnetic Field Effects of the Current.—It has been pointed out in the introductory chapter that the experimental studies of the forces between stationary charges and of the forces between permanent magnets commenced with the work of Gilbert in 1570 and continued for 250 years as quite independent studies. Although several investigators had suspected the existence of a relation between the two sets of forces, and had tried to find it experimentally, no known relation was discovered until 1820. In that year, Oersted, a Danish physicist who was among those seeking the relation, observed that a compass magnet when placed near a wire carrying a current is subject to forces which are clearly due to the current. Previous investigations had been unfruitful because they were attempts to find forces between stationary charges and stationary magnets.

In the previous study of the forces upon magnets, any region in which a **magnet** was acted upon by directive forces had been called a magnetic field of force, or, briefly, a **magnetic field.** At the time of Oersted's discovery, the only known magnetic fields were the earth's magnetic field and the regions surrounding permanent magnets; the only known property of these fields was the directive force upon magnets and upon pieces of iron. Oersted, in effect, discovered that a magnetic field may be produced by an electric current as well as by a magnet. He followed

up his discovery of the force exerted upon a magnet by reasoning that:

> As a body cannot put another in motion without being moved in its turn, when it possesses the requisite mobility, it is easy to foresee that the electric circuit must be moved by the magnet.

This deduction he proceeded to verify experimentally.

Upon learning of Oersted's discovery, Ampere, a French physicist, reasoned:

> When M. Oersted discovered the action which a current exercises on a magnet, one might certainly have suspected the existence of a mutual action between two circuits carrying currents; but this was not a necessary consequence; for a bar of soft iron also acts on a magnetized needle, although there is no mutual action between two bars of soft iron.

To answer this question, Ampere resorted to experiment, and discovered the second magnetic effect of the current, namely, that mechanical forces exist between different portions of a conductor carrying current and between different conductors carrying different currents. He followed this discovery by an analytical study of the forces between elementary portions of the circuit. Within a remarkably short time he formulated a law which makes it possible to compute the force which will be exerted upon any portion of a circuit in the field set up by the current in any circuit of known configuration. He advanced the hypothesis, since confirmed, that all magnetic phenomena are to be accounted for in terms of electric currents, and he attributed the magnetic properties of iron and steel to **concealed** currents perpetually circulating within the molecules of the iron. Following Ampere we will class the phenomena of the magnetic field as **electrokinetic** phenomena.

Eleven years after Oersted's discovery, Faraday discovered (1831) the third important property of the magnetic field, or the third magnetic effect of the current, namely, if a conducting circuit C in a magnetic field is moved relative to the magnet or to the current-carrying coil which is the cause of the field, a current is **induced** in the circuit C. Again, if the circuit C and the current-carrying coil F are fixed with reference to each other,

but if the strength of the current is increased or decreased, a transient current is induced in the circuit C while the current in F is varying.

We have outlined the discoveries of the three magnetic effects of the current, namely,

a. The force exerted upon nearby magnets and upon soft iron.

b. The force exerted upon nearby conductors carrying current.

c. The induction of currents in other circuits by reason of relative motion between the circuits or by reason of variation in the strength of the current causing the magnetic field.

These magnetic effects are the basis of the present-day applications of electricity. The induction of currents in conductors which move in a magnetic field makes possible the electromagnetic generator. Through the agency of these machines electrical energy can be developed in bulk at a cost of less than 1 per cent of the cost from any other source. The induction of currents by variable magnetic fields makes possible the electrical transformer and the high-voltage transmission of power. All motors, telegraph relays, telephone receivers, and most meters for measuring current and potential difference utilize the mechanical force effects. In later chapters, these effects will be studied in a quantitative way, and it will be shown how these three apparently independent effects can all be accounted for in terms of the forces which act upon moving electrons.

Reference may be made to Secs. 223 and 240 for a description of the experiments which show the striking features of the mechanical force effects. The features of the ammeters (amperemeters) in which these mechanical forces are utilized in the measurement of current are described in Secs. 132 to 136.

128. Type Forms of Current Variation.—The type forms according to which the instantaneous values of the electric currents encountered in engineering practice vary in time are shown in Figs. 82 to 86. The instantaneous values of the currents have been plotted as ordinates against the corresponding instants of time as abscissas. A positive value for the current indicates that at that instant the current is in the arrow direction in the wire, and a negative value indicates that the direction of the current is against the arrow.

Thus in Fig. 83 (which is fairly representative of the manner of variation of the 60-cycle alternating current in general use for lighting and power service), the current is zero at the instant t_1. Its value rises to a maximum in the arrow direction in $\frac{1}{240}$ second, and then drops to zero at the end of $\frac{1}{120}$ second. The current then reverses in direction, rises to a maximum in the

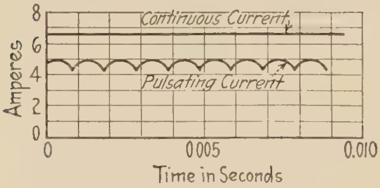


Fig. 82.—Unidirectional currents.

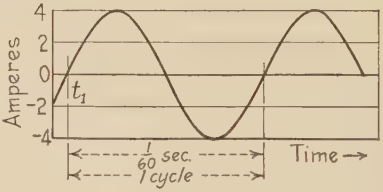


Fig. 83.—Sinusoidal alternating current.

counterarrow direction, and falls to zero again at the end of $\frac{1}{60}$ second. This cycle of values is repeated as long as the current continues, at the rate of 60 cycles per second. The time taken to traverse one cycle of values ($\frac{1}{60}$ second in this case) is called the **period** of the cycle, and the number of cycles per second is called the **frequency** of the current. The curve showing the

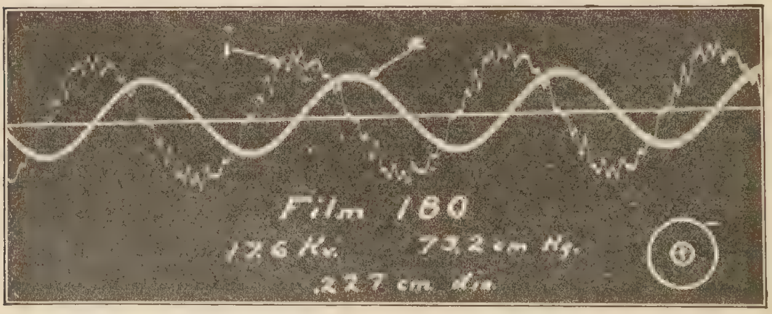


Fig. 84.—Symmetrical alternating current.

sequence of values assumed in one cycle is frequently said to show the **wave form** of the current. "Wave form" is a misleading term to apply to this curve, since the analogy between the rise and fall in the values of a current and real wave motion is very superficial. A better statement is to say that the curve shows the **type form** of the (variation of the) current. These currents are classified in accordance with the type of variation under the following headings:

I. **Direct or unidirectional currents**, currents whose direction of flow does not change:
   1. Pulsating currents (Fig. 82), unidirectional currents which pulsate regularly in magnitude, generally with a period which is a small fractional part of a second.
   2. Continuous currents (Fig. 82), a practically non-pulsating direct current.

II. **Alternating currents**, currents which alternate in direction at regular intervals, generally with a period which is a small fractional part of a second:
   1. With symmetrical half waves. (Unless otherwise specified the term "alternating" signifies symmetrical half waves.)
      a. Of sinusoidal form (Fig. 83 and form *e* in Fig. 84).
      b. Of non-sinusoidal form (Form *i* in Fig. 84).
   2. With non-symmetrical half waves:
      a. Half waves of equal area (Form *i* in Fig. 85).
      b. Half waves of unequal area.

III. **Transient currents.**—The sequence of current values which exist in an electric system after a switching operation and while the current is changing over from the set of **steady state values** corresponding to one set of circuit conditions to the steady state values corresponding to the new conditions is called a **transient current.** Figure 86 illustrates one of the many type forms of transient currents, namely, the current which flows during the discharge of a Leyden jar in an oscillatory manner.

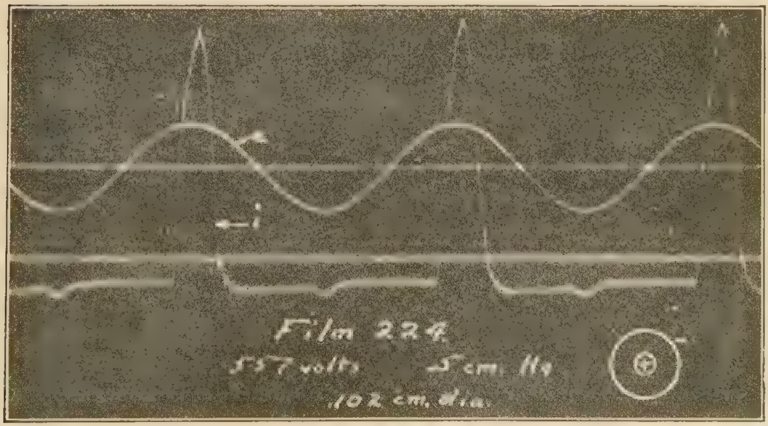


FIG. 85.—Non-symmetrical alternating current.

129. Instruments for Measuring Electric Current.—Each of the effects, heating, electrochemical, and magnetic, may be used to devise instruments for the measurement of electric current. The following classification of meters is based upon the effect utilized to measure the current.

CLASSIFICATION OF ELECTRIC CURRENT METERS

I. Coulombmeters (originally called voltameters)—utilizing the rate of deposition of a metal.

II. Hot-wire meters—utilizing the expansion of a wire or a gas under the heating effect of current in a wire.

III. Magnetic ammeters:
 1. Utilizing forces between one variable and one invariable magnetic system (forces are proportional to the first power of the current);
  a. Movable coil—permanent-magnet type:
   (1) D'Arsonval galvanometer.
   (2) Permanent field magnet ammeters.
   (3) Oscillograph vibrators.
   (4) String galvanometers.
  b. Fixed-coil, movable, magnetic-needle galvanometers.
 2. Utilizing the forces between two variable magnetic systems (forces are proportional to the square of the current).
  a. Electrodynamometer type (having fixed and movable coils):
   (1) Automatically operating (or direct-reading) indicator.
   (2) Manually operated indicator; Siemens dynamometer, Kelvin current balance.
  b. Movable soft-iron vane and fixed-coil instruments:
   (1) Thomson inclined-coil meters.

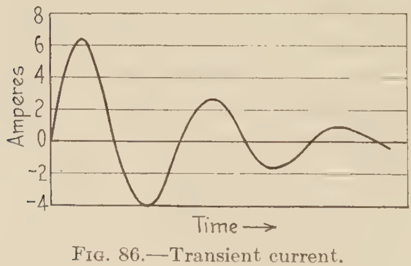


FIG. 86.—Transient current.

130. Coulombmeters, or Voltameters.—The electrochemical equivalent of a metal or of a gas may be determined by the absolute method outlined in Sec. 123 and described in greater detail in Sec. 137. The strength of an unvarying current may then be accurately measured by passing the current through an appropriate electrolytic cell for a measured interval of time *t*, and weighing the metal deposited or measuring the volume of gas

liberated. The strength of the current is given by the equation,

$$I \text{ (amperes)} = \frac{MV}{KWt}, \tag{195}$$

in which M represents the mass (in grams) of the ion deposited.
V represents the valence of the ion.
W represents the ionic weight ($0 = 16$).
K represents the gram-equivalent of the coulomb $= 1.0359 \times 10^{-5}$.

Faraday proposed this method of measuring current, and called an electrolytic cell devised for accurate determinations of weight or volume a **voltameter** (a meter for measuring currents furnished by voltaic cells.) A more descriptive name for the cell is **coulombmeter,** since the mass of the ion liberated is directly proportional to the number of coulombs of electricity which have been passed through the meter. A suitable form for the hydrogen-gas coulombmeter has been illustrated in Fig. 80 (Sec. 122).

130a. Silver Nitrate Coulombmeter.—The measurement of current by the electrodeposition of silver from an aqueous solution of silver nitrate has been exhaustively studied. It has been found to be capable of such accuracy that the civilized governments of the world have defined the legal **standard** which represents the **international ampere** in the following manner:

130b. The international ampere is the unvarying current, which, when passed through a solution of nitrate of silver in water in accordance with standard specifications, deposits silver at the rate of 1.118 milligrams per second.

The silver nitrate coulombmeter prescribed in the standard specifications of the Bureau of Standards consists of a platinum bowl containing the aqueous solution of silver nitrate, in which is suspended an anode in the form of a silver plate. The silver is deposited upon the platinum in the form of an adherent plating, the weight of which is obtained by weighing the dried platinum bowl before and after.

The standard specifications describe in great detail such matters as the features of the electrolytic cell, purification, temperature, and concentration of the solution, etc. To determine the accuracy with which the international unit of current can be

reproduced by silver coulombmeters prepared in accordance with the legal specifications, representatives of different national laboratories have independently made up coulombmeters and have then used them to measure a given current which was passed through all of the coulombmeters in series. The different observers obtained values for the current which did not differ by more than 0.01 per cent.

The most recent determinations indicate that the International ampere is 0.009 per cent smaller than the ampere of the electrostatically derived practical system.

The measurement of a steady current by means of the coulombmeter and a time keeper may be made very accurate, but it is slow and tedious, since the current must be allowed to pass through the cell for several hours. The method gives the average value of the current over the interval and cannot be used at all to obtain the instantaneous values of slowly varying currents. A more rapidly obtained measurement is essential for ordinary purposes. It may be obtained by the instruments described below, in which the heating and mechanical-force effects are utilized.

131. Hot-wire Ammeters.—In these instruments the expansion of a wire under the heating effect of the current in the wire is utilized to measure the current. The features of such an instrument are illustrated in Fig. 87. It consists of a flexible wire ACB stretched between two fixed metal posts A and B. A side pull is exerted on the wire at the point C by a spring S acting through the system of light threads and wires shown. The thread T passes around the pivoted pulley P having attached to it the needle N which, as the pulley turns, moves over the graduated scale.

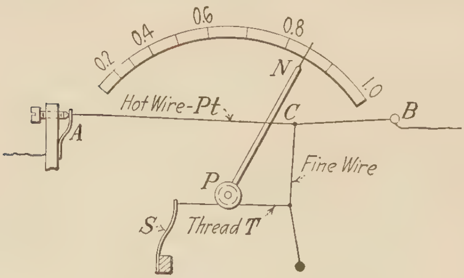


FIG. 87.—Hot-wire ammeter.

The current to be measured is passed through the wire AB. Under the heating effect, the temperature of the wire rises and it elongates, thus permitting the spring to pull the point C farther out of line. As the thread is pulled to the left, the pulley, around which it passes, turns and carries its pointer from the position for zero current to a new position on the scale.

The scale of such an instrument may be graduated to indicate the value of the current in amperes by passing through it unvarying currents whose values are measured by a coulombmeter, and marking the points at which the needle comes to rest (see Sec. 136 for the meaning of the indications of this instrument).

132. Movable-coil, Permanent-field Ammeters.—The feature of this permanent-magnet type of ammeter are illustrated in Figs. 88 and 89. The movable element is a light rectangular-shaped coil of insulated copper wire through which is passed the current (or a definite fractional part of the current) which is to be measured. This coil is provided, above and below, with hardened steel pivots which rest in cup-shaped jewels. It may turn on these pivots in the unvarying magnetic field existing in the air gaps

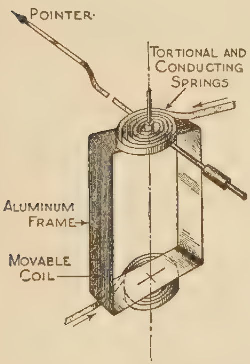


Fig. 88.—Movable coil in permanent magnet meters.

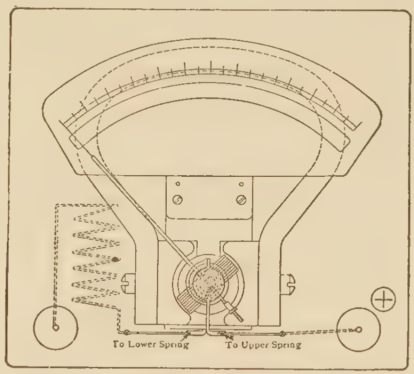


Fig. 89.—A typical direct-current movable coil meter.

between the poles of a carefully aged permanent magnet. The current is conveyed to and from the coil through two light spiral springs which perform the additional function of giving the coil a definite zero position and of exerting a reacting and restoring force whose magnitude is directly proportional to the angular deflection of the coil from the zero position. If the deflection of the coil were not resisted by an element having the characteristic of a spring, any current sufficient to overcome the starting frictional force would cause the coil to deflect to the same end position, the position of stable equilibrium. Fixed to the coil is a long aluminum pointer, or needle, which, as the coil deflects, moves over a scale which has been graduated by empirical means to show the value of the current. Instruments of this type are rugged portable instruments. When properly cared for, their indications may be relied upon to be correct to within 0.2 per cent.

Because of the objectionable heating effect of larger currents on the springs, the movable element is rarely constructed to require more than 0.1 ampere to give full-scale deflection. In instruments for reading larger

currents, a wire of low resistance—a **shunt**—is connected in shunt or parallel with the movable element. The resistance of the shunt path is so proportioned to that of the coil that some definite small decimal part of the current passes through the movable element, say one-tenth, or one-hundredth, or one ten-thousandth. In this case the total current is obtained by multiplying the meter reading by 10 or 100 or 10,000.

In the unipivot microammeters, in which a light coil mounted on a single pivot at its center of gravity is controlled by very weak springs, a scale deflection of 1 millimeter is obtained with a current of only 0.2 microampere. The deflecting torque acting at any instant on a coil in an unvarying magnetic field is directly proportional to the value of the current at that instant, and it reverses in direction if the direction of the current in the coil is reversed. Consequently, if the direction of the current in the coil alternates rapidly, say 120 times per second, the needle trembles slightly but does not deflect, since the alternately directed torques annul each other's effects. Since permanent-magnet instruments cannot be used to measure alternating currents, they are generally called **direct-current** meters.

132a. The d'Arsonval Galvanometer.—The term "galvanometer" is applied to electric meters in which

a. The movable element is suspended by a fine wire, or a fiber of silk or quartz (in the bifilar suspension, two parallel fibers several millimeters apart are used).

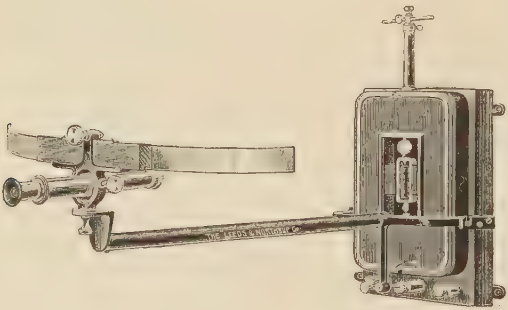


Fig. 90.—A d'Arsonval ballister galvanometer.

b. The reacting force is the torsional force of the fiber suspension (or gravity in the case of the bifilar suspension).

c. The pointer is the beam of light from an incandescent filament, which beam is, by an optical system, caused to fall upon a small mirror on the movable coil and after reflection is brought to a focus on a graduated scale. As the coil and its mirror deflect, the bright image of the wire deflects along the scale.

Other equivalent optical systems are also in use.

The d'Arsonval movable-coil galvanometer illustrated in Fig. 90 differs only in the above respects from the pivoted-coil ammeter just described. Since current must be conveyed to and from the movable coil, it is sus-

pended by a fine phosphor bronze or steel wire which serves as one lead. The other connection is in the form of a very flexible spiral of soft copper ribbon connected to the bottom of the coil, but exerting no appreciable restraint on the coil.

By the use of very fine suspensions and many turns of the fine wire on the coil, galvanometers are made which have a **sensibility** (that is, give a deflection of the spot of light) of 1 millimeter on a scale 1 meter distant from the mirror for a current of 10^{-10} amperes.

132b. The Ballistic Galvanometer.—A galvanometer is used **ballistically** when it is used to measure the quantity of electricity which passes when a transient current is sent through it. The transient current may be over in a second or in a thousandth of a second. Under such conditions there is no steady deflection of the coil, but while the current lasts the coil is subject to an accelerating force (blow) which sets it in motion. Even though the current and the resulting deflecting force have ceased, the coil continues in motion until the increasing torsional force in the suspension brings it to a stop—and then starts it back toward the zero position. The angular deflection at this turning point is read in divisions of the uniformly divided circular scale having its center at the mirror. This reading is called the **throw** of the instrument.

The argument in Sec. 132*c* shows that the **throw** θ will be directly proportional to the quantity of electricity passed through the galvanometer, provided

a. The transient current ceases before the coil has moved appreciably from its zero position.

b. The restraining force which stops the advance of the coil is directly proportional to the angular deflection from the zero position.

$$\int dq = \int i dt = K\theta. \tag{196}$$

A ballistic galvanometer differs from the ordinary galvanometers used for measuring continuous currents only in that the ballistic meter is provided with a circular scale, the period of oscillation of its moving element is made somewhat longer, and the damping of the oscillation by the forces due to friction and induced currents is made as small as possible. The latter feature is not necessary, provided the instrument is properly calibrated over its entire scale by experimental methods.

The longer period of oscillation is essential for two reasons:

1. No precise interpretation can be attached to the readings of the instrument unless the transient current to be measured ceases before the movable element has moved appreciably from its zero position.

2. The slower swing permits of greater accuracy in reading the throw of the instrument. For accuracy in reading, the time of the outward swing should not be less than 5 seconds.

Undamped galvanometers of high sensibility with a quarter period of 8 seconds will give a throw of 1 millimeter on a scale 1 meter from the mirror upon the passage of 3×10^{-10} coulombs.

132c. Ballistic Theory.—If there are no dissipative forces acting on the ballistic galvanometer coil, then the potential energy stored in the twisted suspension at the end of the outward swing must be equal to the energy imparted to the coil by the deflecting forces due to the transient current. By equating the expressions for these two forms of energy, an equation will be obtained expressing the relation between the throw of the coil and the quantity of electricity discharged through it.

The experiments of Chap. X will show that the torque, or turning moment, of the deflecting force is at any instant proportional to the current in the coil

$$\tau = K_1 i.$$

If the current ceases before the coil moves appreciably, the spring suspension exerts no appreciable force on the coil while the current lasts, and the only reacting torque is the inertial force. Therefore during this interval, the expression for the angular acceleration a is

$$a = \frac{\tau}{\text{moment of inertia}} = \frac{K_1 i}{M}.$$

At any instant t_1, after the current starts to flow, the angular velocity is

$$\omega = \int_o^{t_1} a\,dt = \frac{K_1}{M}\int_o^{t_1} i\,dt = \frac{K_1}{M} q,$$

in which q represents the quantity of electricity which has passed through the coil up to the instant t_1.

In the next short interval of time dt the work done by the force on the moving coil is

$$dw = \tau\, d\theta = K_1 i(\omega\, dt) = K_1 i\left(\frac{K_1}{M} q\, dt\right)$$
$$= \frac{K_1^2}{M} q(i dt) = \frac{K_1^2}{M} q\, dq.$$

The total work done on the coil by the deflecting force due to the current is

$$w = \int dw = \int_0^Q \frac{K_1^2}{M}\, q\, dq = \frac{K_1^2}{M}\frac{Q^2}{2},$$

in which Q represents the algebraic value of the quantity of electricity which has passed in a specified direction through the coil. This is the expression for the kinetic energy possessed by the moving mass after the current has ceased and before the coil has deflected appreciably.

When the angular deflection of the coil from the zero position is θ, the restoring torque of the spring suspension is

$$\tau = K_2 \theta.$$

The work done against this force in twisting the suspension through a slight additional angle $d\theta$ is

$$dw = \tau\, d\theta = K_2 \theta\, d\theta.$$

The total work done on the suspension during the angular throw θ is

$$w = \int dw = \int_0^\theta K_2 \theta\, d\theta = \frac{K_2 \theta^2}{2}.$$

Equating the expression for the kinetic energy of the moving coil to the potential energy in the spring when the coil is at rest at the turning point,

$$\frac{K_1^2}{M}\frac{Q^2}{2} = \frac{K_2\theta^2}{2}$$

or

$$Q\left(= \int idt\right) = \frac{\sqrt{K_2M}}{K_1}\theta = K\theta. \tag{196}$$

That is, the net quantity of electricity passed in a given direction through the coil and the throw of the coil are directly proportional, the one to the other.

133. The Oscillograph.—The oscillograph is an instrument for furnishing a photographic record of the type form of any current. Its features are illustrated in Fig. 91. The moving element consists of two fine silver ribbons S, S tightly stretched in the magnetic field in the narrow air gap between the poles N and S of a magnet. A tiny mirror M is cemented across the two strips at their midlength.

The current, or a fractional part of the current, whose type form is desired is passed through the strips, up in one and down in the other, or vice versa. The force of the magnetic field on the currents in the strips pushes one strip forward and the other backward by amounts which are directly proportional to the instantaneous values of the current. Thus when a current flows in the strips the mirror M deflects from its zero position by an angular amount which is directly proportional to the instantaneous value of the current.

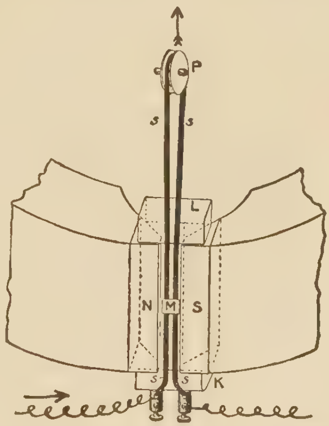


FIG. 91.—Electrical system of the oscillograph.

To obtain a record of the instantaneous values of the angular deflections, a beam of light from the incandescent crater of an arc lamp is by an optical system thrown upon the tiny mirror and after reflection from it is brought to a focus at a horizontal slit in a dark box. As the mirror deflects back and forth with variations in the current, the intense spot of light moves back and forth along the slit. Inside the dark box and immediately back of the slit is the cylindrical surface of a horizontal cylinder around which is wrapped a photographic film. If the cylinder is rotated at uniform speed and if the shutter which normally closes the slit is opened for one revolution, the intense spot of light falls on the moving photographic film and traces a curve. If no current is flowing in the silver strips the spot is stationary, and a straight line, the zero line, is traced. If a rapidly varying current is flowing in the strips, a curve is traced whose ordinates are directly proportional to

the instantaneous values of the current and where abscissas are proportional to the passage of time. Figures 84, 85, and 86 are reproductions of oscillograms obtained this way.

134. Electrodynamometer.—In the movable-coil, permanent-field instruments, the field which acts upon the movable coil is the unvarying field of permanent magnets; in the electrodynamometer type, the field which acts upon the movable coil is the varying field furnished by stationary coils which are excited by the current to be measured, which likewise flows through the movable coil. This is the essential difference which gives to the two types entirely different properties. When a symmetrical alternating current is flowing through the former type, it gives no indication, since the direction of the deflecting force on the movable coil reverses with each reversal in the current. On the other hand, the simultaneous reversal of the current, in both the field and the movable coils of the latter type, leads to a deflecting force on the movable coil which never reverses, and which at any instant is proportional to the square of the value of the current at that instant.

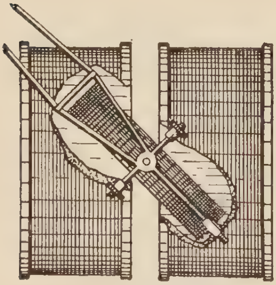


Fig. 92.—Fixed and movable coils of an electro dynamometer.

Since a dynamometer-type meter may be used in measuring alternating currents, it is frequently called an alternating-current meter, although it may also be used to measure direct currents. In fact, it must be calibrated by passing continuous currents of known value through it.

The arrangements of the two sets of coils take many different geometrical forms in the different lines of commercial instruments. With regard to the arrangements for indicating the value of the current, there are two distinct types of instruments; namely, instruments having

(*a*) automatically operating (or direct-reaching) indicators, and

(*b*) manually operated indicators.

In the first type the movable coil with its pivots, springs, needle, and scale may be identical with the same parts in the permanent-field instrument, save that the scale divisions are not of uniform width since the deflecting torque is proportional to the square of the current. Figure 92 illustrates a widely used arrangement between the fixed coils and the stationary coil in its zero position.

134a. The Siemens dynamometer, the features of which are illustrated in Fig. 93, is an example of a meter with a manually operated indicator. The current to be measured is passed through the movable and the fixed coil as shown, being conveyed to the movable coil through mercury cups lying along its vertical axis of rotation. The movable coil is suspended by a fine silk fiber and is controlled by a long helical spring. The upper end of the spring is attached to a torsion head having a pointer which moves over a uniformly graduated circular scale.

When a current is passed through the instrument, the indicator on the movable coil moves from its zero position and strikes one of the stops. The torsion head is then turned by hand until the torque exerted on the coil by the twisted spring just balances the deflecting force and brings the indicator back to the zero. The reading θ of the pointer is proportional to the torque due to the current and, since this varies as the square of the current, the value of the latter may be calculated from the formula

$$I = K\sqrt{\theta}, \tag{197}$$

in which K is a constant of the instrument whose value is readily determined by passing a known current through the instrument and noting the corresponding value of θ.

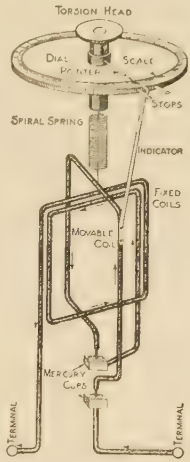


FIG. 93.—Siemens dynamometer.

135. Movable Iron-vane Ammeters.—There are many ways in which vanes of soft iron of various shapes may be so mounted in a fixed coil that the passage of a current through the coil will exert a torque on the vane. For any given position of the vane the torque at any instant is proportional to the square of the value of the current at that instant. This being the case, it is evident that ammeters can be constructed having the same mechanisms as the movable-coil electrodynamometers, save that the iron vane replaces the movable coil. An obvious advantage is that it is not necessary to convey current over the springs to the pivoted vane.

136. Interpretation of Ammeter Readings.—It now becomes necessary to adopt a definition for **the value** of a periodically varying current of the pulsating or alternating type. Obviously, the fundamental definition of value (namely, that it is the equivalent time rate of passage of positive electricity across a surface) defines only the value of a continuous current and the instantaneous values of a varying current. The definition adopted for the value of a periodic current must be such that the "value" has a useful physical application, and that it is readily measurable by some of the meters just described. Let us first determine what aspect of the type form of a periodic current is measured by ammeters of different types.

If a periodic current of moderately high frequency, say 25 cycles or 50 pulsations per second, is passed through ammeters with the heavy movable elements just described, the needle deflects and then remains stationary except for a faint tremor about the new position. That is to say, the angular velocity ω of a movable element which is acted upon by a rapidly varying force is, to all intents and purposes, zero. This enables us to write

$$\omega = 0.$$

But if a body is subject to a variable turning force its angular velocity T_1

seconds after some instant of zero velocity is given by the time-integral of the angular acceleration A.

$$\omega = \int_0^{T_1} A\, dt.$$

The angular acceleration at any instant is, in turn, equal to the sum of the deflecting torque τ_d due to the current at that instant plus the restraining torque of the spring τ_s divided by the moment of inertia M of the movable element

$$A = \frac{\tau_d + \tau_s}{M}.$$

Hence we may write

$$\omega = \int_0^{T_1} A\, dt = \int_0^{T_1} \frac{\tau_s + \tau_d}{M} dt = \frac{1}{M}\left[\int_0^{T_1} \tau_s\, dt + \int_0^{T_1} \tau_d\, dt\right] = 0$$

or

$$\int_0^{T_1} \tau_s dt = -\int_0^{T_1} \tau_d dt.$$

Since the needle is stationary, τ_s is constant and we may write

$$\tau_s \int_0^{T_1} dt = -\int_0^{T_1} \tau_d dt$$

or

$$\frac{\tau_s \int_0^{T_1} dt}{T_1} = \frac{-\int_0^{T_1} \tau_d dt}{T_1}$$

or

$$\tau_s = \frac{-\int_0^{T_1} \tau_d dt}{T_1}. \tag{198}$$

In other words, under the varying torque due to a periodic current, the movable element comes to rest at such a position that the restraining torque exercised by the springs or by the suspension is equal but opposite in direction to the mean value of the deflecting torque averaged over a complete cycle of values. This is the general equation which applies to all ammeters with movable elements so massive that they cannot follow the rapid periodic variations in the deflecting torque.

136a. Let us apply Eq. (198) to the permanent-field ammeters and galvanometers. The study of magnetic fields in Chaps. X and XI will show that in an unvarying field the deflecting torque on the movable coil in any specified position in the field is directly proportional to the instantaneous value of the current in the coil.

$$\tau_d = Ki.$$

Substituting this in Eq. (198),

$$\tau_s = \frac{-K\int_0^{T_1} idt}{T_1} = \frac{-K\int_0^{T_1} dq}{T_1}. \tag{199}$$

Now the first fraction represents the net area under one cycle of the current-time curve (areas above the zero line being + quantities and areas below −) divided by the time of one cycle. This is what is called the mean or average value of the periodic current.

That is to say, **the deflection or reading of a permanent-field, movable-coil instrument is determined by the average value of the current which passes through it**; or, inversely, such an instrument, if properly graduated, will read the average value ($I_{ave.}$) of any periodic current.

$$I_{ave.} = \frac{\int_0^{T_1} i dt}{T_1}. \tag{200}$$

The average value multiplied by the time during which the current flows gives the net quantity of electricity which passes in the circuit in a given direction, and this is a direct measure of the electrolyzing or electroplating value of the current. This is one of the applications of **average value.**

136b. Let us now apply Eq. (198) to the dynamometer and moving-vane types of ammeters. In these instruments, the deflecting torque on the movable element **in any specified position in the field** is directly proportional to the square of the instantaneous value of the current.

$$\tau_d = K_1 i^2.$$

Substituting this in Eq. (198),

$$\tau_s = \frac{-K_1 \int_0^{T_1} i^2 dt}{T_1}. \tag{201}$$

That is to say, the restraining torque, and consequently the deflection, of a dynamometer instrument is determined by the **mean value of the sum of the squares of the instantaneous values taken at short equal intervals over one cycle** (this phrase is abbreviated to **mean-square value**). Stating this inversely, a dynamometer type of instrument measures the **mean-square value** of a periodic current.

Now in calibrating and marking the scales of direct-reading "mean-square" instruments, it is the universal custom to mark the divisions of the scale not with the mean-square value of the calibrating current, but with the **square root of the mean-square value** (this is abbreviated to root-mean-square or r.m.s. value). To make this clear, suppose the location of a few main points for the scale of a dynamometer instrument have been found by passing continuous currents of 1, 2, 2.5, and 3 amperes through the meter and for each position assumed by the needle, ruling a line on the scale card. The mean-square values of the calibrating currents are 1, 4, 6.25, and 9 respectively, but the ruled divisions of the scale are not marked with these values but with the root-mean-square values, namely, 1, 2, 2.5, and 3 amperes. That is to say, direct-reading dynamometer instruments are graduated to give the root-mean-square value of a periodic current, this value being defined by the equation

$$I_{r.m.s.} = \sqrt{\frac{\int_0^{T_1} i^2 dt}{T_1}}. \tag{202}$$

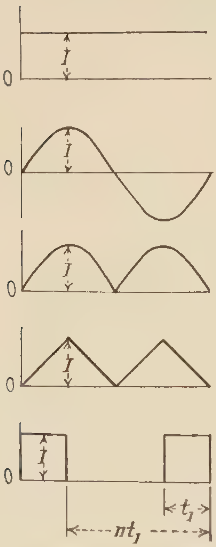
In Sec. 174 it will be shown that the energy converted into heat when a periodic current flows in a given wire for an interval of T seconds is given by the expression

$$W = R\int_0^T i^2 dt,$$

in which R is a constant for the given wire. Now the rise in the temperature of the wire of a hot-wire meter is determined by the average rate P at which energy is expended in it, namely,

$$P = \frac{W}{T_1} = \frac{R\int_0^{T_1} i^2 dt}{T_1}.$$

Therefore, the deflection of the needle of a hot-wire ammeter is likewise determined by the **mean-square** value of the current, but, as in the dynamometer type, the divisions of the scale are always marked with the **root-mean-square** value of the current.



Average Value	R.M.S. Value
I	I
0	$I/\sqrt{2}$
$2I/\pi$	$I/\sqrt{2}$
$I/2$	$I/\sqrt{3}$
I/n	$I/\sqrt{n}$

FIG. 94.—Type forms of current and their values.

From the above it is obvious that one important application of the measured r.m.s. value of a current will be in calculating its heating value. Thus, the heat energy delivered to a heating element when a current of $I_{r.m.s.}$ amperes flows through it for T seconds will be

$$W = RI^2_{r.m.s.}\, T.$$

This is but one of the applications of the r.m.s. value of a periodic current.

Figure 94 shows several type forms and their average and r.m.s. values.

137. An Absolute Method of Calibrating Ammeters.—To calibrate an ammeter is to determine the deflections or the scale indications which currents of known value will cause. By an absolute method of measurement is meant a method which involves nothing but the measurements of length, mass, and time.

Figure 95 illustrates the appliances and the circuit connections for an absolute method of sending a periodic pulsating current of known average value through a meter, and hence an absolute method of calibrating any meter which reads average values. S is a cylindrical rotating commutator with alternate segments of conducting and insulating material. Bearing upon the commutator are three brushes to which are connected a con-

denser D, a generator B, and the meter to be calibrated A. The connections and the setting of the brushes are such that the condenser alternately receives a charge from the generator and then discharges through the meter A. R is a resistance of such value that the condenser becomes fully charged and and fully discharged during the alternate contacts. It may have a great range of values without influencing the results. The type form of the current through the meter is shown in Fig. 95a and its average value is given by the expression

$$I_{ave.} \text{ (amperes)} = CEN \text{ (farads, volts)} \tag{203}$$

in which C is the capacitance of the condenser in farads.

E is the potential difference in volts to which the condenser is charged.

N is the number of discharges per second.

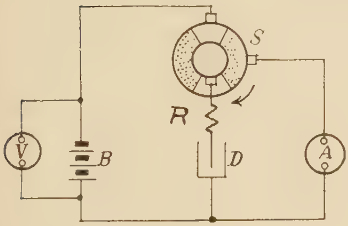


FIG. 95.—Circuit for absolute measurement of current.

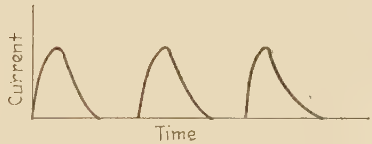


FIG. 95a.—Type form of the calibrating current.

The condenser may be a parallel-plate air condenser whose capacitance has been calculated, or obtained by comparison with concentric spheres. The potential difference to which the condensers are charged may be measured by an absolute guard-ring electrometer V, which is connected across the generator. The generator may be an electrostatic machine, but preferably an electromagnetic generator or a voltaic battery of the types described in the next chapter. The method may be used to calibrate a coulombmeter or any instrument which reads the average value of the current. It cannot be used to calibrate instruments in which the force or the effect is proportional to the square of the current, since the r.m.s. value of the current is unknown. Such meters may be calibrated by first calibrating by the above method a meter which reads average values. Continuous current from a battery may then be passed through the calibrated average-value meter and the uncalibrated r.m.s.-value meter in series. The average value of the current may now be read on the calibrated meter and, since the current is a continuous current, this is also the r.m.s. value.

138. Exercises.

1. A long narrow tank is filled with electrolyte. A cross-section S separates the electrolyte into two regions A and B, each region containing one of the electrodes. Positive ions are carrying charge across the surface

from A to B at the rate of 6 coulombs per second. Negative ions are carrying charge from B to A at the rate of 5 coulombs per second. What is the algebraic value of the current from A to B across the surface? What is the algebraic value of the current from B to A? What is the "direction of the current"?

2. An electrolytic cell is arranged for the electrolysis of water by successively discharging a condenser through it. If the capacitance of the condenser is 0.5 microfarad, the difference of potential to which it is charged is 1200 volts, and the rate at which it is discharged is 10,000 times per minute, what will be the average value of the current through the cell?

3. In exercise 2, if the cell is operated continuously for 2 hours, and if during that interval of time it is found that 0.06 gram of oxygen has been liberated at the anode, calculate the gram-equivalent of the coulomb.

4. How many coulombs of electricity must be passed through a copper refining bath, consisting of two copper electrodes in a solution of copper sulphate, to cause 2 pounds of copper to be taken into solution at one electrode and deposited on the other?

If this deposit is to be made at a uniform rate in 2.5 hours, what value of current must be used?

5. If in exercise 4 the potential drop from one electrode to the other in the direction of current flow is 0.5 volt, how much electrical energy is used in forcing each coulomb of charge through the electrolyte? How much energy is used in depositing the pound of copper? What does this energy cost at 0.5 cent per kilowatt-hour.

6. When a current of 3 amperes is flowing in a conductor, heat is found to be generated at the rate of 8 joules per second. At what rate should heat be generated when the current is 5 amperes?

7. An interrupter is placed in a circuit and operated so that for a period of 0.01 second the current has the value 12 amperes; then for a period of 0.02 second the current is zero. This cycle is repeated indefinitely.

a. What should be the value of current indicated by a permanent-field, movable-coil ammeter? (The movable element is too heavy to follow the fluctuations in the deflecting torque. It deflects and gives a steady reading.)

b. What should be the value of current indicated by a dynamometer type of ammeter?

c. What value would be indicated by a hot-wire meter?

CHAPTER VII

SOURCES OF ELECTROMOTIVE FORCE. ENERGY TRANSFORMATIONS

THEME: The nature of the driving and impeding forces which act upon electrons to cause and to impede their circulation through conducting circuits.

140. Sources of Electric Current.—The preceding chapter has dealt with the effects of the electric current and with the methods of utilizing these effects in the measurement of current. This chapter describes the **sources** from which continuous currents may be obtained, discusses the energy transformations attending the electric current, and presents a qualitative account, in terms of the electron theory, of the nature of the driving and the impeding forces which act upon the electrons to cause and to impede their circulation through conducting circuits.

If two insulated conductors, A and B of Fig. 96, which were initially at different potentials, are connected by a conductor ACB (which may be made up of wires and utilization appliances, such as lamps and electroplating baths), the transient current which flows immediately after the connection is made ceases as soon as it has resulted in a sufficient transfer of electricity between A and B to equalize their potentials. If a continuous current is to be maintained between A and B through the energy utilization path, a potential difference must be maintained between A and B. That is, the bodies A and B must be the terminals, or must be connected to the terminals, of a device G which maintains A at a potential higher than B (notwithstanding the fact that they are connected by a conductor ACB through which electrons are continuously flowing from B to A), by continuously conveying electrons from A to B along some path or channel within itself. Such a device will for the present be

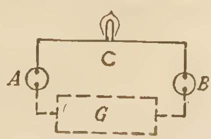


FIG. 96.

called a **source of current.** The important sources of current are:

a. Rubbing-contact machines.
b. Electrostatic induction machines.
c. Voltaic or electrochemical cells.
d. Electromagnetic machines.
e. Thermoelectric couples.

A very brief description of a typical device of each of these five types will be followed by a discussion of the nature of the forces which act upon electrons to cause and to impede their motion. This is preparatory to a more detailed consideration of the properties of these devices.

141. Rubbing-contact and Electrostatic Induction Machines.—The features of machines of these two types were presented in some detail in Chap. IV. In these machines the method of continuously conveying electricity from one terminal to the other is to electrify insulated moving carriers by rubbing contact and by induction respectively. These charged carriers are **driven** by mechanical means up to suitable receptors to which they

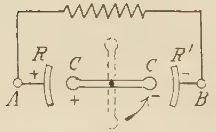


FIG. 97.—Electrostatic alternator.

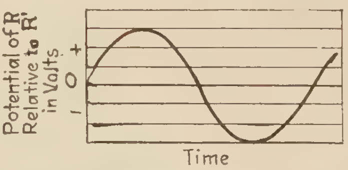


FIG. 98.—Alternating potential difference.

impart their charges, and these receptors are connected one to each of the two **terminals** of the machine.

Figure 97 illustrates the principle of an **alternating-current** machine of the electrostatic induction type. Its essential elements are two oppositely charged carriers C, C and two insulated metallic receptors R, R'. The carriers are mounted upon an insulating member, and are rotated, as indicated by the arrow, at high speed. During the course of each revolution, each charged carrier is carried alternately into the immediate vicinity of each receptor. The carriers do not give up their charges to the receptors, but cause the potentials of the receptors to alternate

between positive and negative values. The manner in which the potential difference between the receptors R and R' varies during one revolution (starting from the dotted position of the carriers) is illustrated in Fig. 98. The potential of R is alternately higher and lower than that of R'. If, now, a conductor is connected from the terminal A to the terminal B, the machine will furnish a small current which will not flow continuously in one direction but will alternate in direction twice during each revolution of the machine. Machines of this type will furnish only very small currents and have not been used. Machines of the rubbing-contact type were the only means available up to the year 1800 for producing continuous currents.

142. Voltaic or Electrochemical Cells.—In 1799, following his discovery and study of the contact electrification of metals (see Sec. 20), Volta invented two devices whose astounding properties excited the widest interest.[1] The first device, known as the **Voltaic pile,** consisted of a series of 60 or more sets of disks

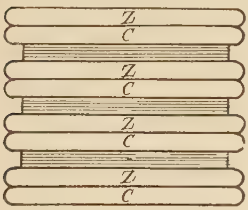


Fig. 99.—Voltaic pile.

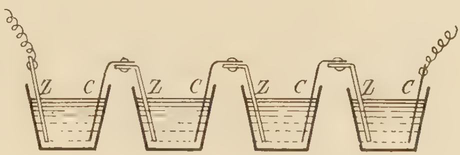


Fig. 100.—Voltaic battery.

—each several centimeters in diameter—of zinc, moistened cloth, and silver (or copper), arranged in the form of a pile in the sequence shown in Fig. 99. Upon simultaneously touching the top and bottom disks of such a pile with the fingers, a perceptible shock was felt each time that the contact was made. This led Volta to prepare an equivalent arrangement which he called a "crown of cups." It consisted of a number of glass cups each containing a strip of silver and a strip of zinc partly immersed in a dilute acid. The cups were connected in series as illustrated

[1] See Volta's letter to Professor Gren, *Observations on Animal Electricity* Phil. Mag., 1799, pp. 68, 163, 306; also Volta's letter to Josiah Banks, *On the Electricity Excited by the Mere Contact of Conducting Substances of Different Kinds*, Phil. Mag., 1800, p. 289.

in Fig. 100, the silver strip of one cup to the zinc of the next and so on. This resulted in the discovery that if the two end terminals of this **battery** of cells, or if the terminals of a single one of these cells, be connected by a wire, a continuous current will flow in the wire. A single one of these cups is called a **voltaic cell,** or a **galvanic cell,** or an **electrochemical cell,** and a group of the connected cells is termed an **electric battery.**

The voltaic cell thus consists of two dissimilar conductors, such as zinc and copper, partly immersed in an electrolyte, such as dilute hydrochloric acid, which will act chemically upon at least one of the conductors. The two conductors are called the **elements** of the cell. The two elements are provided with **terminals** to which connections are made. If the two terminals are connected by copper wires to a voltmeter, the voltmeter will indicate a potential difference of the order of 1 to 2 volts. If the two terminals are connected through a conducting circuit, a continuous current flows from the copper through the external circuit to the zinc. The flow of the current is attended by chemical reactions at the electrodes, as indicated by the evolution of hydrogen at the copper plate and the dissolving of the zinc plate. When a current flows, the potential difference between the terminals is not maintained at its value for open circuit but at some lower value. The greater the current delivered by a given cell to the external circuit the lower will be the potential difference between the terminals while the current is flowing.

143. Electromagnetic Machines.—Electromagnetic machines developed rapidly from the discovery by Faraday in 1831 that an electric current is induced in a coil during its motion in a magnetic field. Figure 101 illustrates the principle of an alternating-current machine of the electromagnetic type. Its essentials are the **field coils** N. S (or permanent magnets) and a loop of insulated copper wire so mounted upon a shaft that it may be continuously rotated in the field of the coils. The two ends of the loop are connected one to each of two brass **rings,** which are mounted upon, but insulated from, the shaft. The connections between the ends of the rotating loop and the external conducting path are made through sliding contacts between these rings and stationary strips of copper or of graphite B (called **brushes**) which bear upon the rings.

If the terminals (brushes) of the machine are connected to a sensitive ammeter and if the shaft with its loop is slowly rotated, the ammeter indicates that the current flows in one direction during one half revolution of the loop, and in the opposite direction during the other half revolution. A sensitive heterostatic electrometer will show that the potential difference between the brushes alternates in the manner illustrated in Fig. 98.

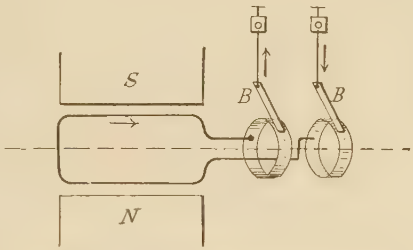


FIG. 101.—Electromagnetic alternator.

If the ends of the loop are connected to the two halves of a single brass ring which is split along a diameter as illustrated in Fig. 102, and if the brushes are properly located, the current in the external circuit always flows in the same direction, and the potential difference between the brushes is always in the same direction, although both pulsate in value in the manner illustrated in Fig. 103. Such a split ring, which serves to reverse the connection of the loop to the brushes at the instant the direction of the induced difference of potential reverses, is called a two-part **commutator.** This construction delivers to the external circuit

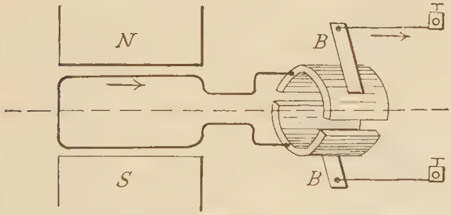


FIG. 102.—Generator with commutator.

FIG. 103.—Rectified electromotive force.

a current which flows always in the same direction but which pulsates in value.

If, now, the single turn loop is replaced by a coil of many turns, or if the field coil is provided with an iron core, or if the rotating armature coil is wound on the surface of (or in slots in the surface of) an iron cylinder, called the armature core,

which rotates in the field, the potential difference generated between the terminals of the armature coil is multiplied many fold (see Fig. 104).

By distributing the turns of the armature coil uniformly around the surface of the iron cylinder, or around an iron ring armature as illustrated in Fig. 105, the large pulsation in the

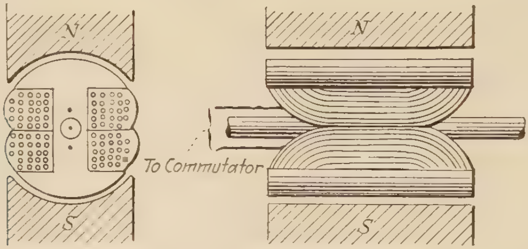


FIG 104.—Shuttle-wound armature.

potential difference is reduced to a small ripple on an otherwise steady potential difference. Each turn still has an alternating potential difference induced between its terminals, but from the manner in which the coils are spaced the potential difference curves are displaced in time with reference to one another, and they sum up to a value which is almost constant in value. It will be observed that the ring armature of Fig. 105 is wound uniformly with a continuous length of insulated wire. This uniform winding is connected at regular intervals to the insulated copper segments of a multisegment commutator C. The whole system, ring, winding, and commutator, is rigidly mounted on the shaft by which it is rotated in the field.

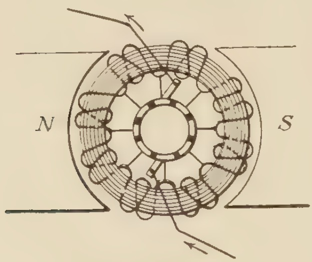


FIG. 105.—Ring armature with distributed winding.

Connection is made between the armature circuit and the external circuit through the fixed conducting **brushes** B, B which make sliding contact with the copper segments of the rotating commutator. Such a machine is called a direct-current (electromagnetic) generator.

The power required to drive a given electromagnetic generator depends upon the magnitude of the current set up in the external circuit. If the circuit is open, so that the current is zero, the power required is largely accounted for by the frictional losses in the bearings and the **windage** losses of the moving armature. The power required to drive the machine when it delivers current to the external circuit is greatly in excess of these open circuit losses. The greater the current the greater is the power required to drive the machine.

Through the agency of these machines, the energy of coal or of the waterfall can be converted into the electrical form at a cost (in large power plants) of 0.3 cent per kilowatt-hour. If the electromagnetic generator were not available, the only commercial source of energy would be the voltaic cell. From such a source the cost of energy is of the order of 30 cents per kilowatt-hour—100 times as great.

144. Thermoelectric Couples.—If a copper and an iron wire are joined to form a circuit, no current flows in this circuit if the two junctions are at the same temperature. If, however, one of the junctions is hotter than the other, a continuous current will flow around the circuit, flowing from the iron to the copper across the cold junction. This effect was discovered in 1821 by Seebeck, and it is called the **thermoelectric,** or the Seebeck effect. Such an arrangement of metals is called a **thermocouple.** Seebeck showed that thermoelectric currents are produced by the unequal heating of the junctions of any two dissimilar metals. The net difference of potential caused by the processes going on at the hot and cold junctions may be measured by cutting either of the wires at some point and connecting the two ends to the terminals of a sensitive quadrant electrometer. If one junction of a copper-iron couple is kept at 0°C., and the temperature of the other junction is raised, the potential difference increases until 275°C. is reached. With a further increase of temperature, the potential difference decreases and becomes zero at 550°C., after which the sign of the potential difference reverses. With other metals the relation between the potential difference and the difference in the temperatures of the junctions is a straight-line relation over a large range of temperature.

The potential difference per degree is of the order of 80 microvolts per degree for a platinum-platinumiridium couple, and 5000 microvolts per degree for an antimony-bismuth couple. This low difference of potential precludes the use of the effect as a source of current in commercial power systems. The main application of the effect is in the measurement of temperature.

145. Forces Which Act upon Electrons.—The previous sections have been confined mainly to a statement of the experimental facts about the sources of electric current. We now propose to discuss the processes occurring in circuits containing sources of current, with the object of rendering an account of the motions of the electrons in terms of force. We propose:

a. To discuss and classify the different forces which appear to be acting upon the electrons in the sources of current, and as they move around the circuit.

b. To present the definitions of the quantities used in expressing these forces.

c. Finally, to present some of the laws which relate the different forces to the motions occurring in the circuit.

145a. Classification of Forces upon Electrons.—The forces which may act upon electrons in different parts of a circuit may be classed as **elemental** forces and **composite** forces.

By an **elemental** or **primitive** electrical force, we mean a force upon electrons which is given by an unresolvable statement of fact, that is, by a statement of fact for which we have no explanation in terms more fundamental or more universal in scope. The elemental electrical forces are:

1. The electrostatic force.
2. The magnetic force.
3. The inertial force.

By a **composite** electrical force, we mean a force upon electrons which is the resultant of the action of more than one elemental force. While the composite forces are conceived to be explainable in a qualitative way in terms of the elemental forces, it may not be feasible to resolve the resultant into its elements.

145b. Electrostatic Force.—Our electrical theory is built upon notions derived from the study of the forces between

charged bodies which are at rest relative to each other. We say that the force observed between the bodies at rest is the force between charges relatively at rest, and we call it the **electrostatic** force. As mentioned in Chap. II, there is experimental evidence which leads us to conceive that the electrostatic force between charges follows Coulomb's inverse square law down to distances which are small in comparison with the radius of the atom. The force of repulsion between electrons, and between positive nuclei, and the force of attraction between electrons and nuclei, are regarded as fundamental or unresolvable facts for which there is no accounting in terms of any experience more elemental or more all inclusive. We regard this force as the means (mechanism) by which a driving force which acts upon a specific group of electrons in one portion of the circuit (for example, the gravitational pulling force on the charged drops of water in the water-dropper generator) is transmitted, or passed on, to cause electrons to move in remote portions of the circuit. The electrostatic force plays a part in the composite forces within the atom, and between atoms. It determines in part the structure of the atom, and the composite electrical forces of chemical affinity, of cohesion, of impact, and the force existing at the surfaces of bodies.

145c. Magnetic Forces.—The experiments described in Sec. 141 and at greater length in Chap. XII show that closed circuits in magnetic fields have currents set up in them when there is relative motion between the circuit and the magnetic field.

The existence of electric currents in the circuits under the conditions named clearly indicates that there is a new set of forces acting on the electrons, which set is related to the magnetic field in ways yet to be discussed. Since magnetic fields themselves are the result of electric currents or of moving charges, we may regard this motional force as a new force acting between one set of charges and another. Thus far we have given very little detail concerning this force, but in later chapters it will be shown that these new forces depend upon the velocities and the accelerations of the various charges and do not act between charges at rest. Therefore this set of forces is entirely distinct from the electrostatic forces which act between charges at rest (as well

as when they are in motion). As in electrostatic force, this force is not accounted for in terms of other known facts. It is taken as an elemental force and it plays an important rôle in the qualitative explanation of the composite structural forces discussed below.

145d. Inertial Forces.—Our qualitative theories of electrons have always pictured the electrons as having a mass which gives rise to inertial forces opposing any accelerations of the electrons. By experiments dealing with streams of electrons through a vacuum, their mass has been determined. It may be that later electromagnetic theory will succeed in explaining these inertial forces in terms of electrostatic and magnetic forces, but at present the inertial force must be classed as an elemental force.

145e. Structural Forces.—By a neutral atom we mean an atom containing just sufficient electrons so that the negative charges are equal to the positive charges contained in the nucleus. The term "neutral" carries the suggestion that there would be no attraction between a neutral atom and any free electron in the vicinity. It is plain that if the atom consisted of a positive nucleus at the center of a sphere, and a negative charge spread uniformly around in a spherical layer, or layers, there would then be no resultant electrostatic force exerted on an electron outside the atom. But the evidence is that **the negative charge is not spread uniformly.** It occurs in lumps—or small particles with comparatively great distances between them. Assume a neutral atom containing four electrons which are spaced evenly around a ring. A free electron approaching this atom along some lines would be repelled from the atom because of its near approach to one of the other electrons; approaching along other lines it would be attracted. Thus it may be taken as a fact that there may be forces of attraction and of repulsion between neutral atoms and nearby electrons. The explanation of this lies in the fact that the negative charge of the atom is not found in a spherical layer of uniform density, but in concentrated particles. At great distances from a neutral atom this segregation is of no effect, the resultant field is practically zero, no matter how the charges are arranged. But at small distances there may be electrostatic

fields of great intensity which depend upon the arrangement of electrons.

The electrons are conceived to be revolving about the nucleus. The forces due to inertia then tend to keep them from being pulled into the center. The motion of the electrons constitutes electric current and thus sets up magnetic fields in which other forces which we have called magnetic forces would also be exerted on any moving electrons. These additional forces make the complete picture less simple than that given above.

It is evident that a neutral atom may also exert forces on a neighboring neutral atom because of the electrostatic and magnetic effect just discussed. The forces of cohesion between similar atoms and of chemical affinity between dissimilar atoms are to be accounted for in terms of these electrostatic and magnetic effects. The chemical properties and other properties of an atom may, then, depend upon the arrangement and motions of the charges within it, or upon the structure of the atom. The composite forces which are exerted upon electrons and atoms by reason of the positions and motions of the parts of the neighboring atoms we will refer to as the **structural** forces. We proceed to discuss several specific examples of structural forces.

145f. Forces of Chemical Affinity between Ions.—The phenomena of electrolytic conduction led to the conception of positively and negatively charged ions; that is, atoms and ions are conceived to be so constituted that the elements known as the electronegative elements have acquired one or more electrons in excess of the number for neutrality, while the electropositive group have lost one or more electrons. It is conceived that the bonds between the atoms in the molecules are, at least in part, the electrostatic forces between the complementary charges of the anions and cations. We conceive that it is partly by reason of these attractions between the ions in the solution and the ions of the electrodes that the voltaic cell acts as a source of current when the electrode and the solution enter into chemical combination.

145g. Surface Forces.—The experiments of electrostatics and of electric conduction show that electrons move readily through the body of a conductor, but do not readily escape through the

surface. For example, if an electric conductor at ordinary temperatures is placed in an electrostatic field in a vacuum, the stream of electrons drawn from the conductor represents an extremely feeble current, even with electric intensities of very high values at the surface of the conductor. The surface forces which prevent the rapidly moving electrons from passing out through the surface of the parent substance into the surrounding empty space (or air) have been frequently referred to in the explanations of electrostatic phenomena.

We conceive that these surface forces are to be accounted for in terms of the forces between electrons and neutral atoms which have been pictured under structural forces. In the body of a conductor, an electron may be simultaneously attracted by several atoms, so that the forces practically neutralize, due to the difference in their directions. This is then a "free" electron. But at the surface an electron which tends to leave is held back by the atoms behind it, since the attractions are all directed inward and cannot neutralize each other. This attraction between electrons and atoms readily accounts for at least a part of the "surface force." It is known that surface forces vary greatly with the condition of the surface, whether clean or oxidized, whether rough or smooth, whether or not a layer of moisture is present, etc. These conditions would all affect the surface force described above.

Now when two conductors of the same material are brought into contact, the atomic attractions described above again balance out, the surface forces disappear, and electrons may easily leave one conductor to pass through the boundary surface into the other. When two conductors of different materials are placed in contact, the atoms on one side of the boundary surface may exert greater attractive forces than the atoms on the other side of the boundary. The joint effect of this difference in the attractive forces on the two sides of the boundary and of the impact forces which drive the electrons across the boundary (see next section) is that electrons pass from one conductor to the other (charging one positively and the other negatively), until the electrostatic forces arising from the two sets of charges balance the difference between the atomic attractions. The excess of electrons of one conductor and the deficit of the other will occur mainly in the form of a double layer on opposite sides

of the surface of separation of the conductors. This difference in the atomic attraction for the electrons may play a large part in the electrification of different substances by contact.

145h. Impact Forces Due to Collisions between Electrons and Atoms.—We conceive that, by reason of collisions between the free electrons in the interior of a substance and the atoms, the free electrons share in the kinetic energy of thermal agitation of the atoms or molecules. By **collision** or **impact** we mean the approach of an electron so close to some part of an atom that the electron is, by the electrostatic or magnetic forces, turned sharply back or to one side. It is through these collisions that the electrons are able to impart energy to, or to receive energy from, the body of the conductor.

A complete account of the forces acting on one electron in the interior of a conductor would require an account of the impact forces and the inertial force at one collision, the impact forces and the inertial force at the next collision, and so on. Such an account would be immensely complicated. But in this discussion, we are not interested in an account of this type. We are interested, rather, in rendering an account of the force which either impedes or causes the flow of a **cloud** or **atmosphere** of electrons, and which is the net or **average resultant force** of the innumerable number of impacts which occur in any short interval of time.

In the interior of a conductor in an electrostatic field, the effect of impacts between atoms and the free electrons is to cause a shooting of electrons here and there in erratic zigzag paths, but there is no tendency to cause a general drift in any particular direction. In the interior of such a conductor, therefore, the average force representing the effects of the impacts is zero. There are, however, at least two cases in which the effect of the impacts between electrons and atoms can be regarded as resulting in a net resultant force upon the atmosphere of electrons, namely,

a. The case in which a stream of electrons (electric current) is passing through a conductor.

b. The case in which a stream of electrons is passing across the boundary surface of a conductor into the surrounding space.

These forces, which are called the **resistance force** of impeding impacts and the **thermionic force,** are discussed in the following sections.

145i. Resistance Force of Impeding Impacts in the Interior of a Conductor Carrying Current.—In the discussion of the nature of metallic conduction, it has been pointed out that, when an electric intensity exists in the body of a conductor, each electron is constantly subject to an accelerating force and has a directed velocity superimposed upon its random velocity. The directed velocity of an electron does not increase indefinitely, however, because in its impacts with the atoms or molecules the electron does work upon the molecules, thereby sharing its increased momentum with the molecules and increasing their molecular velocities of thermal agitation. In greater detail, we picture an inertial force as opposing the accelerating force of the electrostatic field while the electron is accelerating, and as aiding the driving force of the field while the electron is being checked by a collision. But in the final account over a period including hundreds of collisions, all the energy received by an electron as a result of its motion in the direction of the forces of the field is given up to the atoms with which it collides. In an account of the average force acting on the electrons as they stream through the conductor, the inertial forces balance out. Accordingly, we may disregard them, and may picture the driving force of the electrons as being balanced by a steady or average impeding force. The best descriptive name for this force is **impeding impact force,** but in conformity with practice we will call it the **force of electrical resistance.** The force of electrical resistance on electricity is always directed in a direction opposite to the direction of motion of the electricity, and in this is analogous to the frictional force in mechanics and hydraulics.

145j. Surface Impact Force, or Thermionic Force.—A complete account of the forces acting upon an electron at the surface of a conductor would require an account of the opposing of the forces of impact by the inertial force and then an account of the balancing of the inertial force of deceleration against the surface forces discussed above. But in this discussion, we are interested in the average force on the atmosphere of electrons.

In the surface layers of atoms the effect of collisions between electrons and atoms is to cause electrons to shoot in every direction, outward as well as inward. Those electrons shot inward are replaced by others shot back from the inner layers, and thus no net work upon these electrons is chargeable to impact forces. But those shot outward through the surface do not return except under the action of some other force, such as the surface attractive force. Of the electrons which approach the surface, it is mainly the faster moving electrons, possessing a kinetic energy in excess of the work which is done in carrying the electron out against the forces which attract it to the parent substance, which escape. Thus the electrons which escape through the surface are those possessing more than the average kinetic energy, and so we can regard the outwardly directed impact force as doing work on these outwardly moving electrons at the expense of the energy of thermal agitation of the body. The cooling effect at the surface when electrons are emitted, and the heating effect when electrons move inward, have been observed and measured under various conditions.

Thus, in the account of the resultant effects in the surface layers of a body, we may regard the impacts of the atoms and electrons as equivalent in effect to an outwardly directed steady force on the electrons. The work which is done by this force when electrons stream out from the conductor is at the expense of the thermal energy of the conductor. This force may be called the **surface impact force,** or the **thermionic force.** The latter name is suggested by the fact that the emission of electrons from red-hot or incandescent bodies, which is known as thermionic emission, is the most striking example of the working of these surface impact forces.

If an insulated conductor is heated to a red heat, the velocities of thermal agitation and the resulting impact forces become so great that enough electrons escape against the restraining forces of the surface to leave the conductor with a decided positive charge. The passage of electrons from the surface continues at a decreasing rate, and soon the cloud of electrons in the surrounding space and the deficit of electrons on the conductor cause electrostatic forces to act which pull electrons back into the body again. When they come back at the rate at which they

are leaving, a state of equilibrium has been reached. The action of the thermionic vacuum tubes used in radio communication depends upon the above effects.

146. General Method of Applying the Doctrines of the Conservation of Electricity and of Energy to Electric Circuits.—The comprehensive method of rendering a quantitative account of the velocities and deformations in all the parts either of a machine, a hydraulic system, or an electric circuit is as follows:

Let any part of the system, or any combination of parts whatsoever, be singled out. A system of forces is conceived such that the sum of the forces, and the sum of the turning moments of the forces, acting upon this part are each zero for every instant of time. Moreover, this part of the system is conceived to exert upon other parts of the system forces equal and oppositely directed to the forces they exert upon it.

Since the amount of energy transformed from one form to another is defined as equal to the work done by a force, and since work is, in turn, defined as the product of the force multiplied by the distance through which the point of application moves, we see that under the above plan all motion implies the performance of equal quantities of positive and negative work. This signifies the delivery of energy **from** a stock of one type, and the delivery of an equal amount **to** a stock of another type. The method, therefore, embodies the Doctrine of the Conservation of Energy. Moreover, since force has been defined in terms of observed acceleration, we see that the equations expressing the equality of the forces upon and between the parts are, in reality, differential equations in the velocities and the configurations of the parts of the system.

These equations relating to the motions of the parts of the system must be such as to satisfy the other broad generalization of physics, namely, the Doctrine of the Conservation of Matter, or of Electricity, as the case may be.

147. Energy Transformations and Electromotive Force.—The feature common to all sources of current is that the actions going on within the device give rise to a set of forces which cause a separation of positive from negative electricity. Under these

forces the atmosphere of electrons moves **through the device** from the positive terminal to the negative. The direction of motion in the source is **against** the electrostatic forces of the charges which have accumulated during the previous transfer of electrons from one region to the other. The actions going on within the source always involve the expenditure and disappearance (from that region) of energy of the chemical, thermal, or mechanical type.

If a conducting path is provided **outside** the device through which the electrons can return from the negative to the positive terminal, a circulation of electrons takes place. This flow of electrons (current) in the external circuit is in the direction **of** the electrostatic forces. It is invariably attended by the development or reappearance in that region of chemical, thermal, or mechanical energy.

The forces within the device which cause the flow of electricity and the forces in the external circuit which impede the flow have been discussed in a qualitative way in the previous section. The question now arises, how shall we name and define these forces in such a way that the quantities so defined will be readily measurable, and readily usable in electrical calculations?

Let us first note the rôle played by the electrostatic forces of the segregated charges in the general scheme outlined in Sec. 146, the scheme in which the sum of the forces acting on the electricity in each element of the circuit is conceived to be zero. Inside the source of current, the electrons move **against** the electrostatic forces. This means the storage of energy in an electropotential form. In the external circuit the electrons move **in the direction** of the electrostatic forces. This means the electropotential energy is transformed into some other form. The electrostatic forces of the segregated charges are oppositely directed to the driving forces in the source and to the impeding forces of the external circuit. They play a part analogous to the hydrostatic pressures in a hydraulic circuit. We may regard the electrostatic forces as the means (mechanism) by which a driving force which acts upon a specific group of electrons in one portion of the circuit (for example, the gravitational pulling force on the charged drops of water in the water-dropper generator) is transmitted, or

passed on, to cause electrons to move in remote portions of the circuit.

Since all the other forces (the driving forces of chemical affinity, the motional forces of the magnetic field, the impeding forces of resistance impact, etc.) can be balanced against the electrostatic forces, we may measure all these other **non-electrostatic forces** through the electrostatic force. Now, in electrostatics it was found to be far simpler to measure and to deal with the **potential increase** rather than the **electric intensity,** that is, to measure the line-integral of the electrostatic force, rather than its value at a point. Accordingly, in dealing with the non-electrostatic forces in electric circuits, we introduce and define a term which bears the same relation to these forces that **potential increase** bears to the electrostatic forces. The term is **electromotive force.** We define it first qualitatively and then quantitatively.

147a. Electromotive Force (Qualitative Definition).—A *source of current in which energy of the chemical, thermal, or mechanical form is converted into the electrical form is said to be the seat of an intrinsic electromotive force.*

In a qualitative way, the term intrinsic electromotive force signifies an electricity-moving (electromotive) force.

a. Which causes electrons to move through the device from the positive terminal to the negative.

b. Which will maintain a circulation of electrons if an external conducting path is provided, through which the electrons can return from the negative terminal to the positive.

c. Which is an intrinsic or inherent property of the device when it is operating in a specified way.

Regions in which energy of the electrical form is converted into the thermal form are also said to be the seats of (non-intrinsic) electromotive forces; that is, they are seats of electricity-impeding forces, which forces are not developed in the region unless it is connected to a source of intrinsic electromotive force.

147b. ELECTROMOTIVE FORCE OF SPECIFIED NON-ELECTROSTATIC FORCES (Quantitative Definition).—**When a specified kind or type of non-electrostatic force acts upon electricity as it flows along a portion *AB* of a circuit, then the algebraic value (in that portion of the circuit) of the ELECTROMOTIVE FORCE IN THE DIRECTION *AB* of the specified**

type of force is defined to be equal to the work which is done by the specified forces per equivalent unit of positive electricity which flows from A to B. (The flow of electrons in the direction BA is the equivalent of the flow of positive electricity in the direction AB.)

The value of the electromotive force (abbreviation e.m.f.) in a portion AB of a circuit is generally represented by the symbol E. Using this notation, the above definition is expressed by the equation

$$E \text{ (volts)} = \frac{W \text{ (joules)}}{Q \text{ (coulombs)}} \tag{204}$$

in which W represents the work done by the specified non-electrostatic forces during the passage through the region AB of the equivalent of Q coulombs of positive electricity in the direction AB.

147c. Unit of Electromotive Force (DEFINITION).—*The value of the electromotive force of a specified action in a portion AB of a circuit is said to be unity, or "one volt" if the work done by the specified forces is one joule per coulomb of electricity which passes from A to B.*

148. Electromotive Intensity and Its Relation to Electromotive Force.—Let q represent the quantity of free electricity per linear centimeter of length at any point P in the portion AB of a circuit (Fig. 106) and let F represent the force which a specified action exerts on a positive charge at P, per coulomb of electricity in the charge.

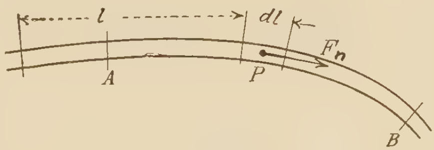


FIG. 106.

Then the quantity of free electricity in the length dl is $q(dl)$, and the force acting on this electricity in the direction of motion is $Fq\ dl \cos (F, l)$. As the quantity Q moves through the circuit, the electricity in dl must move the distance Q/q centimeter. Therefore, the work done by the non-electrostatic force on the electricity in the elementary length dl is

$$dW = Fq\ dl\frac{Q}{q} \cos (F,l) = FQ\ dl \cos (F,l)$$

and the total work done as the quantity Q moves through the portion AB of the circuit is

$$W = \int_{l_A}^{l_B} FQ\, dl \cos (F,l).$$

Therefore, the electromotive force in the length AB is

$$E = \frac{W}{Q} = \int_{l_A}^{l_B} F\, dl \cos (F, l). \tag{205}$$

Let us call F the **electromotive intensity** at the point, defining the term as follows.

148a. ELECTROMOTIVE INTENSITY OF SPECIFIED NON-ELECTROSTATIC FORCES (DEFINITION).—**The ELECTROMOTIVE INTENSITY of a specified non-electrostatic force at a point P in a circuit is defined to be a vector quantity whose direction is that of the specified force on a positive charge at P and whose magnitude is equal to the force exerted on the charge per coulomb.**

$$\mathbf{F} = \frac{\mathbf{f}}{q} \tag{206}$$

We see from the definition, and particularly from Eq. (205), that the use of the term **force** in **electromotive force** is unfortunate, since E is not a force in the technical sense. A force is equal to the work done per centimeter of movement along a path, while E is equal to the work done per coulomb of electricity which moves from A to B. In other words, electromotive force is the line-integral along a path AB of the non-electrostatic force upon a unit quantity of electricity as it moves over the path, or it is the line-integral of the electromotive intensity along the path AB.

149. Electromotive Force. Its Relation to Potential Increase and Its Measurement.—Electromotive force and potential increase have been defined in the same terms, in terms of work done on a unit charge moving in a specified direction along a path. The distinction between them is that e.m.f. is the work done **by** a non-electrostatic force, while potential increase is the work done **against** the electrostatic force. In a given portion of a circuit, there may be two or more types of non-electrostatic action, but only the one type of electrostatic action.

When there is only one non-electrostatic force acting along a path, it must be exactly balanced by the electrostatic force on each element of the path; the work done by the non-electrostatic force must equal the work done against the electrostatic force, and, consequently, **the e.m.f. from A to B is equal to the potential increase from A to B.** Accordingly, when there is but a single e.m.f. along a path AB, its value may be measured by the electrostatic voltmeters devised for the measurement of potential increase (see Chap. IV).

The e.m.f. from A to B is not measured by instruments lying in the part of the circuit which is the seat of the e.m.f., but by voltmeters placed outside this region but connected between A and B. In other words, the measure of the e.m.f. from A to B of a specified region is the work which the electrostatic force will do on unit positive charge as it moves from B to A along an external path.

As a charge moves through a given region, it may be subject to the action of more than one non-electrostatic force. For example, the electrons in passing through the armature of an electromagnetic generator are subject to the driving action of the forces resulting from motion in the magnetic field and to the impeding action of the resistance impacts. Again, it is to be noted that in many cases the processes involve **turbulent** motions of the electricity. For example, in the case of the water-dropper generator, each drop discharges to its receptor in an oscillatory manner with a frequency of oscillation in the billions of cycles per second. This means the dissipation, by heating and radiation, of some of the energy previously converted into an electropotential form by the gravitational forces during the **smooth** descent of the water drop to the receptor.

In the above cases, we say that there are as many electromotive forces in a given region, each having its own descriptive name, as there are distinct types of non-electrostatic force acting therein. The reading of a voltmeter connected to the terminals of a region AB in which turbulent motions are occurring, or in which two or more types of non-electrostatic force are acting, is taken to be the **net** value of the electromotive force of the region, or to be the algebraic sum of all the separate electromotive forces in the direction AB.

The method of resolving the net electromotive force into its component electromotive forces will be to study the manner in which the magnitude of each type of electromotive force varies with the variations in the current and in the proportions of the part of the circuit. From these observations, laws will be derived which express the relation between the value of each type of electromotive force and the factors which determine its value.

From the foregoing discussion it follows that the electromotive forces in a region are to be evaluated, and the laws relating these electromotive forces to the dimensions (in the generalized sense) of the region, are to be determined to conform to the following general principle:

149a. ELECTROMOTIVE FORCE PRINCIPLE (Definition).—**In any given path, the algebraic sum of the electromotive forces in a specified direction along the path is equal to the potential increase from one terminal to the other in the specified direction.**

If the path over which the electromotive forces are to be summed up terminates at the starting point, that is, if it forms a closed path, or a circuit, the potential increase from the beginning terminal to the end terminal is, of course, zero. For such a closed path, the following relation is always satisfied.

149b. LAW OF ELECTROMOTIVE FORCES FOR CIRCUITS (Definition and Deduction).—**At every instant of time, the sum of the algebraic values of the electromotive forces in a specified direction around any closed circuit is zero.**

$$e_1 + e_2 + e_3 = 0. \tag{207a}$$

$$\Sigma e \text{ (around a closed circuit)} = 0. \tag{207}$$

This relation or law is commonly known as Kirchhoff's Law of Electromotive Forces.

150. Expressions for Work and Power.—From the manner in which electromotive force has been defined, it follows that if the equivalent of q coulombs of electricity pass through a region in the direction AB and if the value of a specified e.m.f. (or set of e.m.fs.) in this direction is e volts, the work done by the specified force is equal to the product of the electromotive force times the quantity of electricity.

$$w \text{ (joules)} = eq \quad \text{(volts, coulombs)}. \tag{204a}$$

Suppose that the quantity q passes through in a very short interval of time t. The equality is not destroyed by dividing both members of this equation by t. Therefore,

$$\frac{w}{t}\frac{\text{(joules)}}{\text{(seconds)}} = e\frac{q}{t} \quad \text{volts,} \frac{\text{coulombs}}{\text{seconds}}.$$

But the left member of this equation is the time rate at which work is done by the forces, or it is the power P; and, in the right member, q/t is the instantaneous value of the current i in the direction AB in the region. Therefore,

$$P(\text{watts}) = ei \quad \text{(volts, amperes).} \tag{208}$$

If the value of the algebraic product in the right members of Eqs. (204a) and (208) is positive, it signifies that electricity is moving in the direction in which the force in question tends to move it. This is the case when the force in question represents the transformation of the non-electrical forms of energy (chemical, thermal, or mechanical) into an electrical form, or when it represents a decrease in the electropotential or electrokinetic energy associated (stored) with the region, or, finally, when the region in question is regarded as the "portal" through which energy is being transferred from another system to the circuit of which it is a part.

On the other hand, a negative value for the right-hand member signifies the transformation of energy from an electrical form into the chemical, thermal, or mechanical form, or an increase in the electropotential or electrokinetic energy associated with the region, or the transfer of energy from the circuit of which the region is a part to another circuit.

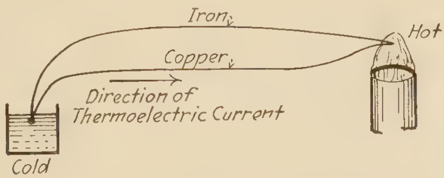


Fig. 107.—Thermoelectric circuit.

151. Electromotive Force of a Thermocouple. The Seebeck E.M.F.—A thermocouple is a circuit composed of wires or strips of two different metals or alloys, the two junctions between the metals being maintained at different temperatures (see Fig. 107).

The junctions between the metals may be made by twisting, clamping, welding, or brazing. By cutting one of the wires and connecting a sensitive galvanometer in such a circuit, Seebeck discovered in 1821 that a current is maintained in the circuit as long as the junctions are maintained at different temperatures. The conclusion is that such a device is a heat engine, or a thermoelectric generator, in which thermal processes result directly in driving forces on the electrons. The **resultant** electromotive force of **all** the driving actions occurring in the thermocouple may be measured by cutting either of the conductors at some convenient point and connecting them to the terminals of a sensitive quadrant electrometer. In practice, the e.m.f. of a couple is rarely measured with a quadrant electrometer, but generally by connecting the couple as in Fig. 108 in series with a sensitive galvanometer and an opposing e.m.f., the value of which is varied until the absence of galvanometer deflection shows that the two e.m.fs. just balance. The value of this balancing e.m.f. is then computed from the known resistances in the balancing circuit and from a reading of the current in that circuit.

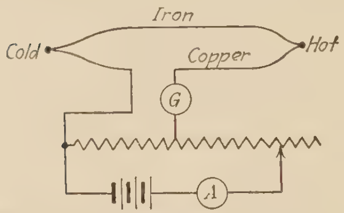


FIG. 108.—Measurement of the e.m.f. of a thermocouple.

151a. THERMAL OR SEEBECK E.M.F.—The thermoelectric e.m.f. of a circuit containing no source of e.m.f. save thermal effects is equal and opposite to the e.m.f. which must be inserted in the circuit to reduce the current to zero.

We will present, first, the experimentally derived laws concerning the magnitude of the Seebeck electromotive force, and, second, the Peltier and Thomson effects. These effects are the thermal aspects of the processes of which the Seebeck effect is the electrical aspect.

151b. Law of Volta.—In a compound circuit, made up of any number of wires of different metals connected in series, all points of which are at the same temperature, there is no current, and therefore the resultant e.m.f. is zero.

151c. Law of Magnus.—In a homogeneous metallic circuit, no matter how the temperature may vary from point to point, there is no current, and therefore the resultant e.m.f. is zero.

151d. Law.—The magnitude of the thermal e.m.f. of a couple,

a. depends upon the metals constituting the couple,

b. for a given pair of metals, depends only upon the temperatures of the two junctions,

c. is independent of the distribution of temperature along the wires between the junctions,[2]

d. is independent of the length and section of the wires,

e. is independent of the manner of making the junctions whether by twisting, welding, soldering, or brazing, provided only that no electrolytic action occurs at the junction.

The relation between the e.m.f. of a thermocouple of given metals and the temperatures of the junctions can be best obtained by maintaining one junction at any convenient fixed temperature, which we will call the **reference temperature,** say, at the melting point of ice, and then measuring the electromotive force corresponding to different temperatures of the other junction. The relation between the e.m.f. of a copper-iron circuit and the temperature of the variable junction is plotted in Fig. 109. In plotting the curve we have taken copper as the **reference** metal, and have plotted the electromotive forces with positive values when their direction around the circuit is along the reference metal from the junction at the reference temperature to the junction at the variable temperature. The curve has been labeled "iron." A curve plotted in like manner for a copper-platinum couple would be labeled "platinum."

An examination of Fig. 109 shows that as the temperature of the variable junction is raised from zero to 284°C. the e.m.f. increases from zero to a maximum of 2.2 millivolts. As the temperature is increased from 284 to 550°C., the e.m.f. decreases in value and becomes zero at 550°. Over this entire range, the direction of the electromotive force around the circuit is from iron to copper across the reference junction. If the temperature

[2] This is not true if the temperatures along the wire are such as to cause a change of allotrophic form or in the state of the metal.

of the variable junction is higher than 550°, the direction of the electromotive force reverses. This temperature at which the direction of the e.m.f. reverses is called the **temperature of inversion with reference to the temperature of the reference junction** T_r.

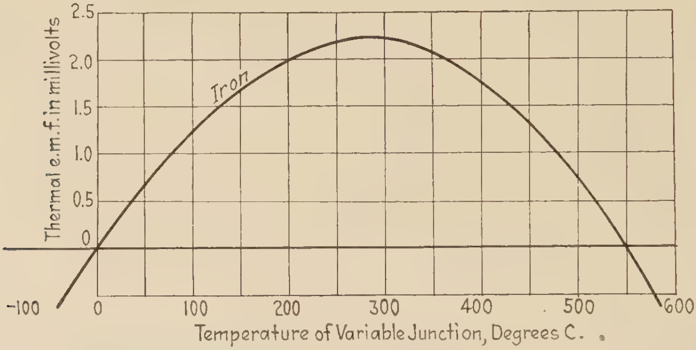


FIG. 109.—Thermal e.m.f.-temperature characteristic of an iron-copper couple. (Positive values indicate that the direction of the e.m.f. around the circuit is along the copper from the junction at the reference temperature to the junction at the variable temperature.)

151e. Law of Successive Temperatures.—The e.m.f. of a given couple between the temperatures T_0 and T_2 is the algebraic sum of the e.m.f. of the couple between the temperatures T_0 and any other temperature T_1 and of its e.m.f. between the temperatures T_1 and T_2.

$$E_{0,2} = E_{0,1} + E_{1,2}. \tag{209}$$

The useful application of this law is that from the curve plotted for a given couple for a single reference temperature, the electromotive forces of the couple can be read off for any two temperatures whatsoever. Thus, from Fig. 109, the following e.m.fs are obtained for copper-iron couples.

$$E_{0,200} = E_{0,100} + E_{100,200}.$$
$$2.0 = 1.25 + E_{100,200}.$$
$$E_{100,200} = +0.75.$$

$$E_{0,500} = E_{0,200} + E_{200,500}.$$
$$0.7 = 2.0 + E_{200,500}.$$
$$E_{200,500} = -1.3.$$

The positive sign in the result indicates that the direction of the e.m.f. around the circuit is along the reference metal from the

junction nearest to the reference temperature to the junction farthest removed from the reference temperature.

151f. Law of Intermediate Metals.—A thermoelectric circuit may be cut at any point and a wire of any other metal inserted without altering the e.m.f. of the circuit, provided the two junctions of the inserted metal are kept at the same temperature.

This law has the following important applications:

a. It is this fact which makes the e.m.f. of a couple between two metals independent of the solder used, or of the alloy formed, in soldering, brazing, or fusing the wires together at the junctions, provided each junction is small enough so that the unalloyed sections of the wires entering it are at the same temperature.

b. The law is applied in using copper wires to connect a platinum-platinumiridium thermocouple to a copper-wire galvanometer. The junctions of the copper lead wires with the platinum are kept at the same temperature by placing them side by side in a suitable bath.

c. By the following argument, the law enables us to predict the thermal e.m.f. of a thermocouple between the metals A and B which have never been tested together, **provided** we have the e.m.f.-temperature curves of each of these metals against some third metal S. This third metal will be called the **standard** of reference.

Argument.—The e.m.f. of the couple AB shown at a in Fig. 110 is not altered by inserting a length of S into it at the hot junction T_1, as at b. Neither is it altered by cooling a portion of S to the temperature of the cold junction, as at c. But clearly in the last case the thermocouples ${}_0A_1S_0$ and ${}_0S_1B_0$ are in series. Therefore, using these symbols also to represent the values of the thermoelectric e.m.fs., we may write

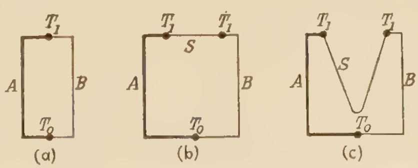


FIG. 110.—Equivalent thermoelectric circuits.

$$ {}_0A_1B_0 = {}_0A_1S_0 + {}_0S_1B_0. \tag{210} $$

The important application of this is that it is not necessary to tabulate or plot the e.m.f.-temperature characteristics of all possible combinations of metals, since the e.m.f. of any couple

can be obtained by taking the algebraic sum of the e.m.fs. given by its component metals with a single standard metal. Some authorities have used copper and others have used lead as the

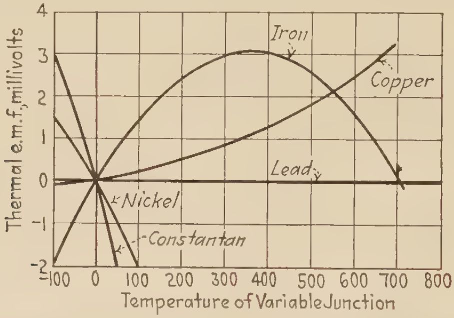


FIG. 111.—Thermal e.m.f.-temperature characteristics against lead. (The direction of the e.m.f. around a circuit of any two metals is from the junction at the reference temperature to the junction at the variable temperature along that metal for which the voltage is algebraically the lower of the two readings.)

standard. Figure 111 contains the e.m.f.-temperature characteristics of a number of metals and alloys with lead as the stand-

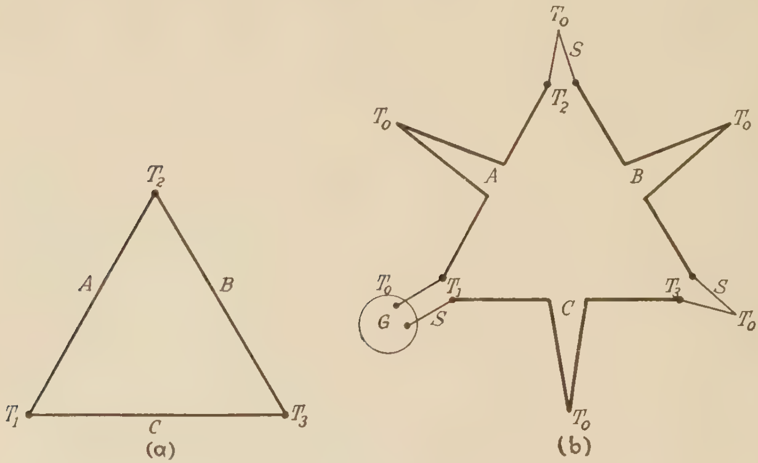


FIG. 112.—Equivalent thermoelectric circuits.

ard metal and 0°C. as the reference temperature. From this figure it is seen that the e.m.f. of an ${}_0\text{iron}_{200}\ \text{copper}_0$ couple is $-2.4 + 0.4 = -2.0$ millivolts.

d. This law also enables us to write the expression for the thermal e.m.f. of a series circuit composed of any number of metals with their junctions at any temperatures. Thus, Fig. 112*a* represents a circuit of three metals A, B, and C with their junctions at the temperatures T_1, T_2, and T_3.

Imagine the standard metal S to be inserted at each junction and to be extended to a bath at the reference temperature T_0, as illustrated at b; also imagine intermediate points on each of the metals A, B, and C to be carried to this temperature. The thermal e.m.f. of the two circuits shown in Fig. 112 will be equal. Therefore, we may write

$$ {}_1A_2B_3C_1 = {}_0S_1A_0 + {}_0A_2S_0 + {}_0S_2B_0 + {}_0B_3S_0 + {}_0S_3C_0 + {}_0C_1S_0. \tag{211} $$

151g. THERMOELECTRIC POWER OF A METAL (DEFINITION).—**If a circuit is formed of two metals P and S with the junctions at the temperatures T and $T + dT$, and dE is the value of the thermoelectric e.m.f. of the circuit (in the direction PS at the cold junction), then this e.m.f. divided by the temperature difference (the e.m.f. per degree of temperature difference) is called the THERMOELECTRIC POWER φ of P with respect to S at the temperature T.**

$$ \varphi \text{ (volts per degree of temp. diff.)} = \frac{dE}{dT} \text{ (defining } \varphi\text{)}. \tag{212} $$

The metal P is said to be thermoelectrically positive with respect to the metal S when the direction of the thermoelectric current is from P to S across the cold junction.

The thermoelectric power of P with respect to S is not a constant, but for a great many pairs of metals its value is found to be a straight-line function of T, which may be written in the form

$$ \varphi \left(= \frac{dE}{dT} \right) = A + BT \tag{213} $$

in which T is the mean temperature of the couple expressed in degrees Centigrade.

The temperature at which the thermoelectric power is zero is called the **neutral temperature** of P with respect to S. For those cases in which the thermoelectric power is expressed by an equation of the form of Eq. (213), the expression for the neutral temperature T_n is

$$ T_n = -\frac{A}{B}. \tag{214} $$

Having defined thermoelectric power in the above manner, it follows from the law of intermediate temperature that the value of the thermoelectric e.m.f. of the couple PS with the junctions at the temperatures T_1 and T_2 will be given by the expression

$$E = \int_{T_1}^{T_2} \varphi\, dT. \tag{215}$$

$$E = A(T_2 - T_1) + \frac{B}{2}(T_2^2 - T_1^2)$$

$$= (T_2 - T_1)\left[A + B\left(\frac{T_2 + T_1}{2}\right)\right]. \tag{216}$$

The following table contains the values of A, B, and $-A/B$ for a number of metals with respect to lead as the standard of reference. From the law of intermediate metals, the thermoelectric power of any metal with reference to any other metal in this table is the difference between their powers with respect to lead. Thus,

if $\qquad \varphi_{PS} = A_1 + B_1T$

and $\qquad \varphi_{NS} = A_2 + B_2T$

then $\qquad \varphi_{PN} = (A_1 - A_2) + (B_1 - B_2)T. \qquad (217)$

151h. Thermoelectric Powers of Metals with Respect to Lead.

$$\frac{dE}{dT} = A + BT.$$

Metal	A in microvolts per degree Centigrade	B in microvolts per degree per degree
Selenium	+800	
Tellurium	+500	
Antimony	+6 to 42	
Iron	+17.1	−0.048
Platinum 90, Iridium 10	+5.9	−0.0133
Copper	+1.34	+0.0094
Silver	+2.12	+0.00147
Lead	0	0
Platinum	−3	−0.02
Mercury	−4.2	
Cobalt		
Nickel	−21.8	−0.0051
Constantan	−37	
Bismuth	−45 to −97	

152. The Peltier Effect.—As stated in Chap. VI under the heading, Peltier Effect, the flow of a continuous current across the junction between two different metals heats the junction if the current is in one direction and cools it if the direction of the current is reversed. In the first case heat energy is evolved and in the second case it is absorbed. This **reversible** effect is in addition to and quite distinct from the ordinary irreversible heating effect associated with the electrical resistance of the metals and with the so-called **contact resistance** due to imperfect contact at the function. The following law is derived from experiment.

152a. (LAW).—**The rate of reversible liberation or absorption of heat at the junction of two different metals caused by a given current depends upon the metals, and upon the temperature of the junction, and is independent of the cross-section of the junction. For given metals at a given temperature, the rate is directly proportional to the current.**

Since the rate of energy transformation occurring at the junction is directly proportional to the current, we may think of a **Peltier e.m.f.** at the junction, defining it in the following manner.

152b. PELTIER E.M.F. (DEFINITION).—**When an electric current is flowing in a specified direction across the junction between two dissimilar metals, the heat energy in joules (reversibly) ABSORBED at the junction per coulomb of electricity crossing the junction will be taken as the value of the Peltier e.m.f. in the specified direction at the junction.**

The value of the Peltier e.m.f. between many metals is of the same order as the Seebeck e.m.f. in a thermocouple composed of the same metals, and is of the order of 1 per cent or less of the value of the contact e.m.f. between the two metals, which is to be discussed in Sec. 156. For closed metallic circuits at a constant temperature throughout, the Peltier e.m.fs. balance out; but with differences of temperature between the junctions this is not the case—the algebraic sum of the Peltier e.m.fs. is not zero.

When both junctions are on the same side of the neutral temperature the absorption and the liberation of heat at the junctions are such as to account qualitatively for the Seebeck e.m.f. of the couple. Thus, if a current be set up in a copper-iron circuit by means of a battery it is found to lead to an evolution of heat at the junction in which the current is from iron to copper and an

absorption at the junction in which the flow is from copper to iron, as illustrated in Fig. 114. Now the direction of the Seebeck current in a copper-iron couple (near room temperature) is from iron to copper across the cold junction, as illustrated in Fig. 107. Thus the Seebeck current must be accompanied by an absorption of heat at the source of heat and a liberation at the refrigerator, as in the ordinary heat engine.

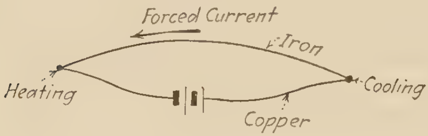


FIG. 114.—Peltier heating and cooling.

If the temperatures of the two junctions of the copper-iron couple be on opposite sides of the neutral temperature (284°C.), however, the Peltier effects of the Seebeck current are as listed in the following table.

152e. Peltier Effects of the Seebeck Current.

Case	Temperature of the junctions		Effects of the Seebeck current	
	Cold	Hot	At the cold junction	At the hot junction
1	Both below T_n		Heating	Cooling
2	Both above T_n		Heating	Cooling
	Below T_n	Above T_n		
3	$T_n - T_1 > T_2 - T_n$		Heating	Heating
	$T_n - T_1 < T_2 - T_n$		Cooling	Cooling

It is thus evident that the Seebeck electromotive force in a circuit cannot be accounted for entirely in terms of the Peltier heating and cooling at the junctions, but that other thermal effects reversible with the current must take place in the circuit. The other effect is the Thomson or Kelvin effect.

153. The Thomson Effect.—After determining from his thermodynamic studies that the Peltier effect is inadequate to account completely for the Seebeck e.m.f., William Thomson (afterwards Lord Kelvin) sought another reversible thermal effect of the current.

In 1845, he demonstrated such an effect with the arrangement shown in Fig. 115. An iron bar was passed through two vessels of boiling water, A and B, and a vessel of cold water M between A and B. The temperatures of the iron bar at a and b were taken with the current from the battery C flowing first in the direction AB and then in the direction BA. The temperature at b was found to be higher than at a, with the current in the direction AB than with the current in the direction BA; while the temperature at b was less when the direction of the current was from B to A. This reversible thermal effect occurring in a homogeneous con-

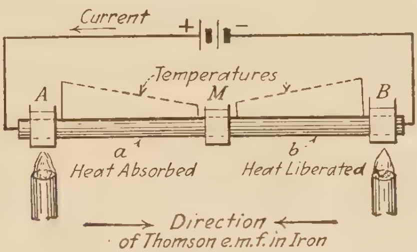


FIG. 115.—Thomson effect.

ductor in which there is a temperature gradient is called the "Thomson effect." An experimental study yields the following law.

153a. (LAW).—**The rate of reversible liberation or absorption of heat in a conductor which is caused by a given current depends upon the metal and its physical condition and upon the temperatures of its ends, and is independent of the temperature distribution and the dimensions of the conductor. For a given conductor with given end temperatures, the rate is directly proportional to the current.**

The Thomson effect proves the existence in the conductor of a set of impact or thermal forces fixed in direction with respect to the temperature gradient. In accord with our general custom, we may define the work done per unit charge by these forces as an e.m.f., and we may call this e.m.f. the Thomson e.m.f.

153b. THOMSON E.M.F. (DEFINITION).—**When an electric current is flowing in a specified direction along a homogeneous conductor whose ends are at different temperatures, the heat energy in joules (reversibly) ABSORBED in the conductor per coulomb of electricity passing through it,**

will be taken as the value of the Thomson e.m.f. in the specified direction in the conductor.

For copper conductors and most metals the Thomson e.m.f. is positive in the direction from cold regions to hot, that is, the impact force acting on electrons is directed from hot regions to cold. But in iron, nickel, bismuth, and a few other metals the e.m.f. is directed from hot to cold, and therefore the force on electrons is directed from the cold regions to the hot (see Fig. 115).

No satisfactory qualitative explanation of this Thomson effect is yet available. On account of the greater violence of agitation of the atoms in the region of higher temperature, one might expect electrons to be forced from the hot to the cold region; that is, one might expect an impact force on electrons to be directed from the hot to the cold region. The actual impact force in copper and in most of the metals is in this direction, but in iron and a few others the force is actually directed from cold to hot regions. It seems evident that some other effect must also be present in these metals, and this other effect may be a decrease in density of free electrons as the temperature increases. If this occurs it would produce a force on electrons directed from cold to hot regions. It may well be that both effects are present in all metals, that in some metals one effect predominates and in others the other effect predominates. But at present this qualitative explanation must be regarded as incomplete.

154. Applications of Thermocouples.—Thermopiles have been constructed for converting heat energy directly into the electrical form. The thermopile consists of many couples in series. It is made up of alternate bars of the two metals assembled in zigzag fashion (frequently in the form of a pile) in such a manner that the odd-numbered junctions can be heated and the even-numbered cooled (see Fig. 116). The Chalmond thermopile, an early device of this kind, consisted of 120 couples made up of a zinc-antimony alloy and a zinc-copper-nickel alloy. It generated an e.m.f. of 8 volts and had an internal resistance of 3.2 ohms. Its maximum power output was thus 5 watts at a current of 1.25 amperes. The heating effect of the gas used in obtaining this output is reputed to have been 1000 watts, or the output of 5 watts was obtained at an efficiency of conversion of only 0.5 per cent.

Since the conduction of heat from the hot junction to the cold is the principal source of waste, and since poor conductors of heat are also poor conductors of electricity (see Sec. 185), it may be that low efficiency of conversion is inherent in the thermopile. Because of its low efficiency, the thermopile finds substantially no application as a source of electric power.

The important engineering applications of the thermocouple are in the measurement of temperatures, of the rates of thermal radiation of distant bodies, and of feeble high-frequency currents.

The thermoelectric thermometer (or pyrometer) is particularly convenient for measuring very high or very low temperatures and for measuring temperatures in places which are inaccessible to ordinary thermometers, as within the windings of machines. This thermometer consists of a single thermocouple of suitable metals, with a bath for keeping one junction at a known temperature (generally at the melting point of ice) and suitable instruments for measuring the e.m.f. of the couple. The e.m.f. may be measured directly with a voltmeter, or preferably by a potentiometer arrangement in which the e.m.f. of the couple is balanced against a known e.m.f., as in Fig. 108. For the measurement of temperatures up to 350°C., the thermocouple is generally of copper and iron, or of copper and the alloy constantan, in the form of wires a millimeter or less in diameter. For furnace temperatures up to 1100°C., nickel-chromium alloys are used, and for temperatures up to 1600°, platinum and an alloy of platinum with 10 per cent of rhodium is used.

For the measurement of the rate of thermal radiation falling upon a surface, or for the measurement of the rate of radiation from a distant body, a thermopile constructed of very fine wires of bismuth and tellurium may be used, the odd-numbered junctions being exposed and the even-numbered junctions screened from the radiation. The terminals of the pile are connected to a sensitive galvanometer.

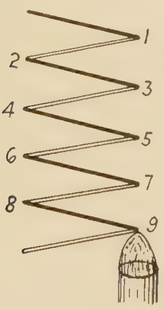


FIG. 116.—Thermopile.

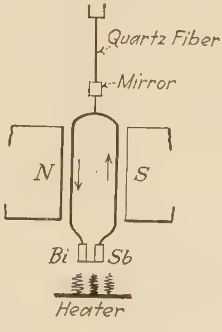


FIG. 117.—Boys' radio-micrometer.

An extremely sensitive thermoelectric arrangement for measuring the rate of incident radiation is **Boys' radio-micrometer,** illustrated in Fig. 117. It consists of a rectangular loop of silver wire suspended by a quartz fiber between the poles of a permanent magnet. The loop ends in two small strips of antimony and bismuth which are in contact at their bottom edges. When this edge is heated by radiation directed upon it, a thermoelectric current flows in the suspended loop, and the resulting deflection of the loop is read, as in the d'Arsonval galvanometer, by the deflection of the beam of light reflected from the mirror.

Duddell has converted the radio-micrometer into a **thermogalvanometer** for the measurement of feeble high-frequency currents. The current to be measured is passed through a small resistance coil (heating element) which is mounted directly beneath the antimony-bismuth junction of the suspended coil. The deflection of the coil under the heat caused by the current in the heating element is a measure of the value of the current. Instruments of this type have been constructed to give a deflection of 1 millimeter on a scale 1 meter distant with a current of 2×10^{-7} amperes.

A less sensitive form of instrument for the measurement of small high-frequency currents is the thermomilliammeter illustrated in Fig. 118. The junction of a fine-wire bismuth-tellurium thermocouple is soldered to a fine constantan wire AB and the whole arrangement is mounted in a highly exhausted bulb. The current to be measured is passed through the heating wire AB, thus heating the junction C of the thermocouple and setting up a thermoelectric current in the sensitive microammeter A. The combination may be readily calibrated by passing known continuous currents through the wire AB and noting the deflection of the meter. Such a combination will give a deflection of 1 millimeter with a current of 1 milliampere.

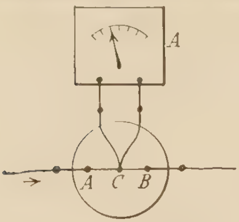


Fig. 118.—Thermo-milliammeter.

155. Conflicting Theories of Voltaic Action.—We now propose to consider the electromotive forces expressing the energy transformations which occur in electrochemical (voltaic and storage) cells. This requires an account of the **electromotive force of chemical action** and also of the **contact electromotive force** which is involved in the electrification of two bodies by contact.

For a century following Volta's description of the voltaic pile and battery, the part played in the action of these devices by the two phenomena of contact electrification and chemical action was a controversial question.[3] Volta, by experiments of the kind described in Sec. 20, had previously demonstrated that two

[3] For an excellent account and bibliography of the controversy, see the report by Oliver Lodge entitled *On the Seat of the Electromotive Forces in the Voltaic Cell*, Reports of the B.A.A.S., 1884, p. 464; reprinted in the Phil. Mag., 1885, Vol. XIX, pp. 152, 254, 340; Vol. XX, p. 372. Also Lodge: *On the Controversy Concerning Volta's Contact Force*, Phil. Mag., 1900, Vol. XLIX, p. 351. See also Langmuir, Irving: *The Relation between Contact Potentials and Electrochemical Action*, Trans. Am. Electrochem. Soc., 1916, Vol. XXIX, p. 166.

See *Contact Electricity of Metals*, Kelvin's Math. and Physical Papers, Vol. V; also Phil. Mag., July, 1898.

plates of different metal when placed in contact acquire opposite charges. Volta seemed inclined to regard the "mutual contact of two different substances as the immediate cause which puts the electric fluid in motion." He demonstrated that, in a closed circuit composed entirely of solid conductors (which he called conductors of the first class) at the same temperature, no current is set up by the contacts, and he considered that the contact electromotive forces balanced out around such a circuit. He considered that when liquids (his conductors of the second class) joined the two metallic conductors, as in the voltaic pile, they allowed the driving force at the metallic junction to become effective. This is the **contact theory** of the electromotive force of the cell.

It was early observed that the delivery of current by the voltaic cell was always accompanied by a chemical action at the surface of the immersed plates, and Fabroni in 1799 and Ritter in 1800 drew the conclusion that the chemical action should be looked upon as the primary cause of the current. This is the **chemical theory** of the electromotive force of the cell.

In their early stages these theories were both qualitative, since they were advanced some 45 years before the principle of the conservation of energy was clearly formulated. The early exponents of the contact theory maintained that the chemical changes which occur in the voltaic cell when current flows are incidental phenomena of secondary importance in accounting for the action of the cell. On the other hand, the exponents of the chemical theory began after some years to question the validity of the notion of contact forces, attributing the electrifications demonstrated by the Volta experiments (of Sec. 20) to oxidations or "tendencies" to oxidation either at the junction of the metals or at the surfaces of the metals.

Two things contributed to discredit the notion of contact electromotive forces between metals:

1. After the acceptance of the principle of the conservation of energy, it was shown that the electrical energy delivered by the cell is derived from the chemical energy of the reactions at the electrodes.

2. Peltier, in 1834, found that, when current passes through the junction between two metals, heat is liberated or absorbed, depending upon the direction in which the current flows. In studying the Peltier effect in 1866, Le Roux advanced the following theorem: "If, in a circuit, an absorption

or evolution of heat occurs which is proportional to the current and changes sign with the direction of the current, then these effects correspond to and are proportional to electromotive forces of the same or opposite sign, located at the places at which the absorption or evolution of heat takes place." Le Roux calculated the electromotive force at the junction by dividing the rate (in watts) at which heat energy is absorbed or liberated by the current causing the effect He obtained values of the order of a few thousandths of a volt—1 or 2 per cent of the values claimed as the contact electromotive force—and this he took to be the true contact electromotive force between metals. Edlund in 1870 and Maxwell in 1873 independently drew the same conclusion. On the other hand, both Helmholtz and Kelvin from thermodynamic considerations maintained that the electromotive force E_p so obtained is not the contact electromotive force E_c, but is related to it by the equation

$$E_p = T\frac{dE_c}{dT}, \tag{219}$$

in which T represents the temperature of the metals.

From the study made, during the past 20 years, of the emission of electrons from hot metals into space evacuated to the highest degree attainable, very convincing evidence has been obtained that the contact effect—the passage of electricity from one metal to a second when they are placed in metallic contact—is not to be attributed to chemical action in the customary meaning of the term. The evidence does not, however, sustain the contention of the contact adherents that the electromotive force of the voltaic cell is to be accounted for in terms of the contact effect. The source of the electrical energy obtained from the voltaic cell is the chemical energy which disappears in the reactions occurring at the electrodes, and the electromotive force of the cell is to be traced to these reactions.

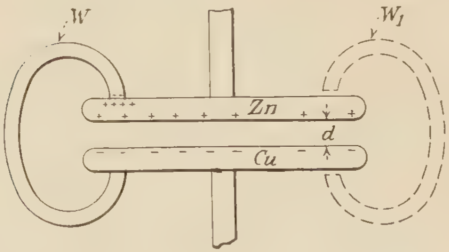


FIG. 119.—Volta's contact electrification.

The following sections contain a somewhat qualitative discussion of contact and chemical electromotive forces.

156. Contact Electromotive Force between Metals.—Volta's experiments with plates of zinc and copper, as outlined in Sec. 20, demonstrate that if these plates are mounted on insulating

stems in air (or in evacuated space), as illustrated in Fig. 119, and are connected (either momentarily or permanently) by a **metallic** wire W of any material, they become equally and oppositely charged, the zinc positively and the copper negatively. During the contact, forces act which cause a transfer of electrons from zinc to copper through the bridging wire. The electromotive force of these forces is called the **contact e.m.f.**, and may be defined as follows:

156a. CONTACT E.M.F. (DEFINITION).—**When a charge moves across a surface of separation between two metallic conductors, the net work done by the surface forces of the two metals per unit charge is called the CONTACT E.M.F. in the direction of the equivalent motion of positive charge.**

Upon forming a complete circuit by bridging from the zinc to copper with another wire (W_1 of Fig. 119) of any material, Volta found no evidence of any steady current in the circuit. This finding is generally stated in the following form.

156b. Volta's Law of Successive Contacts.—In a series circuit made up of any number of wires of different metals in metallic contact (that is, without electrolytic contacts), all points of which are at the same temperature, no steady current is set up.

From this, the conclusion may be drawn that the electromotive forces tending to cause a transfer of electrons from zinc to copper via any two wires are the same in value regardless of the metals used to bridge between the zinc and copper.

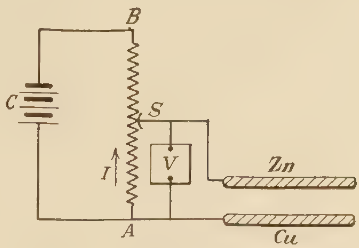


FIG. 120.—Circuit for measuring contact e.m.f.

Figure 120 shows an arrangement for measuring the value of the electromotive force tending to cause electrons to transfer from one metal to another. By means of a sliding contact S, a greater or less length of the wire AB may be included in the conductor bridging between the plates. Current from the battery C is sent through the wire AB in such a direction that the e.m.f. of resistance in the length AS is opposite in direction to the contact e.m.f. The e.m.f. of resistance required to balance the contact e.m.f. is determined by moving the slider along

until such a point is reached that the zinc and copper plates, after being disconnected and tested for charge, are found to be uncharged. The balancing e.m.f. is then read by a voltmeter V connected between A and S.

From measurements of the contact e.m.f. between different metals made with the above apparatus or its equivalent, the following laws have been determined.

a. The contact e.m.f. between any two metal plates is affected by the nature of the finish of their surfaces and by the condition of the finish. For example, the contact e.m.f. from a polished zinc plate to a polished copper plate is 0.75 volt, but, by burnishing the surface of the zinc plate with a steel tool, the e.m.f. may be increased to 1 volt. In all the statements which follow, it is presupposed that surface conditions of any given metal under test are uniform over the entire surface.

b. The value of the contact e.m.f. between any two metals is independent of the spacing and shape of the samples, and of the composition of the metallic wires used in making contact between the test samples.

c. The contact e.m.f. as measured between any two metals is found to be equal to the difference between the contact e.m.fs. of these metals as measured with any third metal. This fact makes it possible to obtain the contact e.m.f. between any two metals whatsoever from a brief table containing the measured contact e.m.fs. between the metals and some single reference metal. Thus, from the table printed below, the contact e.m.f. from zinc to tin is $0.75 - 0.45 = +0.3$ volt, and from zinc to platinum $0.75 - (-0.24) = +0.99$ volt.

156c. Contact Electromotive Forces between the Metals and Copper.—(A positive value denotes that the metal becomes positively charged and the copper negatively. Any metal is said to be electropositive to those below it in the table.)

Metal	Contact E.M.F. in Volts
Zinc (burnished)	1
Zinc	0.75
Lead	0.54
Tin	0.45
Iron	0.15
Copper	Reference metal
Platinum	−0.24

The contact e.m.fs. given in the above table are for samples of metals of commercial purity having freshly polished surfaces. Other samples and other methods of polishing would yield values differing somewhat from these. The phenomena which account for the Volta contact effect are discussed in the next section.

157. Electron Affinities of the Metals.—The experimental studies[4] of the emission of electrons from hot metals into space evacuated to the highest degree attainable demonstrate that

a. The emission of electrons against the surface forces pictured in Sec. 145*g* is an intrinsic property of all bodies, and that the emission occurs independently of any chemical action.

b. The measured emission increases with the temperature of the body at a rate which is in fair agreement with predictions based upon thermodynamic theory and the kinetic theory of gases.

c. The steady emission of electrons from a metal is accompanied by a steady loss of the heat energy of the emitting body. This loss is in addition to the loss by radiation which occurs when the metal is not emitting electrons.

The loss (or cooling effect of emission) is accounted for by the fact that only the more rapidly moving free electrons approaching the surface are able to escape through the surface. The work done by the electrons in escaping **against** the surface force is thus derived from the kinetic energy of thermal vibration of the more rapidly moving free electrons in the metal. We thus conceive of the **surface impact** or **thermionic** force as acting on electrons in the outward direction, and of the surface attractive force, which we may call the force of **electron affinity** of the metal, as acting in the inward direction.

From the measurements made under *b* and *c* the work φ done against the surface force per coulomb of escaping electrons has been determined for a number of pure metals. Since this quantity (expressed in joules per coulomb) is of the dimensions of

[4] This study will be outlined in Sec. 191. The article by Langmuir referred to in footnote (3) is an excellent resumé containing references to the original studies.

See also, Richardson, O. W.: *The Emission of Electricity from Hot Bodies*, 1916.

electromotive force, it may be called the **electromotive force of electron affinity** of the metal.

157a. E.M.F. OF ELECTRON AFFINITY (DEFINITION).—**The work done by the surface attractive forces on a coulomb of electrons as they cross the boundary surface of a conductor in a specified direction is called the E.M.F. OF ELECTRON AFFINITY φ in the opposite direction (the direction of equivalent motion of positive charge).**

$$\varphi \text{ (volts)} = \frac{W}{Q} \frac{\text{(joules)}}{\text{(coulombs)}}. \tag{220}$$

This e.m.f. is positive in the outward direction for all metals and has values which depend upon the composition and condition of the surface of the conductor. The absolute value of the e.m.f. is very often called the **electron affinity** of the metal.

The following table contains the values of the electron affinities of several pure metals with clean surfaces, that is, with surfaces free from gases, oxides, and other impurities. Such surfaces are obtained only in the very best vacua. The table was compiled by Langmuir[4] from measurements made by himself and others. These values have been found to be independent of the temperature of the metal within the limits of experimental error.

157b. Electron Affinities of the Metals.

	Volts		Volts
Tungsten	4.52	Iron	3.7
Platinum	4.4	Zinc	3.4
Tantalum	4.3	Thorium	3.4
Molybdenum	4.3	Aluminum	3.0
Carbon	4.1	Magnesium	2.7
Silver	4.1	Titanium	2.4
Copper	4.0	Lithium	2.35
Bismuth	3.7	Calcium	2.24
Tin	3.8	Sodium	1.82
		Potassium	1.53

Let us now consider plates of zinc and copper to be mounted on insulating stems in air with their plane faces parallel but separated by a short distance d, as in Fig. 119 (but not connected by W). At room temperature the rate of emission of electrons from these plates is so low that no appreciable transfer of charge would occur across the space from one metal to another in a week's time.

Let a metallic connection be made between the two plates by means of the copper wire W, and let us suppose that the density of the free electron atmosphere is the same in both metals. Since the electron affinity of the copper is 4 volts and that of the zinc is only 3.4 volts, we would expect that, upon

making contact between the copper wire and the zinc, electrons would cross the junction at a greater rate from the zinc to the copper than from the copper to the zinc.

The excess negative charge thus acquired by the copper and the electron deficit (or positive charge) of the zinc should distribute in such a manner as to keep each metal an equipotential region. That is, the charges should be found mainly on the closely adjacent surfaces of the plates. The electric forces from the double layer of charge at the junction would oppose the flow of electrons from the zinc to the copper, and the difference in the rates of transfer should grow less and finally become zero when the work done (upon the electrons which cross the junction) by the forces of the double layer is just equal and opposite to the net work done by the surface attractive forces.

The experimental observations and measurements of the Volta contact effect confirm these expectations. The copper does become negatively charged and the zinc positively. Moreover, the contact e.m.f. between any two metals as listed in Table 156*c* is in rough agreement with the difference between the electrons affinities of the two metals as given in Table 157*b*.[5]

It remains to reconcile the conception of a contact e.m.f. of the order of a volt at the junction of two metals with the fact that the Peltier heating or cooling which occurs when a steady current from a battery flows across the junction corresponds to an electromotive force which is only a few hundredths of a volt. We see, however, that after contact has been made and equilibrium has been attained there are two sets of forces acting upon electrons which cross the junction—the contact e.m.f. forces which are the net result of the two sets of surface attractive forces, and the oppositely directed forces due to the double layer of charge. Upon the assumption that the densities of the electron atmospheres in the two metals are equal, the work done by these two sets of forces on crossing electrons should be equal and opposite in sense, and there should be no Peltier heating at the junction. The small Peltier effect which is observed is probably to be traced to the difference in the volume densities of the electron atmospheres in the two contacting metals.

While Volta's law indicates that the contact electromotive forces cannot of themselves give rise to steady currents, yet it should be pointed out that the phenomena occurring at the junction of two metals may play an essential rôle in the processes going on in cells which are sources of steady currents. For example, suppose that positive and negative ions are continuously produced by any agency in the air or other medium separating the copper and

[5] It is not at all certain that the e.m.f. from one metal to another with which it is in contact is to be accounted for entirely or precisely in terms of an electromotive force arrived at from the emission of electrons into evacuated space. When zinc and copper are in contact, the fields of the contact layers of the copper and zinc atoms overlap, and, as suggested by Hall, it is possible that the interchange of electrons between the colliding atoms may play a large rôle in the transfer of electrons from one metal to another.

zinc plates in Fig. 119. By virtue of the charges on the zinc and copper plates, a field exists in the region between the plates, and the + ions will be impelled to move through the medium to the copper plate, and the − ions to the zinc plate. If the ions can come in contact with and discharge to the plates, they will continuously neutralize some of the charge on the plates, and this charge will, in turn, be steadily replaced by the forces acting at the junction of the copper and the zinc. Thus there will be a steady flow of electrons from the zinc to the copper, and, by placing resistances or motors in the wire connecting the zinc with the copper, this current can be caused to produce heating effects or to do mechanical work. The source of this energy is not at the junction, but it is the agency which produces the ions between the plates. For example, the ions may be produced from the air by passing a beam of X-rays or a beam of sunlight through the air between the plates, or they may be electrons ejected from the positive plate under the action of ultra-violet light. Or the ions may be the ions existing in an electrolytic solution into which the zinc and copper plates of Fig. 119 dip. The energy is derived from the beam of X-rays, or from sunlight, or from ultra-violet light or from the chemical reaction by which the electrolytic ions discharge to the plates.

158. Simple Voltaic Cell.—If a strip of commercial sheet zinc is placed in a solution of hydrochloric acid in water, a brisk evolution of hydrogen gas and of heat energy occurs at the surface of the zinc plate as the zinc goes into solution. The reaction is expressed by the formula

$$Zn + 2HCl = ZnCl_2 + H_2 + 166{,}000 \text{ joules per gram-molecule of Zn.} \quad (221)$$

The voltaic cell described in Sec. 142 is an arrangement for carrying out the same chemical reaction in such a manner that the chlorination of the zinc occurs at the surface of the zinc plate, the evolution of the hydrogen occurs in a second region (at the surface of the copper plate), and the evolution of the energy of the reaction occurs mainly in the metallic circuit joining the two plates.

Let us consider the phenomena in a simple cell consisting of strips of zinc and copper dipping into an electrolyte of 1 part (by volume) of fuming hydrochloric acid (HCl) to 10 parts of water (Fig. 121). When the zinc strip (not connected to the copper) is immersed in the electrolyte, we may suppose that zinc dissolves in the bordering layer of electrolyte in the form of positively charged zinc ions (Zn^{++}). This leaves the zinc strip nega-

tively charged, and faced with a film of electrolyte containing positively charged zinc and hydrogen ions.

The positively charged hydrogen ions in this border film are drawn to the negatively charged zinc strip. Upon contact with it they take up electrons from the strip and then combine into neutral hydrogen molecules. These molecules collect as the gaseous film which so rapidly forms even on pure zinc when it is immersed in the electrolyte. The zinc, if very pure, dissolves very slowly, partly because of the protective effect of the bubbles of hydrogen gas adhering to its surface, and partly because zinc ions can continue to enter the solution only as the previously dissolved zinc ions diffuse outward from the zinc. The slow diffusion of the zinc ions outward must be accompanied by the diffusion of hydrogen ions from these regions to the zinc plate

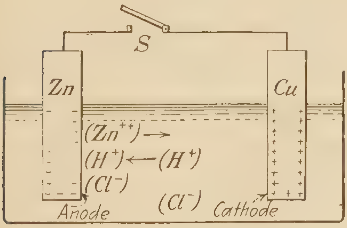


FIG. 121.—Zinc and copper not in contact.

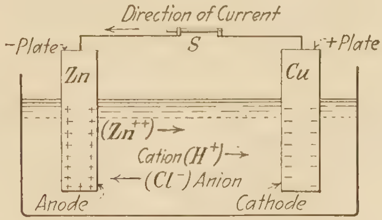


FIG. 122.—Zinc and copper in contact.

If, now, the zinc strip is connected to the copper by closing the circuit at S as in Fig. 122, the contact e.m.f. at the junction of the copper and the zinc leads to the transfer of electrons from the zinc to the copper; the copper becomes negatively and the zinc positively charged, as indicated in Fig. 122. These two layers of charge give rise to electric intensities in the electrolyte between the plates whose direction is from the zinc to the copper. Under these intensities the positively charged Zn and H ions in the electrolyte are impelled toward the copper plate and the negatively charged Cl ions move toward the zinc plate. The H ions upon reaching the copper take up electrons from it, combine into neutral molecules, and collect in bubbles at the surface of the copper plate. The evolution of the hydrogen gas at the copper becomes evident almost immediately after the switch at S is closed, and continues as long as current flows in the circuit.

The current, which is said to flow from copper to zinc in the external circuit, thus consists of the movement of electrons from zinc to copper in the wire and of the drift of the ions in the electrolyte in the directions indicated in Fig. 122. The electrons which flow to the copper plate through the wire are constantly removed to supply the electron deficit of the H ions which are drawn to the plate. On the other hand, the electrons which leave the zinc by way of the wire are continuously replaced by the electrons which are left behind when the zinc goes into solution as positively charged ions. In this manner the flow is maintained until the zinc ions have replaced all the hydrogen ions of the acid.

159. Local Action.—The rapid action of dilute acid on commercial zinc (with copious liberation of hydrogen at the zinc) as contrasted with the slow action of the acid on pure zinc is attributed to the fact that the commercial zinc contains such impurities as small particles of iron and carbon. These impurities when imbedded in the surface of the zinc, and thus in contact with the zinc and the electrolyte, form minature cells having closed metallic circuits. Local currents flow in these minature cells whether the external circuit of the cell is closed or not. These local currents are accompanied by the rapid and useless solution of the zinc. This wasteful chemical action is termed **local action.**

Local action may be greatly diminished by amalgamating the zinc with mercury. To amalgamate the zinc, it is cleaned by dipping it into dilute acid, and then, while it is still wet with the acid, several drops of mercury are poured on the surface. The spread of the mercury over the cleaned surface may be hastened by rubbing with a cloth.

160. Polarization.—Let the simple cell described above be connected as shown in Fig. 123 to supply current to an external circuit containing a considerable length of wire W. The current in the circuit and the electromotive force between the terminals of the cell may be read on the ammeter A and the voltmeter V. If the switch S is closed and if readings of the current and electromotive force are then taken at frequent intervals, the readings for a particular cell and circuit plot as shown in Fig. 124.

When the switch is first closed, the cell delivers a current of 50 milliamperes, but this current decreases in value at first rapidly and then more slowly, dropping to 60 per cent of its initial value in 5 minutes. Before closing the switch, the terminal electromotive force of the cell is seen to be 0.86 volt. Immediately after the closing of the switch, and the establishment of the current, the terminal electromotive force drops to 0.78 volt. It continues to drop and decreases to 60 per cent of its initial value in 5 minutes. Upon opening the switch, the terminal voltage rises as shown.

The decrease in the net electromotive force from terminal to terminal which occurs immediately upon closing the external circuit is due to the

internal resistance of the cell. That is to say, when the circuit is open, the electromotive force between the terminals is the line-integral through the cell of the driving forces. When the circuit is closed and current flows, the resistance forces of impeding impacts come into play and the electromotive force is reduced by the line-integral of these forces. The less the impeding resistance of the external circuit the greater will be the current through the cell; the greater the current through the cell the greater will be the impeding forces within the cell, and the less will be the terminal electromotive force.

The gradual decrease in the net electromotive force of the cell which occurs as the cell supplies current, and the failure of the electromotive force to rise immediately to the initial value when the circuit is opened, are due to the

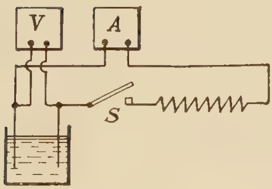


FIG. 123.

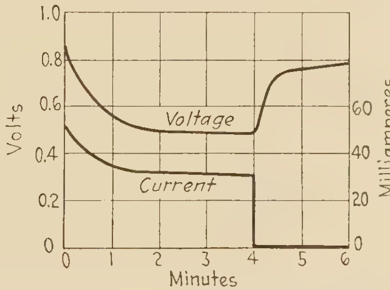


FIG. 124.—Drop in terminal voltage due to polarization.

changes which the ionic currents have produced, primarily in the composition or concentration of the electrolyte near the electrodes but also in the surface condition of the metal electrodes. A cell in which such changes have occurred is said to be **polarized.** The phenomenon of the decrease in the electromotive force of the cell accompanied by changes in concentration is known as the phenomenon of **polarization.** In the zinc-HCl-copper cell, the concentration of the chlorine ions in the electrolyte around the copper plate is reduced, leaving the electrolyte with an excess of positively charged H ions; and the concentration of the chlorine ions increases around the zinc plate. Since the processes of diffusion tend to annul this difference of concentration, they may be thought of as giving rise to forces on the ions in a direction opposite to the direction in which the ions move under the predominating driving forces.

The opposing electromotive force arising from changes in the concentration of the solution near the electrodes may be demonstrated most simply by immersing two electrodes of the same metal in a single solution (for example, copper plates in a solution of $CuSO_4$, or platinum plates in a solution of H_2SO_4) and then passing a current through this electrolytic cell by connecting it in the circuit of a primary cell or a dynamo. As the current flows through

the copper sulphate cell, the concentration of the copper sulphate solution increases around the anode and decreases around the cathode. If, after this process has gone on for some time, the copper sulphate cell is disconnected from the primary source and is closed through a high resistance as in Fig. 123, it is found that the cell will deliver a current whose direction through the cell is the reverse of the direction of the current which brought about the state of polarization. (A cell having identical electrodes immersed in two solutions of the same substance but at different concentrations is called a **concentration cell.**)

The electromotive force which the cell shows after it has been polarized and disconnected gradually decreases in value and finally disappears as the processes of diffusion annul the differences in concentration around the electrodes. If the circuit is closed, as in Fig. 123, the electromotive force more rapidly decreases to zero, since the ionic currents more rapidly annul the differences in concentration.

160a. Remedies for Polarization.—The polarization of a cell is the result of the exhaustion (in the region adjoining the electrodes) of the substances used in the electrolytic reaction more rapidly than they are replaced by the processes of diffusion, or of the accumulation of the products of the reaction more rapidly than they can be removed.

The measures taken to lessen polarization may be mechanical, chemical, or electrochemical in nature. Thus:

a. The electrolyte may be stirred or circulated in such a way as to sweep away the films adjoining the electrodes.

b. The cathode may be surrounded by an oxidizing agent which readily reduces and oxidizes the hydrogen to water. Compounds used in this manner to lessen the polarization are called **depolarizers.** Examples of depolarizers are the crushed manganese dioxide surrounding the carbon cathode in the dry cell, the sodium bichromate or chromic acid added to the sulphuric acid in the bichromate cell, and the copper oxide cathode used in the Edison-Lalande cell.

c. In the electrochemical method two solutions are used, one about each electrode, the solution around the cathode being such that it does not liberate polarizing products. The Daniell cell is an example of this method.

161. Classification of Voltaic Cells.—Based upon the manner in which they are used, voltaic cells may be classified as follows:

I. Primary cells: used to supply electrical energy from the chemical reaction of substances which are consumed in the process.
   1. Closed-circuit cells: cells suitable for delivering a current steadily or continuously, for example, the Daniell and the Edison-Lalande cells.
   2. Open-circuit cells: cells suitable for the intermittent supply of energy for short intervals, normally left standing with the external circuit open; for example, the dry cell.

II. Standard cells: used as standards of electromotive force, constructed to careful specifications of chemically purified materials which have been found to yield a very definite e.m.f., for example, the Weston cadmium cell.

III. Storage cells: used alternately to store energy in the chemical form and to deliver it in the electrical form, by means of a chemical action which is reversible with a reversal in the direction of the current through the cell, for example, the lead cell and the Edison nickel-iron-alkaline cell.

161a. The Daniell Cell—Gravity Form.—This two-fluid cell is illustrated in Fig. 125. The oxidizable element, or anode, is a massive zinc casting in the form of a crowfoot suspended near the top of a glass jar. The cathode, of sheet copper, rests on the bottom of the jar and is partly surrounded by crystals of copper sulphate. The lower half of the jar is filled with a saturated solution of copper sulphate, and the upper part of the jar is then filled with a solution of zinc sulphate by carefully pouring the latter solution upon the former. The latter solution because of its lower density tends to float on the former. If the cell is left standing on open circuit, however, the copper sulphate slowly diffuses into the zinc sulphate and upon reaching the anode copper is deposited on the zinc plate. This diffusion of the copper sulphate toward the anode is annulled by the ionic current if the cell is left on closed circuit; consequently, the cell finds its commercial application in situations in which a small constant current must be supplied, as in railway semaphore systems.

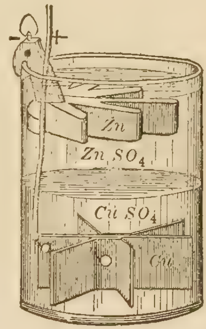


FIG. 125.—Daniel cell—gravity type.

When current flows, the reaction is

$$Zn + CuSO_4 = ZnSO_4 + Cu + 210{,}000 \text{ joules.} \qquad (222)$$

At the anode, zinc goes into solution as zinc sulphate; at the interface between the liquids, zinc replaces the copper; at the cathode, copper is deposited on the copper plate, the SO_4 ion migrates toward the anode, and the impoverished solution again becomes enriched by the diffusion of the copper sulphate from the region of the crystals.

The cell has an open circuit e.m.f. of 1.09 volts and an internal resistance of about 1.2 ohms.

161b. Edison-Lalande Cell.—The Lalande cell in the form developed by Edison consists of a depolarizing cathode in the form of a flat plate of compressed copper oxide (CuO), which is suspended between two amalgamated zinc plates in a solution of 1 part (by weight) of sodium hydroxide to 3 parts of water. The solution is covered with a layer of mineral oil to prevent access of the CO_2 of the atmosphere to the sodium hydroxide.

When current flows the reaction is

$$Zn + 2NaOH + CuO = Na_2ZnO_2 + H_2O + Cu. \qquad (223)$$

There is little local action and little polarization, since the hydrogen traveling to the cathode readily reduces the copper oxide, with the formation of metallic copper and water. The cell may, therefore, be used for open- or closed-circuit work. The e.m.f. of the cell is 1 to 1.1 volts but when current is delivered this drops rapidly to 0.9 volt and then more slowly to 0.7 volt. The internal resistance is very low—0.03 to 0.05 ohm.

161c. Bichromate Cells.—The bichromate or chromic acid cells make use of chromic acid in solution as the depolarizing agent. The two-fluid form known as the Fuller cell is illustrated in Fig. 126. The oxidizable anode is an amalgamated zinc casting resting on the bottom of a porous earthenware cup. A little mercury is placed in the bottom of the cup to keep the zinc well amalgamated and the cup is filled either with a solution of common salt or with dilute sulphuric acid. The cup rests in a glass jar which is partly filled with a solution made by adding chromic acid (CrO_3) (1 part by weight) to a solution of sulphuric acid (3 parts) in water (9 parts). The cathode, a plate of carbon, dips into the chromic acid solution. When current flows, the reaction is

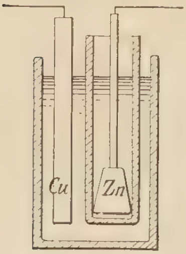


FIG. 126.—Fuller cell.

$$3Zn + 2CrO_3 + 6H_2SO_4 = Cr_2(SO_4)_3 + 3ZnSO_4 + 6H_2O. \qquad (224)$$

There is little local action. The cell on open circuit has an e.m.f. of 2 volts, and an internal resistance of the order of 0.5 ohm.

161d. Leclanche Cell.—In this cell, which takes many forms, the anode is amalgamated zinc, generally in the form of a rod which dips into an unsaturated solution of ammonium chloride (NH_4Cl—sal ammoniac) contained in a glass jar. The cathode is of carbon in the form of a rod or plate or in some cases a hollow cylinder surrounding the zinc rod. The depolarizing agent is powdered manganese dioxide. In the earlier forms, the carbon rod was embedded in a mixture of the powdered dioxide and crushed coke in a porous earthenware cylinder. In the later types, the porous cup is replaced by a canvas sack or is dispensed with by molding the mixture of carbon and dioxide into suitable forms.

When current flows, the reaction is

$$Zn + 2MnO_2 + 2NH_4Cl = Mn_2O_3 + H_2O + 2NH_3 + ZnCl_2. \qquad (225)$$

The e.m.f. on open circuit is between 1.5 and 1.6 volts, and the internal resistance varies from 0.7 ohm in the zinc rod-porous cup type to 0.10 in cells having an anode of sheet zinc in the form of a hollow cylinder surrounding the canvas sack.

There is little local action, so that the cell does not deteriorate rapidly when unused. The depolarizing takes place very slowly, so that the cell is

quite unsuited to furnish currents of the order of an ampere for an hour, but is well suited to supply continuous currents of the order of a few milliamperes, or to furnish current at occasional intervals during the course of a year, as in the operation of bells and local telephones.

161e. The Dry Cell.—The dry cell illustrated in Fig. 127 may be regarded as a Leclanche cell in which the free space for the electrolyte has been eliminated, and in which the electrolyte is retained in the porous mass of granular carbon and manganese dioxide which surrounds the central carbon cathode, and in an absorbent layer of blotting paper, paste, or plaster of paris which separates the depolarizing and conducting powder from the zinc anode. The latter is in the form of a hollow cylinder and it also serves as the container.

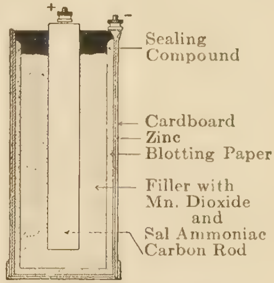


FIG. 127.—Dry cell.

The electrolyte, which saturates the powdered mass and the absorbent lining of the zinc container, is a solution of ammonium chloride and zinc chloride.

When new, the e.m.f. of the cell is between 1.5 and 1.6 volts, and the internal resistance of the 2.5 in. $\times$ 6 in. size is from 0.05 to 0.1 ohm. A cell in good condition should deliver from 15 to 30 amperes when momentarily closed through an ammeter having a resistance of 0.01 ohm or less.

Even when unused the cells deteriorate because of local action which finally corrodes through the zinc and allows the electrolyte to evaporate. The shelf life of a cell is of the order of a year.

161f. Weston Cadmium Standard Cell.—By a standard cell is meant a specially designed voltaic cell, the electromotive force of which is known to a high degree of accuracy and which, therefore, may be used to measure unknown electromotive forces by comparison methods. Such a cell must possess the characteristics of **reproducibility** and **constancy** of electromotive force. Now the electromotive force of cells is, in general, affected by impurities in the elements, surface condition of the electrodes, concentration of the electrolyte, temperature, polarization effects, etc. Only a few types of cells are suitable as standards. Of these the Cadmium Standard Cell, as developed by Weston, is now in general use.

The construction of a portable form of the Weston cell is shown in Fig. 128. The container is an H-shaped glass tube having platinum leading in wires sealed into the bottom of each leg. The anode is the mercury shown at the bottom of one leg. This is covered with a layer of mercurous sulphate paste. The cathode is a liquid amalgam of 1 part of cadmium in 7 parts of mercury in the bottom of the other leg. These materials are held in position at the bottoms of the legs by porcelain tubes with enlarged bases about which is a packing of asbestos fiber. The electrolyte, a solution of cadmium sulphate, connects the two legs, which are corked at the top and then sealed with wax.

The cell is made in two types, the saturated or so-called "normal cell," and the unsaturated or "secondary" cell. In the normal cell, cadmium sulphate crystals are left in the tubes so that the solution is saturated at all temperatures. This cell has a rather definitely reproducible e.m.f. of 1.01830 volts at 20°C., the e.m.fs. of different cells differing by only a few parts in 100,000. It has a fairly large variation of e.m.f. with temperature. Its e.m.f. at any ordinary room temperature t is

$$E = E_{20} - 0.0000406(t - 20) - 0.00000095(t - 20)^2.$$

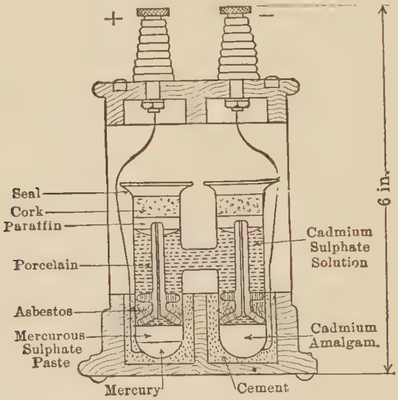


FIG. 128.—Weston standard cell.

In the unsaturated cell, the solution is saturated at 4°C. and is unsaturated at room temperature. The e.m.f. of these cells averages 1.0186 volts and the internal resistance 200 ohms. While the e.m.f. cannot be exactly predetermined but must be determined by actual measurement for each cell, this type has the advantage that its e.m.f. remains very constant with age and is substantially unaffected by temperature variations between 4 and 40°C.

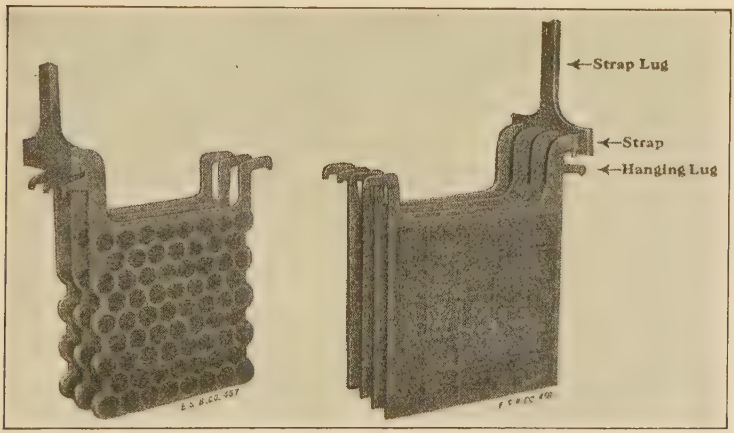
To avoid polarization effects, standard cells are so used that they deliver current never in excess of a few microamperes, and then only momentarily.

163. Storage Cells.—In a primary voltaic cell, the elements and the electrolyte are replaced by new materials when the oxidizable element has been completely oxidized. By a **secondary** cell, or **storage** cell, or electric **accumulator,** is meant a voltaic cell in which the chemical processes are reversible with the reversal in the direction of the current through the cell, and in which the elements after becoming oxidized or otherwise exhausted during the delivery of current (the **discharge** of the cell) are not discarded but are deoxidized and returned to their initial condition by the process of **charging.** The charging

process consists in connecting the cell in series with a steam-driven electric generator (or its equivalent) which generates a higher opposing e.m.f., and thus causes current to flow through the cell in the reverse direction for the necessary length of time.

The terminal or plate of a cell which is at the higher potential is called the **positive plate.** In a given cell, the same plate is always the positive plate whether during charge or during discharge. During discharge the direction of the current **through** the cell is from the − to the + plate, and the anode (surface) is therefore the surface of the − plate. During charge, the surface of the + plate is the anode. There are only two types of storage cells in commercial use: the lead-acid cell and the nickel-iron-alkaline cell in the form developed by Edison.

164. Lead-acid Storage Cell.—In the lead-acid storage cell the oxidizable element, or negative plate (or anode during dis-



Positive group Negative group

Fig. 129.—Plates of lead storage cell.

charge), is of lead, and the depolarizing cathode is of lead peroxide (PbO_2). The electrolyte is a solution of 1 part (by volume) of fuming sulphuric acid and 4 parts of distilled water.

The plates hang vertically in glass or hard rubber or lead-lined wooden containers holding the electrolyte. Positive plates alternate with negative plates, all positives being burned to one lead strap and all negatives to another, as in Fig. 129. The adja-

cent positive and negatives are kept from touching by separators in the form of glass or hard-rubber rods or sheets of wood or of perforated hard rubber. The two outside plates are always negatives, since the swelling of the peroxide during the charging process has a tendency to buckle the positive plate if it is active on only one side.

When current flows during discharge, the SO_4 ion migrates to the anode and combines with the lead plate, the reaction being

$$Pb + SO_4 = PbSO_4.$$

The lead sulphate is not soluble in the electrolyte and adheres to the lead plate.

The H ion migrates to the cathode and there reduces the peroxide, with the formation of lead sulphate, which remains in place on the plate. The reaction at the cathode may be written

$$2H + H_2SO_4 + PbO_2 = PbSO_4 + 2H_2O.$$

The combined reaction may be written

$$Pb + PbO_2 + 2H_2SO_4 = 2PbSO_4 + 2H_2O + 374{,}000 \text{ joules.} \tag{226}$$

During the charging process this reaction is reversed. It follows that during the discharge sulphuric acid is withdrawn from the solution and water is formed, while during charge acid is returned and water is withdrawn. The density of the solution thus decreases during discharge and increases during charge, and is a good indication of the extent to which the cell has been discharged. Thus, in one type of stationary cell, the density of the solution of the fully charged cell should be 1.210 at 21°C.; the cell should not be discharged below the density 1.17.

164a. Types of Plates.—The plates consist of (*a*) the chemically active porous material having a large surface exposed to the action of the electrolyte, and (*b*) the framework which is necessary to support the active material and to conduct current to it. The framework is generally a casting of an alloy of lead with about 7 per cent of antimony. Two methods of preparing the active material are in use.

In the Plante or "formed" plates, plates having a large surface area exposed to the action of the electrolyte, are made by such methods as passing a lead plate under a row of rapidly rotating steel disks which convert the plane surface into a succession of deep furrows and projecting lead fins, or by pressing spirals of corrugated lead ribbon into circular holes in an antimony-lead frame. The outside layers of the extensive lead surfaces

thus formed are then converted into lead peroxide by passing current from them for long intervals, using them as anodes in a dilute solution of sulphuric acid with certain accelerating agents, such as nitric acid. If such plates are to be used as negatives, the peroxide is converted into sponge lead by connecting them as negatives in a battery and charging it for a long interval.

In the Faure or "pasted" plates, the interstices in a lead-antimony lattice are plastered full with a paste consisting of litharge (PbO) and other ingredients mixed with sulphuric acid. The paste sets into a hard mass of lead sulphate and is then electrochemically converted into sponge lead and lead peroxide respectively by immersing the plates in a solution of magnesium or aluminum sulphate and passing current between them.

In service, Plante (peroxide) positives gradually shed their active material. New lead is then automatically converted into active material and eventually the frame becomes mechanically weak. Good Plante positives have a life of 1800 to 2400 cycles of charge and discharge at the 8-hour rate. The active surface of the sponge lead of Plante negatives decreases in service. This occurs more rapidly if the cell is allowed to stand for long intervals in a discharged condition. New negatives have an excess capacity of 75 to 100 per cent to allow for this and should have a life of from 2400 to 3000 cycles.

Faure positives lose active material by shedding. Thus, positives 6 millimeters thick have a life of from 600 to 800 complete cycles. The pasted negatives also decrease in capacity with life, due to the shrinkage in the surface of the sponge lead.

The **pasted** plate, for a given discharge rate, costs less, weighs less, and takes up less space than does the Plante plate. On the other hand, its life, expressed in complete cycles, is less than that of the Plante plate. In practice, the negative (lead sponge) plates are generally of the pasted type. The positive (peroxide) plates may be of either type. In a stationary battery which is to be subject to continued heavy demands for current, the heavier, more costly, and longer lived Plante plate has the advantage. On the other hand, in motor vehicle service, where weight and bulk are objectionable, and for **standby service,** in which the battery is called upon to furnish current only in emergencies at long intervals, the pasted positive has a distinct advantage.

164b. Capacity of a Storage Cell.—The capacity of a cell may be expressed either as the number of watt-hours or the number of ampere-hours which the completely charged cell is capable of delivering before its voltage falls below some selected value when discharging at some specified constant current and at 21°C. For stationary lead batteries the limiting voltage is taken as 1.8 volts and the "normal rated current" of a cell is taken as that current which will cause it to discharge to the limiting voltage in 8 hours.

The larger the current drawn from the cell during discharge, the lower is the ampere-hour capacity of the cell. This is illustrated in the following table:

164c. Ampere-hour Capacity of Lead Cells in Per Cent of Capacity at 8-hour Rate.

Discharge current, per cent of normal	100	160	266	800
Capacity in per cent, Plante type	100	88	75	50
Capacity in per cent, Faure type	100	93	83	60

At the high currents, the polarization e.m.f. for a given ampere-hour delivery is far greater than that at the low currents, since there is far less time for the processes of diffusion to neutralize

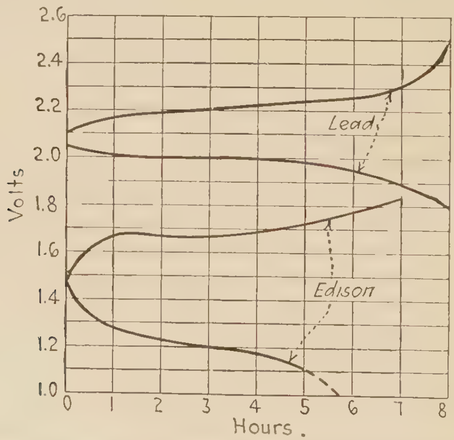


FIG. 130.—Terminal voltage of storage cells during charge and discharge at normal rates.

the changes in acid concentration occurring within the pores of the active material.

Figure 130 contains curves showing the manner in which the terminal voltage of a typical stationary lead cell varies with hours of charge or discharge at the normal 8-hour rated current. The open circuit voltage of the charged cell is about 2.1 volts. The drop in voltage in the internal resistance caused by the rated 8-hour current is about 3 per cent of the cell voltage. The average value of the terminal voltage during discharge is about 1.95 volts, and during charge about 2.25 volts. The difference

between the two values is due only in part to the *IR* voltage in the cell, which during discharge subtracts from and during charge adds to the intrinsic e.m.f. of the cell. The difference is due more largely to polarization effects.

Toward the end of the charge, the terminal voltage rises to 2.5 or 2.6 volts. At this voltage water is electrolyzed and hydrogen and oxygen are given off copiously. The water thus lost must be replaced from time to time.

The **ampere-hour efficiency** of the charge and discharge cycle at the 8-hour rate is about 95 per cent, while the **energy efficiency** is from 75 to 85 per cent. A lead cell standing charged will lose only 0.5 per cent of its energy content per day by reason of local action.

By applying Faraday's law, it may be determined that the delivery of one ampere-hour in accordance with the chemical reaction expressed by Eq. (226) will require the use of 3.86 grams of lead, 4.46 grams of PbO_2, and 3.66 grams of H_2SO_4, or a total of 12 grams of active material. Taking the average discharge voltage as 1.95 volts, this means that 12 grams of active material is capable of supplying 1.95 watt-hours, or 1 kilogram is capable of supplying 162 watt-hours of electrical energy. Since the cell consists not only of active material, but of frames, containers, reserve material, etc., the capacity per kilogram of cell will be far less than this. The capacity per kilogram of cell (complete with electrolyte) obtained in different types is shown in the following table.

164d. Capacity of Storage Cells in Watts and Watt-hours per Kilogram of Cell Weight.

Type	Lead	Lead	Lead	Edison
Service	Stationary, continued demands	Stationary, emergency demands	Motor vehicle	Motor vehicle
Plates	Plante + Pasted −	Pasted	"Iron-clad" + Pasted −	See Sec. 165
Watts, 8-hour rate	0.83	1.3	3	3.9
Watt-hours, 8-hour rate	6.6	10.4	24	31
Watts, 1-hour rate	3.3	6.4	13	20.4
Watt-hours, 1-hour rate	3.3	6.4	13	20.4

These energy-storage capacities may be compared with the energy of combustion in a kilogram of good coal, namely 8800 watt-hours, of which 25 per cent is converted into electrical energy in the best central-station practice.

165. The Edison Storage Cell.—In the storage cell commercially developed by Edison, the oxidizable material of the negative plate is finely divided iron, and the depolarizing material of the positive plate is nickel hydroxide $Ni(OH)_3$. The electrolyte is a 21 per cent solution of potassium hydroxide (KOH, density 1.20) containing a small amount of lithium hydroxide.

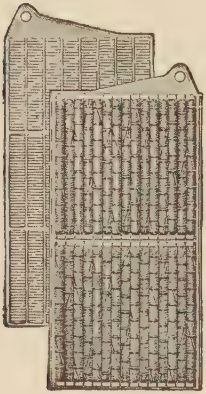


Fig. 131.—Positive and negative plates of Edison storage cell.

The negative plate is a nickel-plated steel grid in the openings of which are pressed perforated nickel-plated steel boxes which contain the finely divided iron mixed with some mercury.

The positive plate is likewise a nickel-plated steel grid into which are pressed a number of perforated nickel-plated steel tubes tamped full of alternate layers of nickel hydroxide and flake nickel. (See Fig. 131.)

The commercial cells are made up with from 2 to 12 of these positives interleaved with negatives and mounted in closed nickel-plated sheet steel boxes.

When current flows during discharge, the (OH) ion migrates to the anode and the iron is oxidized, possibly with the reaction

$$Fe + 2(OH) = Fe(OH)_2.$$

The potassium ion travels to the cathode and reduces the nickel hydroxide to a lower oxide, possible with the reaction

$$2Ni(OH)_3 + 2K = 2Ni(OH)_2 + 2KOH.$$

The combined reaction may be written

$$2KOH + Fe + 2Ni(OH)_3 = Fe(OH)_2 + 2Ni(OH)_2 + 2KOH. \tag{227}$$

The reaction does not change the density of the solution as a whole, although changes in concentration occur in the pores of the active material.

Figure 130 shows the manner in which the terminal voltage of the typical cell varies during charge and discharge at the normal rated current for 5-hour discharge.

The open-circuit voltage of the charged cell is 1.5 volts, but this drops in any hour to 1.3 volts, then drops slowly to 1.15 volts at the end of 4 hours, and then rapidly to low values.

During discharge the terminal voltage averages 1.2 volts. On charge the terminal voltage rises rapidly from 1.4 to about 1.65 and then to 1.8 volts at the end of 7 hours, averaging 1.7 volts during charge. The evolution of oxygen and hydrogen becomes rather brisk toward the end of the charge. The water thus lost must be replaced from time to time. Unlike the lead cell, the Edison cell is not injured if allowed to stand for long periods in a discharged condition.

The ampere-hour efficiency of the cell on the "7-hour charge, 5-hour discharge cycle" is about 82 per cent and the energy efficiency about 60 per cent.

166. Calculation of the E.M.F. of Voltaic Cells.—If the reactions which take place in a voltaic cell are carried out as ordinary chemical reactions outside the cell, a certain amount of energy is evolved in the form of heat. If upon carrying out the same reactions in the cell, the heat energy were all converted into electrical energy, the intrinsic e.m.f. of any cell could be readily calculated from the known heats of reaction in the following manner:

Let E represent the intrinsic e.m.f. of the cell in volts.
K represent the gram-equivalent of the coulomb (see Eq. (192).
H represent the heat (in joules) of all the reactions which accompany the oxidation of one gram-atom of the anode.
V represent the valence of the anode ion.
Q represent the quantity of electricity passing through the cell in the oxidation of one gram-atom of the anode.

Then
$$Q = \frac{V}{K}$$
and
$$E = \frac{H}{Q} = \frac{KH}{V}$$
$$E \text{ (volts)} = \frac{1.035H}{10^5 V} \text{ joules per gram-atom.} \tag{228}$$

This relation, known as the Kelvin or as the Helmholtz-Thompson relation, gives approximately correct values for a number of cells—notably the Daniell cell.

The argument ignores the possibility of thermoelectric effects. By an analysis in which a reversible cell is carried through a thermal cycle, Helmholtz and Gibbs have derived the following expression for the e.m.f. of such a voltaic cell:

$$E = \frac{1.035H}{10^5V} + T\frac{dE}{dT}, \tag{229}$$

in which T is the absolute temperature of the cell, and $dE/(dT)$ is the temperature coefficient of its e.m.f.

This is known as the Gibbs-Helmholtz relation. If the temperature coefficient of the cell is positive, the e.m.f. is greater than computed from the heat of reaction, or the discharge must be accompanied by the absorption of heat and the cooling of the cell.

Applying Eq. 228 to the reaction in the Daniell cell as expressed by Eq. (222),

$$E = \frac{1.035}{10^5} \times \frac{210{,}000}{2} = 1.086 \text{ volts.}$$

The observed voltage of a Daniell cell at 0°C. is 1.096 volts and its observed temperature coefficient is 0.000034 volt per degree.

167. Exercises.

1. Define carefully the following units, and identify each as a unit of energy or of power: horsepower, horsepower-hour, watt, watt-second, watt-hour and kilowatt-hour.

Which of these are units of the Practical System of electrical units?

2. An electromagnetic generator receives energy from the engine at the rate of 5 horsepower. Assume that 85 per cent of this energy is converted into the electrical form. If the current through the generator is 29 amperes, what is the value of the generated e.m.f. in the machine?

3. Assume that the cost of electric energy for cooking and heating is 2 cents per kilowatt-hour; the cost of gas having a heating value of 600 B.t.u. per cubic foot is $1.15 per 1000 cubic feet; and the cost of anthracite coal having a heating value of 13,500 B.t.u. per pound is $15 per ton.

Calculate the cost of heating water for a bath, 55 liters from 12 to 38°C., by each agency, on the assumption that all heat is delivered to the water in each case.

4. The e.m.f. of a Leclanche cell is 1.5 volts, and the cost of zinc may be taken as 18 cents per pound. What would be the cost of the zinc consumed, if such a cell were used to furnish 1 kilowatt-hour of electric energy?

5. Heat is generated in a conductor at the rate of 30 joules per second when a current of 5 amperes passes through it. What is the e.m.f. of the resistance forces along the conductor?

6. *a.* If the chemical energy of combustion of coal could be converted into the electrical form with no loss, what would be the cost of the coal necessary to generate 1 kilowatt-hour? Assume that coal whose heating value is 13,500 B.t.u. per pound costs $3.50 per ton.

b. The efficiency from the coal pile to the switchboard bus bars of the most modern turbo-alternator plants is about 25 per cent. What is the cost of coal per kilowatt-hour for energy delivered at the switchboard?

7. At the hydroelectric plant of the sanitary district of Chicago, which is located at Lockport on the drainage canal, the average "head" (fall from head water to tail water) is 34 feet. Above the power house at the point of minimum width of canal where the canal is approximately rectangular in section, the width is 160 feet, and the average depth of water is 20 feet. At this point the average velocity of flow is 3.1 feet per second. If the overall efficiency of the plant is 70 per cent, what is the maximum kilowatt output of the plant under the above conditions?

8. A 2000-kilowatt transformer, having an efficiency at full load of 98.5 is kept cool by passing water through a cooling coil located in the oil in the top of the transformer casing. When the transformer is delivering full load, at what rate in liters per minute (also gallons per minute) must water be passed through the cooling coil if the temperature of the escaping water is not to exceed that of the water supply by more than 10°C. Neglect the heat radiated from the transformer casing.

CHAPTER VIII

PROPERTIES OF ELECTRIC CONDUCTORS

168. Classification of Conductors.—The experiments recited below and in Chap. VI on the electrical conducting properties of materials, and on the mode of conduction of electricity through them, lead to the classification of all materials, as regards their conducting properties, under the following headings:

1. Metallic conductors.
2. Electrolytic conductors.
3. Pyroelectric conductors.
4. Non-conductors or insulators.
5. Gaseous conductors.
6. Electronic conduction through evacuated space.
7. Conductors with asymmetrical conducting properties.

The physical and chemical phenomena which attend the passage of electricity through metallic and electrolytic conductors have been described and accounted for in a preliminary way in Chap. VI. The present chapter is to present the relations between the current and the electromotive force in conductors of each of the above types, and is to deal in particular with the engineering aspects of the dissipation of energy which goes on in conductors when they carry current.

169. Voltage-current Characteristic of a Conductor.—The important electrical characteristics of a conductor may be obtained by connecting the conductor C to a souce of intrinsic electromotive force B, through an ammeter A, as shown in Fig. 133. The source of e.m.f. may be a battery or an electromagnetic generator. The current flowing through the conductor may be made to assume a number of different values by either of the following methods:

a. The intrinsic e.m.f. of the source may be varied; for example, by using 1, 2, 3, 4, etc. voltaic cells connected in series.

b. Or the intrinsic e.m.f. of the source may be kept constant, but a control conductor or resistor D of variable length may be connected in series with the conductor C. The control conductor is generally suitably mounted in a frame which is provided either with a sliding contact or with switches to permit of readily changing the length of the control conductor connected between the terminals of D. Such a variable resistance is called a **rheostat.**

The electromotive force of the actions in the conductor C for each value of the current through it may be read on an electrostatic voltmeter V connected across the terminal or end surfaces of C.

If the current through the conductor is to have the same value at all cross-sections of the conductor, it is necessary that the conductor be insulated on all surfaces save the two which serve as the terminal or end surfaces. Again, if the electromotive force is to be a definite quantity, each end surface should be an equipotential surface. This condition is not always precisely attained in experimenting with conductors, but it is fulfilled for all practical purposes when the variation in potential over each terminal surface is small in comparison with the potential difference between the surfaces.

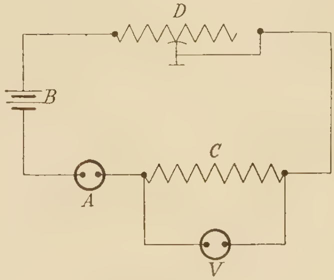


FIG. 133.—Circuit for obtaining voltage-current characteristics.

The properties of any material as a conductor of electricity are obtained by taking simultaneous measurements of the current through the conductor and the electromotive force between its end surfaces. When a number of pairs of simultaneous readings are plotted, current as abscissas and voltage as ordinates, the resulting curve through the points is called the **voltage-current characteristic** of the conductor.

As the current through a conductor is increased, the energy dissipated in the conductor increases, and the rise in the temperature of the conductor above its surroundings becomes greater and greater. The conductor may reach an incandescent temperature (as in the incandescent lamp), or it may pass from a solid to a molten state. Now the conducting properties of a

given substance almost invariably change with its temperature. Some substances become better conductors and others poorer conductors as their temperatures rise. Therefore, if the term **voltage-current characteristic** is to be used precisely, it must be supplemented by a qualifying phrase or statement which conveys information as to the thermal environment of the substance under which the characteristic was obtained. For example, we may take the voltage-current characteristic of the tungsten wire in an incandescent lamp under the conditions of usage of the tungsten in the evacuated bulb; or we may break the bulb and take a second characteristic with the tungsten wire immersed in an oil bath which is maintained at a temperature of, say, 20°C. The oil conducts the heat away from the fine wire so readily that a current which will cause the wire to attain an incandescent temperature in the evacuated bulb of the incandescent lamp will cause the temperature of the same wire, when oil immersed, to exceed that of the oil by only a few degrees. The striking difference between these two characteristics for the same wire is shown in Fig. 134.

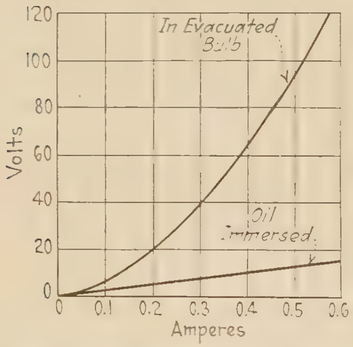


FIG. 134.—Voltage-current characteristic of tungsten wire of 60 watt lamp.

At this time we may distinguish between the **voltage-current characteristic for a constant temperature of ( )°C.**, and the **voltage-current characteristic for normal operation** of the substance or the appliance. By the former we mean the characteristic obtained when such special cooling measures are used that the temperature of the substance is held constant at some specified value throughout the series of readings. By the latter, we mean the characteristic obtained when the cooling action of the surroundings is the same as that which prevails during the normal conditions of usage of the appliance.

170. Voltage-current Characteristics of Metallic Conductors. Ohm's Law.—The first class of conductors contains as its most

important members all the metals and their alloys. Therefore conductors of this class are called **metallic** conductors. It also includes all other conducting materials, such as carbon, which are thought to conduct electricity in the same manner as the metals. The nature of metallic conduction has been explained in terms of the electron theory in Sec. 121. The distinguishing feature of metallic conduction is that no chemical or physical changes take place in the conductor, save such secondary changes as may occur from the rise in the temperature of the conductor. The experiments indicate that in metallic conduction the only moving charges are the free electrons in the body of the conductor, and that the positively charged atoms do not migrate, but remain fixed in the atomic structure of the substance.

If the voltage-current characteristic of a metallic conductor is taken at any specified constant temperature, say 20°C., the curve is found to be a straight line passing through the origin. The straight-line characteristic shown in Fig. 134 for oil-immersed tungsten wire is typical of metallic conductors. This remarkably simple relation which was experimentally discovered in 1827 by Ohm, a German scientist, is known as Ohm's law. It may be stated in the following form.

170a. OHM'S LAW (EXP. DET. REL. 1827).—**When an unvarying current flows through a homogeneous metallic conductor which is kept at a constant temperature, the electromotive force between the terminals of the conductor is directly proportional to the value of the current.**

The straight-line relation between the two quantities is expressed by either of the following equations:

$$E \text{ (e.m.f. of resistance)} = -RI. \tag{230}$$

$$I = -GE \tag{231}$$

It should be noted that the law is a **conditional law.** The direct proportionality between electromotive force and current applies only to metallic conductors, which are homogeneous throughout, which are maintained at a constant temperature, and which are not in a changing magnetic field.

The electromotive force E in the conductor is called **the electromotive force of resistance.** It represents the work done by the impeding forces of the atomic impacts upon the moving electrons per unit quantity of electricity which passes through

the conductors. These impeding impact forces are called into play only when the electrons move through the conductor under the driving forces of some source of intrinsic electromotive force (as a battery) which is connected in a circuit containing the conductor.

The negative sign appears in the equations which express Ohm's law because the preceding definitions of current and e.m.f. include the following conventions.

171. CONVENTIONS AS TO POSITIVE AND NEGATIVE QUANTITIES. -1. For convenience in specifying directions, an arrow will be drawn in an arbitrarily selected direction along the conductor, and the direction so indicated will be called the ARROW DIRECTION.

2. Any symbol used to represent the current in the conductor will be understood to represent the algebraic value of the current in the ARROW direction. (This convention is nothing but a precise and rapid method of defining the symbol.)

3. Any symbol used to represent the electromotive force of specified forces in the conductor will be understood to represent the ALGEBRAIC value of the electromotive force in the ARROW direction.

That is to say, an e.m.f. which is the line-integral of forces which would drive + charges in the direction of the arrow, and electrons against the arrow, is to be taken as a positive quantity. We know that if the current flows in the arrow direction, the impeding impact forces would, if acting alone, send current in the direction against the arrow. That is to say, if the current is a positive quantity, the e.m.f. of resistance is a negative quantity, and we may write the equations for Ohm's law in either of two forms:

a. We may write the equations with the negative sign as in Eqs. (230) and (231). In this case, the proportionality constants R and G are positive quantities.

b. We may omit the negative sign from Eqs. (230) and (231). In this case the proportionality constants R and G (which are constants of the conductor) must be regarded as negative quantities, or the equation must be regarded as arithmetical and not algebraic in character.

The first plan seems the more natural and we adopt it. The algebraic signs of the current and voltage may be neglected, and they generally are neglected, in simple calculations dealing with

a single conductor or a simple series circuit. However, in setting up the equations from which the currents in complicated networks may be computed, it is of the utmost importance that these conventions as to the algebraic signs of the quantities be kept clearly in mind and rigidly adhered to. If the wrong algebraic sign is used for any one quantity, the computed currents will not only be erroneous as to sign, but also as to absolute value.

172. Resistance and Conductance of Conductors.—The proportionality constants R and G between the current in a conductor and the electromotive force of resistance are used so frequently that names have been coined for them. They are called the **resistance** and the **conductance** of conductor, respectively. These terms may be defined as follows.

172a. RESISTANCE (DEFINITION).—**The ratio of the electromotive force between the end surfaces of a homogeneous conductor to the steady current through the conductor is called the RESISTANCE of the conductor BETWEEN THE SPECIFIED END SURFACES. Resistance is invariably represented by the symbol R.**

$$R \text{ (ohms)} = \left|\frac{E}{I}\right| \frac{\text{(volts)}}{\text{(amperes)}} \text{(defining } R\text{)}. \tag{232}$$

172b. Unit of Resistance—the Ohm (DEFINITION).—A conductor has a resistance of unity (or of 1 **ohm**) if the ratio of the electromotive force, in volts, to the current, in amperes, is equal to unity. The descriptive name of the unit is the **volt per ampere,** but a short name has been coined for the unit, namely, the **ohm.**

172c. CONDUCTANCE (DEFINITION).—**The ratio of the steady current through a homogeneous conductor to the electromotive force between its end surfaces is called the CONDUCTANCE of the conductor. Conductance is represented by the symbol G.**

$$G \text{ (mhos)} = \left|\frac{I}{E}\right| \frac{\text{(amperes)}}{\text{(volts)}} \text{ (defining } G\text{)}. \tag{233}$$

The conductance of a conductor is the reciprocal of its resistance.

$$G \text{ (mhos)} = \frac{1}{R} \frac{1}{\text{(ohms)}}. \tag{234}$$

172d. Unit of Conductance—the Mho (DEFINITION).—A conductor has a conductance of unity (or of 1 mho) if the ratio

of the current, in amperes, to the electromotive force, in volts, is equal to unity. The descriptive name of this unit is the **ampere per volt,** but a short name, the **mho,** has been obtained for the unit by spelling ohm in reverse order.

173. The International Ohm.—The ohm defined above is the unit of resistance of the electrostatically derived practical system (see Appendix F). Partly because of the precision with which standards of resistance can be made up and intercompared, and partly because of the facility with which the standards of other derived units can be calibrated in terms of the ohm and the ampere, the ohm has been selected as one of the independent units of the international units. The standard of resistance as legally defined for this purpose is known as the **international ohm.** It is defined as follows.

173a. International Ohm (DEFINITION).—*The international ohm is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice,* 14.4521 *grams in mass, of a constant cross-sectional area and of a length of* 106.300 *centimeters.*

The most recent determinations indicate that 1 international ohm = 1.00052 ± 0.00004 ohm of the practical system.

The column of mercury specified above is approximately 1 square millimeter in cross-sectional area. This mercury standard is used as a primary standard only. For secondary and working standards carefully mounted coils of wire made of special non-aging alloys are used. These wire standards are adjusted to have resistances which are exact decimal multiples or submultiples of the international ohm, that is 0.01, 0.1, 1, 10, 100 ohms, etc.

174. Heat Developed in a Metallic Conductor by an Electric Current. Joule's Law.—When a current of strength I flows in a conductor of resistance R, the electromotive force E of the actions (impeding impacts) occurring in the conductor is

$$E = -RI. \tag{230}$$

The rate P at which energy is converted into the electrical form in the conductor is

$$P = EI = (-RI)I = -I^2R. \tag{235a}$$

The negative sign signifies that the conductor is a receiver of electrical energy at the rate of I^2R joules per second. This energy is supplied to the conductor by the source of intrinsic e.m.f. to which the conductor is connected.

The question now arises, What becomes of the electrical energy thus applied to the conductor? We know that at least a part of it is converted into heat energy, because a current in a conductor always causes a rise in the temperature of the conductor. But may not a portion of the energy be dissipated in some form outside the conductor? By the only method capable of furnishing the answer to the question, Joule obtained the answer in 1841. By precise calorimetric measurements of the amount of energy converted into the form of heat when a current traverses a conductor, he established the fact that the energy so delivered is all converted into heat in the body of the conductor. His conclusions may be stated in the following form.

174a. JOULE'S LAW (EXP. DET. REL., 1841).—**The rate P at which heat is developed in any metallic conductor is equal to the resistance of the conductor multiplied by the square of the current traversing the conductor.**

$$P \text{ (watts)} = RI^2 \text{ (ohms, amperes).} \qquad (235)$$

This law holds whether the current is steady or is varying, provided the variation in the current takes place so slowly that the current density is the same over the entire cross-section of the conductor. If the current is varying, P and I represent instantaneous values of power and current.

175. Variation of Resistance with the Temperature of the Conductor.—As pointed out in Sec. 169, the resistance of a conductor depends upon the temperature of the conductor. The relation between the resistance of a conducting material and its temperature must be determined originally by maintaining the material at many known temperatures and measuring the values of the resistance at these temperatures. The resistance may be measured by the voltmeter-ammeter method of Fig. 133, or by the **bridge** comparison methods of the next chapter. If the measured resistances are plotted as ordinates against the temperatures as abscissas, the curve drawn through the points is called the **resistance-temperature characteristic** of the material.

The resistance-temperature characteristic for the metallic conductors is for all ordinary purposes a straight line between the limits -50 and $+200°C$. Figures 134 and 135 are typical of the relations found for the metallic conductors.

The relation between the resistance and the temperature shown by the straight line of Fig. 135 is expressed by the equation

$$R = R_1[1 + \alpha_1(T - T_1)], \tag{236}$$

in which R_1 is the resistance at the reference temperature T_1.
R is the resistance at the temperature T.
α_1 is a factor whose value depends upon material of the conductor, the value of the reference temperature T_1, and the units in which T and R are expressed.

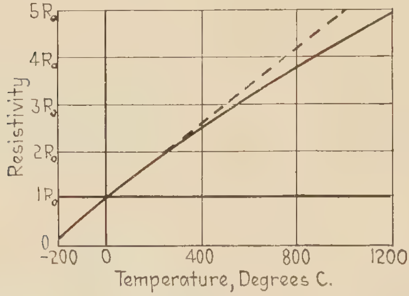


FIG. 135.—Resistivity-temperature characteristic of platinum.

The coefficient α_1 is called the **resistance-temperature coefficient** of the material **for the reference temperature** T_1. We see that it may be defined as follows:

175a. RESISTANCE-TEMPERATURE COEFFICIENT (DEFINITION). **The resistance-temperature coefficient of a specified substance for a reference temperature T_1, is equal to the increase in resistance per degree rise expressed in decimal parts of the resistance at the reference temperature T_1. In other words, the resistance-temperature coefficient is equal to the increase in the resistance of the substance per degree rise in temperature, divided by the resistance R_1 at the reference temperature.**

For the case in which 0°C. is used as the reference temperature, Eq. 236 reduces to

$$R = R_0(1 + \alpha_0 T). \tag{237}$$

The relation between the coefficient α_1 and α_0 is

$$\alpha_1 = \frac{\alpha_0}{1 + \alpha_0 T_1}. \tag{238}$$

Equations 236 and 237 may be used for a range of temperature of about 250° only. For a greater range of temperature, a formula of the type

$$R = R_0\,(1 + \alpha_0 T + \beta_0 T^2) \tag{239}$$

may be made to express with greater accuracy the resistance-temperature relation for metallic conductors.

The temperature coefficient for a given metal as obtained from different samples is found to vary slightly with such factors as the degree of purity of the sample, the method in which the sample has been worked, the annealing, etc. In Secs. 175*b* and 178 will be found the temperature coefficients of a number of substances. In Sec. 175*b* the coefficients are given for two reference temperatures, namely, 0 and 20°C.

175b. Resistance-temperature Coefficients.

Material	α_0	α_{20}
International annealed copper standard	+0.00427	+0.00393
Hard-drawn aluminum	+0.00423	+0.0039
Platinum	+0.0039	+0.00362
Very pure iron	+0.00625	+0.00555
Soft steel	+0.00424	+0.0039
Manganin alloy	+0.00002	+0.00002

An inspection of these tables will disclose the fact that the metallic elements all have positive temperature coefficients. For all of the common metals (save the magnetic metals) the temperature coefficient for a reference temperature of 0°C. is +0.0042. That is, for each degree rise in temperature, conductors made of commercially pure metals increase in resistance by approximately 0.4 per cent of their resistance at 0°C. If this straight-line relation held precisely for lower and lower temperatures, the resistance would become zero at −236°C. The absolute zero is −273°C. Experiments carried on at temperatures within 4° of the absolute zero show that the resistance of the conductors of the pure metals does approach zero as the absolute zero is approached. The ordinary range of temperature of the copper wires used in electric appliances is from a room temperature of

20 to 80°C. (the maximum permissible operating temperature for cotton insulation). At 80°C. the resistance of the wire is 24 per cent higher than at 20°. For use in resistance standards, ductile alloys have been developed which have temperature coefficients as low as +0.000005. For a range of 20°C. such a wire would not change in resistance by more than 0.01 per cent.

While carbon is classed as a metallic conductor, its temperature coefficient is negative. The resistance of the carbon filament of a carbon incandescent lamp at the operating temperature is about 0.3 of its resistance at room temperature. In striking contrast to this, the resistance of the tungsten filament at the operating temperature of the tungsten lamp is from 9 to 12 times its resistance at room temperature.

We will find that electrolytic, pyroelectric, and non-conductors invariably have negative temperature coefficients. In insulators like glass and porcelain, the resistance falls so rapidly with rise of temperature that these materials become fair conductors at temperatures in the neighborhood of 1000°C.

Consider the effect of the resistance-temperature coefficient of a wire upon the shape of its voltage-current characteristic. Imagine three wires a, b, and c having +, zero, and − temperature coefficients respectively. The greater the current passing through any of these wires the greater its temperature—unless special means are used to keep the wire at a constant temperature. Consequently, with increasing currents, the resistance of wire a increases, that of wire c decreases, and that of wire b remains constant. It may be readily seen from this that the voltage-current characteristic of the three wires will have the shapes shown in Fig. 136. The voltage-current characteristic of the wire with a positive temperature coefficient will be concave upward (toward the voltage axis), that of the wire with zero coefficient will be a straight line, and that of the wire with the negative coefficient will be concave downward (toward the current axis).

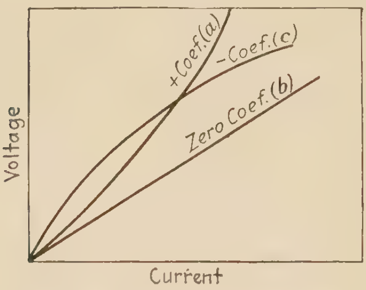


FIG. 136.—Effect of resistance-temperature coefficient on the shape of the voltage-current characteristic.

176. Resistance Thermometers.—Suppose the temperature coefficient of a small coil of wire is known, and that the resistance of the coil has been measured at some known temperature. Let this coil be inserted in a region of unknown temperature and allowed to come to the temperature of the region. If lead wires of low resistance extend from the coil to instruments outside the region, the resistance of the coil may be measured. By substituting, in Eq. (236) or (237), the new resistance and the previously measured reference resistance, the temperature of the coil may be computed. Although the resistance of the coil might be measured by the voltmeter-

ammeter method, it is usually measured by the comparison methods of the next chapter, because of the greater precision of the comparison method. A coil of wire used to measure temperature in this manner is called a **resistance thermometer.** The wire is usually of platinum, since this may be used at any temperature from -260 to $+1200°C$.

The change in the resistance of the windings is the method used to determine the average rise of temperature of the inaccessible copper windings of electric machines; that is, the winding itself is made the coil of the resistance thermometer.

177. Relation between the Resistance of a Cylindrical Conductor and Its Dimensions.—By experimental measurements, the law stated below has been found to express the relation between the resistance and the dimensions of any conductor which may be described geometrically as a **right cylinder.** A right cylindrical conductor is a conductor whose end or terminal surfaces are parallel planes, and whose lateral or insulated surfaces may be generated by the motion of a straight line which always remains perpendicular to the end surfaces. Rectangular blocks of metal are right cylinders, no matter what faces are considered as end surfaces. Any perpendicular cross-section of a right cylinder has exactly the same shape and area as the end surfaces.

177a. RESISTANCE-DIMENSION LAW (EXP. DET. REL.).—**The resistance R between the bases of a right cylindrical homogeneous conductor of length, l and of cross-sectional area a, is directly proportional to the length, inversely proportional to the cross-section, and also depends upon the nature of the material of which the cylinder is made, and upon its temperature.**

This law is embodied in the following equations:

$$R \text{ (of a cylinder)} = \frac{\rho l}{a} \tag{240}$$

$$G \text{ (of a cylinder)} = \frac{\gamma a}{l} \tag{241}$$

in which ρ and γ are constants whose values depend upon the material of the conductor and its temperature. It is evident that ρ and γ represent, respectively, the resistance and the conductance of a cylinder of the material of unit length and unit

cross-section. These constants are called the **resistivity** and the **conductivity** of the material (at the specified temperature).

177b. RESISTIVITITY AND CONDUCTIVITY (Definitions).—**The RESISTIVITY ρ of a given material at a specified temperature is the resistance, in ohms, between opposite faces of a centimeter cube of the material at that temperature. The unit of resistivity is called the OHM, CENTIMETER.**

The CONDUCTIVITY γ of a given material at a specified temperature is the conductance, in mhos, between opposite faces of a centimeter cube of the material at that temperature. The unit is called the MHO-CENTIMETER.

The unit of resistivity is sometimes called the "*ohm per centimeter cube.*" This name should not be used because the "per" seems to lead to the grossly erroneous notion that the resistance of a conductor may be calculated by dividing the resistivity of the material by the volume of the conductor in cubic centimeters!

It is evident that the conductivity and the resistivity of a given material are reciprocals one of the other.

$$\rho \text{ (ohm-cms.)} = \frac{1}{\gamma} \text{ (mho,cm)}. \tag{242}$$

It is likewise evident that the same empirical formulas which enable us to calculate the resistance of a conductor at any temperature from its known resistance at some reference temperature may be used to compute the resistivity of a material at any temperature from its known resistivity at a reference temperature. Thus, for use in resistivity calculations, Eqs. (236), (237), and (239) may be written in the forms

$$\rho = \rho_1[1 + \alpha_1(T - T_1)]. \tag{243}$$

$$\rho = \rho_0(1 + \alpha_0 T). \tag{244}$$

$$\rho = \rho_0(1 + \alpha_0 T + \beta_0 T^2). \tag{245}$$

There is no definite or sharp line of division between the materials classed as conductors and those classed as insulators, but there are all degrees of conductivity from the extremely high conductivity of silver and copper to the extremely low conductivity of air. This range is illustrated in the resistivities tabulated in Sec. 178.

178. Electrical Resistivities and Resistance-temperature Coefficients of Materials.

Material	Resistivity, ohm-centimeters at 20°C. except as noted	Resistance-temperature coefficient for 20°C.
Metallic conduction class:		
Silver	1.59×10^{-6}	+0.0038
Copper	1.724×10^{-6}	+0.00393
Aluminum	2.828×10^{-6}	+0.0039
Zinc	5.8×10^{-6}	+0.0037
Tungsten	5.6×10^{-6}	+0.0045
Iron	10×10^{-6}	+0.0059
Platinum	10×10^{-6}	+0.0036
Lead	22×10^{-6}	+0.0039
Mercury	95.78×10^{-6}	+0.00089
Tellurium		
Carbon (metallized lamp filament)	4.7×10^{-4}	+0.001
Carbon (amorphous lamp filament)	3.5×10^{-2}	−0.001
Graphite	3×10^{-4}	
Carbon (arc lamp)	5×10^{-3}	
(Resistor alloys)		
Manganin (Cu, 84; Mn, 12; Ni, 4)	4.4×10^{-5}	+0.000006
Constantan (Cu, 60; Ni, 40)	4.7×10^{-5}	+0.00001
Excello	9.2×10^{-5}	+0.00016
Nichrome	10.0×10^{-5}	+0.0004
Pyroelectrolyte class:		
Cast silicon	0.5	
Nernst filament (incandescent)	2	
Electrolytes:		
Sodium chloride (fused 750°C.)	0.29	
Potassium chlorate (fused 355°C.)	2.2	
HNO_3 (6 normal)	1.3	
KOH (7 normal)	1.9	
NaCl saturated	4.4	
$CuSO_4$ saturated	29	
Sea water	30 about	
River water	1000 to 10,000	
Ethel alcohol	3×10^{5}	
Water distilled	5×10^{5}	
Water distilled in vacuo	3.7×10^{6}	
Gaseous conductors:		
Mercury tube		
Carbon arc		
Insulators:		
Ice, −0.2C.	2.8×10^{8}	
Transformer oils	10^{13} to 10^{15}	
Paraffin oil	10^{16}	
Glass	10^{14} to 10^{16}	
Porcelain	10^{14}	
Hard rubber	10^{15} to 10^{18}	

179. Stock Sizes of Wire. The American Wire Gage.—Electric conductors are manufactured and carried in stock in certain standard sizes. These sizes, when less than ½ inch in diameter, are usually specified by gage numbers. Many arbitrary systems of gage numbers have been used by manufacturers, but at present only two of these systems are still in use in this country. The stock sizes of iron and steel conductors are still those of the Birmingham wire gage (B.w.g.). The diameters of successive sizes of wire in the B.w.g. change in an arbitrary manner. There is no simple relation between the diameter of one size and the diameter of any other. As a consequence, the wire tables in the engineering handbooks must be consulted for the properties of wires expressed in this gage.

The stock sizes in conductors of copper, aluminum, and the high-resistance alloys are those of a gage known as the American wire gage (A.w.g.), or the Browne and Sharpe gage, (B. & S.g.). The diameters of wires of the different gage numbers are usually listed in **mils** and the cross-sectional areas in **circular mils.**

The **mil** is a unit of length equal to 0.001 inch.

The **circular mil** is a unit of area equal to the area of a circle 1 mil in diameter. Wires have circular cross-sections, and the area of a wire D mils in diameter is D^2 circular mils.

$$1 \text{ circular mil} = \frac{\pi}{4} \text{ square mils} = \frac{\pi}{4} \times 10^{-6} \text{ square inches.}$$

$$1 \text{ square mil} = 10^{-6} \text{ square inches.}$$

Wires ½ inch in diameter and larger are specified in terms of their cross-sectional area in circular mils. A cylinder whose length is 1 foot and whose cross-sectional area is 1 circular mil is called a **circular mil-foot.**

1 *circular mil foot of copper, whose conductivity equals that of the international annealed copper standard, has a resistance of* 10.371 *ohms at* 20°*C.*

It follows that the resistance at 20°C. of any length of wire may be computed from

$$R \text{ (ohms)} = 10.371 \frac{l \quad \text{(feet)}}{a \text{ (circular mils)}} \qquad (246)$$

The electrical engineering handbooks contain copper wire tables showing the diameters, the cross-sectional areas, and the resistance and weight per thousand feet of the various sizes of wire in the A.w.g. These wire tables should be consulted for accurate information,[1] but when they are not available, the approximate values for any size may be determined from the law of the A.w.g., as explained below.

In the American wire gage, the numbers extend from 0000 (the largest size) through 000, 00, 0, 1, 2, etc. to 40 (the smallest). The diameter of 0000 was taken as 460 mils, and the diameter of No. 36 as 5 mils. The diameter of the other sizes were then fixed so that a constant ratio exists between the cross-sectional area of one size and that of an adjacent size. This ratio is very nearly the cube root of 2 (actually the cube root of 2.005), or 1.26. For a change of three sizes, then, the area and the weight per 1000

[1] See also *Copper Wire Tables* in Circular 31, Bureau of Standards.

feet will be multipled or divided by 2, the resistance will be changed inversely in the same ratio, and the diameter will vary as the square root of the area. By means of these relations, the constants of a wire of any size in the A.w.g may be readily calculated, if the constants of any one size are known. The constants of the No. 10 size are the most convenient to remember.

The diameter of a No. 10 *A.w.g. is approximately* 100 *mils* (*actually* 101.9), *the approximate cross-sectional area is* 10,000 *circular mils, the resistance per thousand feet at* 20°C. *is* 1 *ohm, and the weight per* 1000 *feet is* (*by chance*) 10π *pounds.*

In calculating from No. 10 to any other size, it is possible to jump by steps of three sizes, and factors of 2, either to the desired size or to one adjacent to it. Another "short-cut" relation may be remembered and made use of, namely, for any change of 10 sizes, the area and the resistance change by a factor of 10.16.

180. Distribution of Current in a Cylindrical Conductor.—From the fact that the conductance of a cylindrical conductor is directly proportional to the cross-sectional area, the conclusion may be drawn that each unit area of the cross-section carries as much current as any other unit area, or that **the current density is uniform over the whole cross-section of a cylindrical conductor carrying a steady current.**

Students often assume that electric current consists only of the motion of the excess charges of the conductor, and that, since the excess charges have been shown to reside on the surface, therefore an electric current is essentially a flow of electricity along the surface of conductors. This is erroneous. It is probably true that the excess electrons on the surface drift along the conductor under the electrostatic driving forces at the same speed as do the free electrons in the body of the material, but the number of excess electrons on the surface is negligibly small in comparison with the number of free electrons in the body. Consider a section 1 centimeter long of a conductor which, for example, we may take as 1 square centimeter in cross-section. Calculations show that the number of electrons on the surface cannot exceed 6×10^{10}, without a brush discharge into the air, while the estimated number of free electrons in the section is 10^{19} (Sec. 33), or at least 160,000,000 times as great.

Electrostatic effects are due to the excess electrons which reside on the surface of conductors, while electric currents in metallic con-

ductors are almost entirely the motion of the free electrons which are uniformly distributed throughout the body of the conductor.

181. Electric Power Conductors.—The desirable properties for the wires used to carry the currents in electric power circuits and in power generating, transforming, and utilizing machinery are as follows:

1. High tensile strength, accompanied by ductility in contrast to brittleness.
2. A hard surface which will not wear away under the abrading processes at the points of support.
3. Ability to resist corrosion or deterioration under the customary weathering processes.
4. High melting point, to withstand accidental arcs.
5. High volume conductivity, in order that the surface to be covered with insulating material, or that space occupied by a wire of given conductance per unit length may be low.
6. High mass conductivity (conductance per meter-gram), in order that the weight of a conductor of given length and conductance may be a minimum.
7. Low cost per mho unit length, in order that the investment in conductors may be a minimum.

The materials which best meet the above requirements are copper, aluminum, iron and steel, and some of the brasses and bronzes (copper alloys). Silver is the only material which has a conductivity greater than copper (17.5 per cent greater). The great cost of silver ($2700 per mho-kilometer, as against $50 for copper) precludes its use as a conductor. Iron and steel have a very limited field of use, namely, for "third-rail" conductors, which are subject to the abrading action of trolley shoes, or for river crossings requiring spans of such great length that the tensile strength of copper is inadequate, or for bare overhead transmission conductors where the current to be transmitted is so unusually small that the small copper conductors would have insufficient mechanical strength to withstand storm conditions.

181a. Copper Conductors.[1]—The conductor resistances tabulated in the copper wire tables are computed for conductors of the specified diameters having a certain standard resistivity. The standard resistivity now in use was adopted by the International Electrotechnical Commission in 1914 after extensive determinations of the resistivity of commercial annealed copper wire. It is called the **international annealed copper standard** and is defined as follows:

181b. INTERNATIONAL ANNEALED COPPER STANDARD (Definition).—*At a temperature of* 20°C., *the resistance of a wire of* **standard annealed copper** 1 *meter in length and of a uniform cross-section of* 1 *square millimeter is* $\frac{1}{58}$ *ohm, or* 0.017241 *ohm. The resistance-temperature coefficient is* 0.00393

at 20°C. *and* 0.00427 *at* 0°C. *The density of the copper is* 8.89 *grams per cubic centimeter at* 20°C.

Expressed in the various units of mass resistivity and volume resistivity, and of conductivity, the international annealed copper standard has the values

1.7241 microhm-centimeters at 20°C.
0.5800 megamho-centimeters at 20°C.
1.5328 ohms (meter, gram) at 20°C.
10.371 ohms (mil, foot) at 20°C.

181c. Per Cent Conductivity.—The conductivity of actual conductors of copper, aluminum, or steel is frequently expressed in per cent of the annealed copper standard, preferably for the temperature of 20°C. The highest conductivity found in any sample of annealed copper by the Bureau of Standards is 101.88 per cent. The average conductivity of samples representing 100,000,000 pounds of wire bar was 100.4 per cent. The conductivity of hard-drawn copper is about 97.3 per cent of annealed copper.

181d. Aluminum Conductors.[1]—The constants of commercial hard-drawn aluminum conductors at 20°C. are as follows:

Volume resistivity (microhm-centimeters)	2.828
Volume conductivity (megamho-centimeters)	0.3535
Volume per cent conductivity	61.0 per cent
Mass resistivity (ohms-meter, gram)	0.0764
Mass conductivity	200.7 per cent
Density, grams per cubic centimeter	2.7
Resistance-temperature coefficient	0.0039

181e. Iron and Steel Conductors.—All iron and steel conductors must be well galvanized to prevent rapid rusting. The constants of steel conductors are as follows:

Iron and steel	Ultimate tensile strength, pounds per square inch	Volume conductivity, per cent
E. B. B. iron	53,000	15.9
B. B. iron	60,000	13.9
Steel telegraph	66,000	11.8
Plow steel	200,000	9.5
Ordinary galvanized strand	51,000	14.
Siemens-Martin strand	86,000	10.6
High-strength strand	140,000	9.3
Extra-high-strength strand	210,000	8.2

181f. Comparison of Power Conductor Materials.—In the purchase of bare conductors for a power line of a given length, the buyer is primarily in the market for the electrical quantity, conductance. Of course, the superior mechanical properties of one conducting material over another for certain purposes may be such that the purchaser is warranted in paying more for one line than another, even though both have the same conductance. In the following table, the constants of the different conducting materials are listed, and a comparison is made of the relative weights, cost, and ultimate tensile strengths of lines of copper aluminum and steel, all having the same conductance as the copper line, which is taken as the standard of comparison. The engineering handbooks should be consulted for the properties of composite conductors, such as the copper-clad steel, and the steel-core aluminum cable.

COMPARISON OF POWER-CONDUCTOR MATERIALS

Material	Copper	Aluminum	B.B. iron	Steel telegraph
Density	8.89	2.7	7.75	7.79
Percentage conductivity	97	61	13.9	11.8
Cost per pound	$0.19	$0.35	$0.065	
Elastic limit	30,000	14,000		
Ultimate tensile strength, pounds per square inch	60,000	26,000	60,000	66,000

The relative properties of lines having a conductance equal to that of a copper line are (the prices fluctuate and may vary from those given by ± 30 per cent):

Section	1	1.59	7.0	9
Diameter	1	1.26	2.75	3
Weight	1	0.48	6.1	7.9
Cost	1	0.90	2.6	
Ultimate tensile strength	1	0.69	7	9.9

181g. Insulated Conductors.—In the purchase of insulated conductors the buyer is in the market for two electrical quantities, **conductance** and a certain type of **insulation** for a specified operating voltage between the wires. Except for the cheapest grade of insulation, the purchaser must pay more for **insulation** than he does for the conductance. This is shown in Fig. 137, in which curves are plotted, one for each type of insulation. The curves show as ordinates the **cost of a 1000-foot length per mho of conductance** for insulated conductors of the sizes indicated by the abscissa scale. The abscissa scale shows both the gage number of the wire and the con-

ductance in mhos of a 1000-foot length of each gage number. For example, the No. 4 A.w.g. code rubber-covered wire costs $14.10 per mho for a 1000-foot length, while the bare No. 4 wire costs $7.20 per mho; that is to say, in purchasing No. 4 code rubber-covered wire, the buyer pays $6.90 for insulation for each $7.20 he pays for conductance. The abscissa scale shows that the conductance of No. 4 wire is (approximately) 4.0 mho per 1000 feet. Hence the cost of No. 4 wire is 4.0 × $14.10 = $56.40, per 1000 feet.

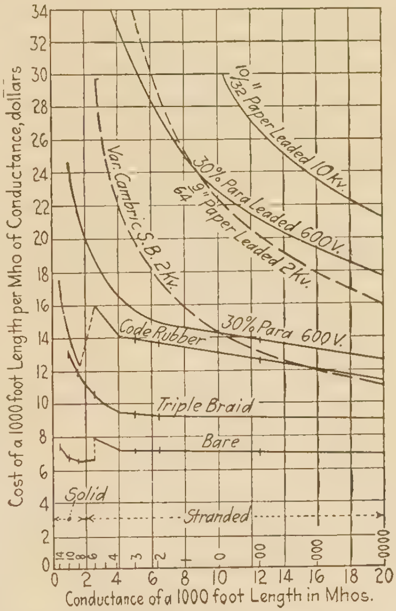


FIG. 137.—Wire costs (February, 1925).

182. Electric Resistor Materials. The desirable property in the wires used in resistance coils, rheostats, and heating appliances is high resistivity, and not high conductivity, as in power conductors. The wires used in resistance coil standards must have an extremely low temperature coefficient and must be electrically stable, that is, the resistivity must not drift with the passage of years. The wires used in heating appliances must have a high melting point and must be chemically and structurally stable at high temperatures; that is, they must not become brittle when operated for long periods at high temperatures. The properties of some of the commercial resistor wires are listed in Sec. 178. These wires are alloys of Ni-Cu or Ni-Cu-Zn, or Ni-Cr, or Ni-Mn-Cu or Ni-Fe-Cr. No ductile alloys have been developed which have a resistivity greater than 70 times that of copper.

Carbon or graphite blocks are used as the resistance materials in the **carbon compression rheostat.** In this rheostat the resistance is partly localized in the vicinity of the limited areas of contact between adjacent blocks. By varying the mechanical pressure on a column of blocks, the area of contact is varied, and thus the resistance of the column is controlled.

183. Considerations Serving to Determine the Size of the Conductor to Use for a Specific Purpose.—In every specific case where power is to be transmitted from one point to another, the following question arises: Power is obtained at a point A and costs at this point a definite amount per kilowatt-hour (and the cost per kilowatt-hour at A may be a constant or it may be a variable, depending upon the amount of power delivered at A). The power is to be transmitted to the point B to render a definite service—

for example, to run a motor, to light a building or a specific lamp, or to be resold. Considerations foreign to this inquiry have resulted in the selection of a definite transmission voltage which is to be maintained approximately constant at either A or B. The question that arises is, What size of conductor should be used?

A specific example of this is the following:

Problem.—What size of wire should be used to feed a 250-volt direct-current motor located 750 feet from the point where energy is received and paid for, the motor delivering 7.5 kilowatt mechanical power 8 hours per day, 300 days in the year, at an efficiency of 85 per cent. Power is delivered at 250 volts and costs 2 cents per kilowatt-hour. Assume weatherproof wire costing 15 cents per pound is to be used, and that the annual charges upon the investment in the conductors are 6 per cent for interest, 1 per cent for taxes, and 3 per cent for depreciation.

Like examples on a larger scale are encountered by the power-transmission companies in the vicinity of Niagara Falls which purchase power at a definite rate per kilowatt-year at the high-tension bus bars of the generating companies and transmit it to various points, where it is resold to public utilities or to communities. The more complex questions encountered in proportioning feeders and networks for transmitting and distributing power over extensive areas resolve into elementary problems of the nature of the problem outlined above.

The proper size of conductor to use to perform a **specified** service (by a specified service is meant the supply at known points and at a specified voltage of specified amounts of power, which power may vary in a specified way throughout the day, the week, and the year) is the smallest conductor which will satisfy **all** of the following requirements:

1. The mechanical strength must be adequate for the service.

2. The current-carrying capacity of the wire must be such as to avoid a hazardous rise in the temperature of the wire.

3. The variation in the wire-to-wire voltage

a. between the extreme points of supply, and

b. at any point of supply (due to variations in the load) must not exceed the percentage limits set for voltage variation by good engineering practice.

4. The annual cost of rendering the service must be the minimum which is consistent with requirements 1, 2, 3, and 5.

5. The practical requirements of operation and construction must be met.

The following brief comments on each of these requirements will be illustrated by considering their application to the problem stated above.

183a. Adequate Mechanical Strength.—Very rarely, except in short lighting circuits, is the size of the wire fixed by this consideration. The requirement in the National Electric Code[2] that no wire smaller than No.

[2] The National Electric Code is a set of regulations relating to electric wiring issued by the National Board of Fire Underwriters. Many cities have passed ordinances requiring that all new electric work shall conform to the requirements of the Code (see the engineering handbooks).

14 A.w.g. shall be used in interior wiring is based largely on the consideration that the mechanical strength of smaller wires is inadequate. The committee on overhead construction of the National Electric Light Association has recommended that no wire having a breaking strength less than No. 6 soft-drawn copper shall be used for pole-line spans.

The calculation of the tension and the sags allowed in stringing overhead lines, which is based on the consideration of adequate mechanical strength, is outside the scope of this text.

Application.—If the line supplying the motor is strung from joists indoors, No. 14 A.w.g. conductors will meet the mechanical strength requirement; if strung on a pole line out of doors, nothing smaller than Nos. 6 or 8 should be used.

183b. No Hazardous Temperature Rise.—In the case of bare conductors, the hazard referred to is the danger of setting fire to inflammable materials which may come into contact with overheated wires. The size of bare conductors is rarely fixed by this consideration. In insulated conductors the hazard is the deterioration of the insulation which occurs when the temperature of the wire exceeds certain limits. This deterioration occurs at a temperature much lower than that constituting an actual fire hazard. The allowable current-carrying capacities of insulated conductors of each size are given in a table entitled "Allowable or Safe Current-carrying Capacities of Insulated Copper Wires" embodied in the National Electric Code. The allowable currents listed therein for rubber-covered wires will cause the copper to rise 15° above the surrounding air. Such a small rise constitutes in itself neither a fire hazard nor a menace to the integrity of the insulation. The rise at the rated current is limited to 15°, because circuits are so frequently overloaded by the connection of additional flatirons, etc., which were not contemplated in the original layout of the circuit. If the circuit is

ALLOWABLE OR SAFE CURRENT-CARRYING CAPACITIES OF INSULATED COPPER WIRES

Size, A.w.g.	Amperes	
	Rubber insulation	Other insulations
14	15	20
12	20	25
10	25	30
8	35	50
6	50	70
4	70	90
2	90	125
0	125	200
000	175	275

overloaded until the current reaches twice the rated current, then the I^2R loss will be four times as great as the contemplated loss, and the temperature rise will be $4 \times 15°$, or $60°$.

Application.—Since the motor delivers 7.5 kilowatts at an efficiency of 85 per cent, it must receive $7.5 \div 0.85 = 8.8$ kilowatts of electric power. If we assume that the drop in voltage in the 750-foot line is 5 per cent, or 12.5 volts, the power is delivered to the motor at $250 - 12.5 = 237.5$ volts. Therefore, the current must be $8800 \div 237.5 = 37$ amperes. Referring to the carrying-capacity table of the Code, it will be found that the allowable carrying capacity of No. 10 weatherproof wire is 30 amperes and that of No. 8 is 50 amperes. Therefore, no stock size of wire smaller than No. 8 will meet the temperature hazard requirement.

183c. Voltage Variation Must Not Be Excessive.—If the voltage at the generator end of a system is held constant, the voltage at the appliances at the different outlets on the system varies because of the variable IR drop in the wires from the generator to the outlet, the variation being due to the variation in the current caused by the variation in the demand for power. Now the performance of the power-utilizing devices, such as lamps, motors, and heating devices, is dependent upon the voltage impressed across the terminals of the device. For example, if a tungsten lamp is operated at a voltage which is 5.0 per cent above its rated voltage, its candlepower rises 20 per cent above the normal and its life is reduced to 50 per cent of its normal life. In good engineering practice, the attempt is made to use large enough conductors so that the drop in voltage in each part of the system from the central station to the farthest lamp of the most distant customer does not exceed the following per cent of the wire-to-wire voltage:

	Per Cent
High-tension feeders from station to a load center....	3–8
High-tension mains..............................	1
Step-down transformer..............................	1.3
Low-tension mains..............................	2
Service wires from pole to house....................	0.5
Customers' mains..............................	0.5
Customers' branch circuits........................	1.5

Application.—In the motor circuit (with no lamps on the same circuit) a voltage drop as large as 5 per cent would be permissible. If the drop in a circuit carrying 37 amperes (approximately) is not to exceed 5 per cent or 12.5 volts, the resistance of the 1500 feet of wire must not exceed $12.5 \div 37 = 0.34$ ohms. The resistance of 1000 feet must not exceed $0.34 \times 1000/1500 = 0.22$ ohms. The resistance of 1000 feet of No. 10 wire is 1 ohm, of No. 7 it is 0.5 ohm, of No. 4 it is 0.25 ohm, and of No. 3 it is 0.2 ohm. Therefore, to meet the requirement for good voltage regulation, nothing smaller than a No. 3 wire should be used.

183d. Economic Considerations.—The items which go to make up the annual cost of power at the utilization end B of a circuit are:

1. The annual charge for interest, tax, and depreciation on the original investment.

2. The annual amount paid for the power lost in transmission due to the I^2R loss.

3. The annual amount paid for useful power, that is, for power received at A and actually delivered at B.

4. The annual labor expenses incurred in operating and maintaining the circuit.

Of these four items, the values of the last two are substantially unaffected by the size of wire. Only the first two items are functions of the size of the wire used to transmit the power. The larger the wire used to transmit a given current the smaller will be the I^2R loss, or the lower will be the cost per annum of lost power, but the greater will be the annual charge for interest and depreciation on the investment. The smaller the conductor, the lower will be the interest and depreciation charge, but the greater will be the cost of lost power. Eonomy in transmission would dictate the use of a conductor of the size for which the sum of interest, tax, and depreciation charge plus the cost of the lost power is a minimum.

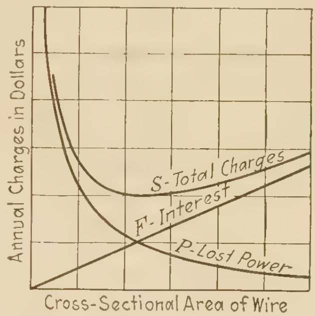


FIG. 138.—Relation between the annual charges in transmitting power and the area of the wire.

Figure 138 shows the manner in which items 1 and 2 will vary with the cross-sectional area of the wire which is used. The weight and the cost of a bare conductor of a given length are directly proportional to the cross-sectional area of the wire. Therefore, the annual charge F for interest, depreciation, and taxes will be directly proportional to the cross-sectional area a, as shown by the straight line marked F.

$$F = Ka.$$

Since the resistance of the line is inversely proportional to the cross-sectional area of the wire, the energy lost as I^2R loss, and its cost, will be inversely proportional to the area a, as shown by the hyperbola marked P.

$$P = \frac{K_1}{a}.$$

The sum of these two items is shown by the curve marked S, which rises to infinity as the cross-sectional area becomes infinitesimally small or infinitely great, and has a broad minimum. For bare conductors, we may in the following manner derive a law which gives the cross-sectional area for which S is a minimum.

$$S = F + P = Ka + \frac{K_1}{a}.$$

To find the value of a which will make S a minimum, we take the derivative of S with respect to a, equate the derivative to zero, and solve the equation for a

$$\frac{dS}{da} = K - \frac{K_1}{a^2}.$$

By equating the derivative to zero, we obtain the following statement of the condition under which S will be a minimum:

$$K - \frac{K_1}{a^2} = 0$$

or

$$\frac{K_1}{a} = Ka \quad \text{or} \quad F = P \tag{247}$$

The relation expressed in Eq. (247) was pointed out by Kelvin, and it is generally known as Kelvin's law. It may be stated in the following words:

183e. Kelvin's Law for the Most Economical Size of Bare Conductor.—*The total annual charges for transmitting electric power by bare conductors from one point to another will be a minimum if the cross-section of the wire is such as to make the annual cost of the lost power just equal to the annual charge for interest, depreciation, and taxes on the investment in the conductors.*

Kelvin's law does not apply rigorously to insulated wires, because the cost of insulated wire is not directly proportional to the cross-sectional area. However, it is a good guide as to the most economical size of insulated wire.

Application.—The annual charges may first be computed for some size of wire which may seem reasonable (say No. 3) in the following manner:

1500 feet No. 3 has 0.3-ohm resistance and weighs (from manufacturer's tables) 309 pounds.
Investment = 309 × \$0.15 = \$46.40.
Annual charge on investment, 10 per cent = \$4.64
Power lost per annum = $37^2 \times 0.3 \times 300 \times 8 \div 1000 = 980$ kilowatt-hours.
Cost per annum of lost power = \$19.60.

The cost of the lost power is approximately four times the fixed charge, or the No. 3 wire is too small. By making use of the simple relations between the resistances of the different sizes in the A.w.g., the items appearing in the first line of the following table may be written without further computations. The items in the second line must be computed from a table of weights of weatherproof wire.

The annual charge is seen to be a minimum for the No. 0 wire. It should be noted, however, that wires one or two sizes on either side of the most economical may be used without causing the annual charges to exceed the minimum charges by more than a few per cent.

Comparison of Annual Charges

	A.w.g. No.				
	3	1	0	00	0000
For lost power.	$19.60	12.35	9.80	7.77	4.90
For interest, depreciation, and taxes.	4.64	7.38	9.55	11.75	18.00
Total.	24.24	19.73	19.35	19.52	22.90
Per cent of minimum.	125	102	100	101	118

183f. Practical Considerations.—In fixing on the size of wire to use, one must not be guided by the above considerations alone, but must bear in mind such practical considerations as the following:

a. Wires of the odd gage numbers higher than No. 6 are rarely carried in stock.

b. If the wire is being used in a new enterprise which may fail because of lack of capital to carry it through the early development period, it is advisable to use a wire smaller than the most economical. This leaves more capital available for emergencies, without greatly increasing the annual operating expense.

c. On any job it is advisable to keep the number of different sizes of wire small by adopting a few sizes as standard.

184. Economic Current Densities.—The current density which is the most economical to use in bare copper may be calculated in the following manner:

A cubic centimeter of copper weighs 8.89 grams and costs (with copper at $0.15 per pound) 0.29 cent.

The annual charge on a cubic centimeter (at 6 per cent for interest, 1 per cent for taxes, and 3 per cent for depreciation) is 0.029 cent.

If the current density in copper is J amperes per square centimeter, and the copper carries this current density 24 hours per day, or 8760 hours per year, the loss per cubic centimeter per annum will be

$$W \text{ (kilowatt-hours)} = J^2\rho\frac{8760}{1000} = J^2\frac{1.724}{10^6} \times \frac{8760}{1000}$$
$$= J^2\, 1.51 \times 10^{-5}.$$

In practice, the loss per annum in transmission-line conductors would rarely exceed 30 per cent of the value given above, because the current does not remain constant at the peak-load value but drops off during a part of the day. Assuming only 30 per cent of the above loss, and assuming further that energy costs 1 cent per kilowatt-hour, the value of the energy lost per cubic centimeter of wire is

Cost of lost power per annum $= J^2\, 4.5 \times 10^{-6}$ cents.

To find the value of the current density which is most economical, we equate the above expression for the cost of the lost power in a cubic centimeter to the annual interest and depreciation charge on a cubic centimeter of copper, and solve the equation for J:

$$J^2\, 4.5 \times 10^{-6} = 0.029$$
$$J = 73 \text{ amperes per square centimeter}$$

That is to say, for the assumed conditions, namely, 15-cent copper, 10 per cent fixed annual charges, 1-cent power, and an average loss equal to 30 per cent of the loss at peak load (or a 30 per cent loss factor), the economic current density based on the peak-load current is 73 amperes per square centimeter.

If the cost of copper or the percentage charge is greater, or if the cost of power or the loss factor is lower, the economical current density will be higher. Likewise, if the wire is insulated or is used as a coil around an iron core, the economic current density will be higher. Under extreme and unusual conditions, the economic peak-current density may be as low as 25 or as high as 500 amperes per square centimeter. A current density of 73 amperes would mean a current of 1.5 amperes in a No. 14 wire and of 78 amperes in a No. 0000 wire. The allowable carrying capacities of rubber-covered wires of these sizes are 12 and 210 amperes, respectively. From this, it is evident that the above economical current density would cause a rise of temperature of $(1.5/12)^2$ of 15, or 0.24°C. and $(78/210)^2$ of 15, or 2.0°C., respectively. From this it appears that only for the conductors necessary to carry large currents will the size be determined by the heating rather than by the economic consideration.

185. Relation between Thermal and Electrical Conductivities. As a result of an experimental comparison of the thermal and electrical conductivities of materials, Wiedemann and Franz announced the following empirical relation.

185a. Wiedemann-Franz Relation (EXP. DET. REL., 1853).—*At any given temperature, the ratio of the thermal to the electrical conductivity has the same value for all good conductors.*

From theoretical considerations similar to those presented below, Lorenz in 1872 announced the following relation:

185b. Relation of Lorenz (DEDUCTION, 1872).—*For good conductors, the ratio of the thermal to the electrical conductivity is proportional to the absolute temperature at which the ratio is taken.*

The following table (from the experiments of Jaeger and Disselhirst) contains in the second column the ratio of the conductivities

at 18°C. as determined for a number of metals, and in the third column the temperature coefficient of the ratio for a reference temperature of 0°C. The temperature coefficient as predicted by the Lorenz relation should have the value $1/273 = 0.0036$. Thermal conductivities are expressed in joules, cm., degree centigrade, and electrical conductivities in mho-cms.

Material	Ratio of thermal to electrical conductivity at 18°C.	Temperature coefficient of this ratio for 0°C.
Silver	6.86×10^{-6}	0.0037
Copper	6.65×10^{-6}	0.0039
Aluminum	6.36×10^{-6}	0.0043
Zinc	6.72×10^{-6}	0.0038
Nickel	6.99×10^{-6}	0.0039
Platinum	7.53×10^{-6}	0.0046
Mercury	7.9×10^{-6}	
Iron	8.02×10^{-6}	0.0043
Steel	9.03×10^{-6}	0.0035
Constantan (60 Copper, 40 nickel)	11.0×10^{-6}	0.0023
Carbon	2×10^{-4}	
Glass	10^{12}	

These relations hold only in good metallic conductors. The correspondence between the thermal and electrical conductivities leads us to attribute the two effects to the same cause. Since we conceive that the electrical conductivity is to be accounted for in terms of the flow of the free electrons, we infer that the conduction of heat in good thermal conductors is due almost entirely to the interdiffusion of electrons between hot and cold portions of the metal. The following approximate analysis, based on the assumptions that all electrons have the same free path L, the same velocity of thermal agitation V, and that directed velocities of drift acquired in the electric field in one path do not carry over into the next, enables us to derive expressions for the two conductivities.

If the electric intensity in the conductor is F, the force acting on the electrons in the direction of the intensity is Fq, and the acceleration a is Fq/m; in which q and m represent the charge and mass (in gram-sevens) of the electrons. Therefore, the directed velocity of drift V_d which is superimposed on the random velocities of thermal agitation is $\frac{1}{2}at$, in which t is the time taken to transverse the path L. But if the velocity of thermal

agitation V is large in comparison with the superimposed velocity, $t = L/V$. Whence

$$V_d = \frac{at}{2} = \frac{Fq}{2m}\frac{L}{V}. \tag{248}$$

Hence if N represents the number of free electrons per cubic centimeter, the net number which in unit time drift across a unit area taken perpendicular to F is $\frac{NFqL}{2mV}$.

The quantity of electricity crossing unit area per second, or the current density J, by reason of this drift is

$$J = \frac{NFq^2L}{2mV} \tag{246}$$

and the conductivity γ of the material is

$$\gamma \text{ (mho-cms.)} = \frac{Nq^2L}{2mV} \frac{\text{(coulombs, cms.)}}{\text{(gram-sevens, seconds)}}. \tag{250}$$

The following analysis, equally naive in its assumptions, yields an expression for the thermal conductivity. Consider a square centimeter taken perpendicular to the temperature gradient in a metal. Let the temperature at this isothermal surface be T, and let the temperature gradient be represented by G. The number of electrons crossing this area per second in each direction is $\frac{1}{6}NV$ (see Meyer's *Kinetic Theory of Gases*).

Those crossing from hot to cold had their last collisions at a point where the temperature is $T + LG$, and those moving in the opposite direction, where the temperature is $T - LG$.

Now from the kinetic theory of gases, the energy per unit volume in a gas $= 3P/2 = 3RT/2$. Therefore, the kinetic energy w of an electron coming from a region of the temperature T will be obtained thus

$$N(w) \text{ or } N\left(\frac{1}{2}mV^2\right) = \frac{3RT}{2}. \tag{251}$$

The kinetic energy W_h carried per second from the hot region to the cold across the square centimeter will be

$$W_h = \frac{1}{6}NVW = \frac{NV}{6}\frac{3R}{2N}(T + GL),$$

the energy carried from the cold to the hot region will be

$$W_c = \frac{NV}{6}\frac{3R}{2N}(T - GL),$$

and the net amount of energy W transported per second across unit area will be

$$W = W_h - W_c = \frac{RVGL}{2}$$

and the thermal conductivity K in joules per sq. cm. per degree Centigrade will be

$$K = \frac{W}{G} = \frac{RVL}{2}. \tag{252}$$

The ratio of the thermal to the electrical conductivity at the temperature T is

$$\frac{K}{\gamma} = \frac{RmV^2}{Nq^2}.$$

Upon substituting the value of mV^2 from Eq. (251), this becomes

$$\frac{K\,\text{(joules, cms.)}}{\gamma\ \text{(mho-cms.)}} = \frac{3R^2T\ \text{(degree Kelvin)}}{N^2q^2\ \text{(coulombs)}} \tag{253}$$

in which $R = 3.718 \times 10^{-4}$ (dyne-sevens, cms., degrees Centigrade)
$N = 2.705 \times 10^{19}$
$q = 1.591 \times 10^{-19}$ (coulombs).

Whence $$\frac{K}{\gamma}\ (\text{for } 18°\text{ C.}) = \frac{6.5}{10^6}.$$

The experimentally determined values given in the preceding table will be seen to be in remarkably close agreement with this value.

186. Voltage-current Characteristics of Electrolytic Cells.—The chemical effects which occur at the electrodes when current passes through an electrolyte have been discussed in Secs. 122–126 and 158–165. The conducting properties of electrolytes may be studied by means of the voltage-current characteristics of electrolytic cells. It is found that there are characteristics of the three different types illustrated by the curves A, B, and C of Fig. 139. Each one of these characteristics is typical of an important commercial application of electrolysis.

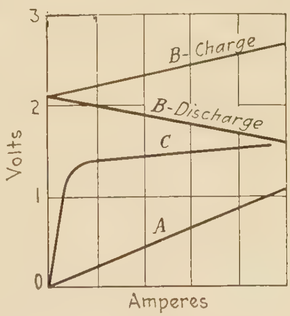


Fig. 139.—Voltage-current characteristics of electrolytic cells.

Characteristic A is that of an electrolytic refining bath or an electroplating bath. This case is distinguished by the fact that the chemical change produced at one electrode is the opposite of that produced at the other and there is very little change in the electrolyte. For example, in a copper sulphate ($CuSO_4$), copper refining bath, copper is deposited on one electrode and taken from the other. The concentration of the solution remains about the same. If the electrolyte is stirred, the voltage-current characteristic of such a refining bath is found to be practically a straight line similar to the characteristic of a metallic conductor. The ratio E/I is, therefore, again called the **resistance** of the cell.

The second typical voltage-current characteristic as shown by curve B is that of an electric battery. In this case the electrodes are dissimilar, and because of the chemical reactions, the cell is a source of intrinsic e.m.f. The characteristic curve is shown for current flowing through in the direction of its e.m.f. (battery discharge) and for current in the opposite direction (battery charge). The curves are found to be practically straight lines over the useful range of current values. The equation of the characteristic curves may be written as the equation of a straight line with an intercept E_0 and a slope $\pm R$.

$$E = E_0 \pm RI. \tag{254a}$$

The absolute value of the current is represented by I.

In Eq. (254a) the same positive value is assigned to I, no matter what the direction of flow, and the effect of the direction of flow is taken into account by using the proper sign between the terms. A better way to take directions into account is to indicate arbitrarily one direction through the cell as the arrow direction and to let the symbols E and I stand for the e.m.f. and the current in the arrow direction. With these conventions (see Sec. 171 and 198) Eq. (254) is always an expression of the relations.

$$E = E_0 - RI. \tag{254}$$

In Eq. (254) the intercept E_0 is the value of the e.m.f. due to the chemical forces, or actions in the battery. The coefficient R is called the resistance of the battery; the term $-RI$ then is the e.m.f. of resistance. With this interpretation Eq. (254) is a statement in a special case of the general law that "the potential increment E along a path is equal to the algebraic sum of the e.m.fs."E and $-RI$.

The straight-line relations shown in curves A and B are obtained by closing the switch with the circuit adjusted to send a current approximating some desired value through the cell, and then taking a reading of the current through the cell and of the voltage across it before the flow of the current has given rise to any appreciable change in concentration at the electrodes. The circuit is then opened, adjusted for another current value, a second reading is taken, and so on. If the current is allowed to flow for some time between readings, the characteristics depart

somewhat from the straight line. This departure is due to the change with time in the concentration of the electrolyte near the electrodes. Such changes in concentration produce not only a change in the e.m.f. of the cell, but also a change in the conductivity of the electrolyte.

The third voltage-current characteristic C is that of a cell containing similar electrodes but used for the electrolytic production, let us say, of hydrogen and oxygen. It may contain electrodes of nickel or of iron in a solution of sodium hydroxide, or of platinum in dilute sulphuric acid.

The characteristic is obtained not by taking readings before polarization has occurred, but only after the current has continued long enough at each value to bring the cell into the "steady state" in which further change in concentration does not occur, because the processes of diffusion just balance the effect of electrolytic transport. The portion of the curve above the knee is substantially a straight line, which, if projected back, intercepts the voltage axis at a voltage E_0 which is somewhat higher than the e.m.f. necessary to supply the energy required to dissociate water, namely, 1.45 volts.

Upon first closing the circuit through the cell with, say, a 1-volt source in series with it, a current perhaps 20 or more times that shown by the curve for 1 volt flows. As the cell polarizes, this current decreases to the steady state value. The currents which flow at voltages below the voltage E_0 may be called "diffusion" currents, since an infinitesimal current would be sufficient to keep the cell polarized at these voltages were it not for the processes of diffusion. For the portion of the characteristic above the knee of the curve, the ratio $\Delta E/\Delta I$ is a constant. This ratio is taken to be the resistance of the cell.

When the resistance of an electrolytic cell is defined as outlined above, it is found that "the rate P at which heat is developed in the cell is equal to the resistance of the cell times the square of the current traversing the cell." That is to say, Joule's law applies to electrolytic as well as metallic conductors.

187. Conductivity of Electrolytes.—When the resistance of an electrolytic conductor is defined as in Sec. 186, it is found that the **resistance-dimension** law of Sec. 177*a* applies likewise to

electrolytic conductors. Consequently, the terms "resistivity" and "conductivity" are also used in connection with electrolytes.

The customary way of avoiding polarization effects in measuring the resistance of an electrolyte is to place it in a glass cell having platinum electrodes, and then to determine its resistance by the Wheatstone bridge method of Sec. 208, using an alternating current of such high frequency that no appreciable changes in concentration occur in any half cycle.

187a. Effect of Temperature.—It is found that the conductivity of electrolytes for quite a range of temperature not too close to the freezing or boiling points of the solutions increases with temperature in accordance with the straight-line relation

$$\gamma_T = \gamma_{18}[1 + \alpha(T - 18°)] \qquad (255)$$

in which, the conductivity-temperature coefficient has the value 0.02 to 0.025 for salts and bases, and 0.01 to 0.016 for acids.

187b. Effect of Concentration.—The manner in which the conductivity of some of the best conducting electrolytes varies

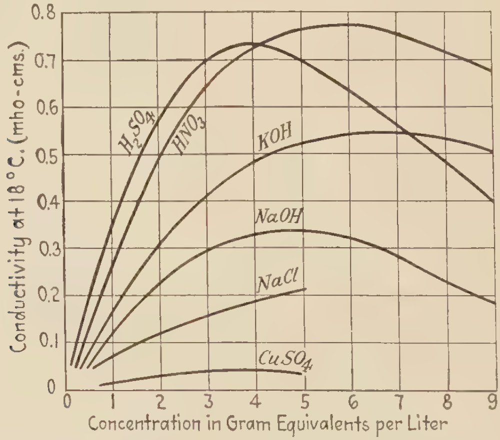


Fig. 140.—Variation of conductivity with concentration.

with the concentration is shown in Fig. 140. It is seen that in dilute solution the conductivity is roughly proportional to the concentration. The relation between the conductivity and the concentration of any substance is brought out most clearly through the notion of **equivalent conductivity,** defined as follows:

187c. DEFINITION.—By the EQUIVALENT CONDUCTIVITY Λ of a given electrolytic solution is meant the conductivity of the electrolyte per gram-equivalent of the salt, acid, or base contained in a centimeter cube of the solution.

$$\Lambda \text{ (mho-cms. per gram-equiv. per cu. cm.)} = \gamma v_g, \tag{256}$$

in which v_g represents the volume of the electrolyte containing one gram-equivalent of the solution.

In Sec. 187*d* are tabulated the equivalent conductivities of a number of substances at different concentrations. As the concentration decreases, the equivalent-conductance of a given substance increases, and approaches as a limit a definite value which is characteristic of the substance. This limiting value Λ_∞ is called the **equivalent-conductivity** of the substance at **infinite dilution.**

187d. Equivalent-conductivites at Different Concentrations.

Volume in cubic centimeters containing 1 gram-equivalent	KCl	$AgNO_3$	H_2SO_4	HCl	KOH	Acetic acid
∞	130.1	115.8	383	380	238.7	350
10^6	127.3	113.2	361	377	234	41
10^5	122.4	107.8	308	370	228	14.3
10^4	112	94.3	225	351	213	4.6
10^3	98.3	67.8	198	301	184	1.32
2×10^3	92.6	55.8	183	254	160.8	0.80

An intercomparison of the equivalent-conductivities of a number of substances having the same anion but different cations, and of substances having the same cation but different anions, shows that it is possible to assign to each ion equivalent-conductivities which are characteristic of it at each concentration and temperature (of solution) and which conform to the following principle:

187e. (LAW).—**The equivalent-conductivity of a substance at any concentration is the sum of the equivalent-conductivities of the ion constituents at that concentration.**

$$\Lambda = \Lambda_a + \Lambda_c. \tag{257}$$

The **equivalent-conductivities** at **infinite dilution** of a number of ions are listed in Sec. 187*f*.

187f. Equivalent-conductivities at Infinite Dilution of Ions.

Cations	18°C.	25°C.	Anions	18°C.	25°C.
H^+	313	347	OH^-	173	195
Na^+	43.2	50.6	Cl^-	65.3	75.2
K^+	64.3	74.2	NO_3^-	61.7	70.6
Ag^+	53.9	62.6	SO_4^{--}	67.7	78.4
Zn^{++}	46	54			
Cu^{++}	46	54			

188. Voltage-current Characteristics of Pyroelectric Conductors.—A number of substances which at room temperatures are non-conductors, or very poor conductors, become fair conductors when heated to a red heat or higher. Such substances have been termed **pyroelectric conductors.** They include silicon, boron, magnetite, glass, and many porcelain-like substances made up of suitable vitrified mixtures of basic and acid oxides. The features of the voltage-current characteristic of these materials are illustrated by curves *A* and *B* of Fig. 141.

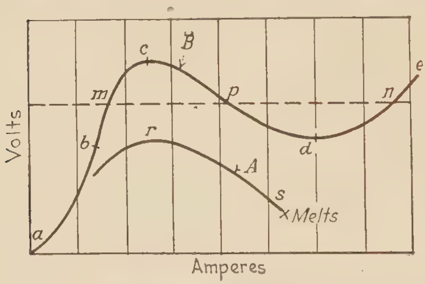


FIG. 141.—Voltage-current characteristics of pyroelectric conductors.

It should be understood (*a*) that the materials show these characteristics only when they are allowed to heat under the action of the current, (*b*) that with increasing current values the temperature of the conductor becomes higher and higher, and (*c*) that over the portion of the characteristic corresponding to the higher current values the substance may be at a red heat or even at an incandescent temperature. In fact, characteristic *A*, which is that of the glower, or light-emitting filament, of the Nernst lamp, can be obtained only by heating the filament with a flame to a yellow heat and then maintaining it at an even higher temperature by the continued passage of current through it.

Characteristic *B*, which is obtained with only a few substances, such as rod made up mainly of magnetite (Fe_3O_4), exhibits the following features: Over the rising portion *ab* the resistance-temperature coefficient is +; at *b* the temperature-coefficient passes through zero and at increasing temperatures assumes higher and higher negative values until at *c* the resistance is decreasing so rapidly with the increased temperatures which result from increased currents that any further increase in current is accompanied by a decrease in the e.m.f of the impeding impacts. At temperatures higher than *c*, the temperature coefficient first assumes higher negative values and then starts to decrease in value until at *d* it has become so small that the voltage again increases with increasing current. In many substances the rising portion *de* is not observed but the characteristic continues as a falling characteristic until the conductor is destroyed by melting or vaporizing.

Over the rising portions *ac* and *de*, the operation of such a conductor when connected across the terminals of a source which **tends to maintain a constant terminal voltage** is stable. That is to say, if the conductor is operating at any point, as *m* or *n*, on the rising portions of the characteristic, and if the voltage of the source increases or decreases by a slight amount, the current through the conductor likewise increases or decreases by only a slight amount.

Over the falling portion *cd* of curve *B* or *rs* of curve *A*, the operation of the conductor **when connected across a constant voltage source** is unstable. That is to say, if the conductor is operating at the point *p* and if the voltage of the generator decreases momentarily, or if the conductor is cooled slightly by blowing on it, the current starts to decrease. This causes a further cooling of the conductor, with an accompanying increase in resistance, and the current continues to decrease until the point *m* is reached on the rising part of the characteristic. On the other hand, if, when operating at *p*, the voltage of the generator rises momentarily, or if the temperature of the conductor is caused to increase slightly by shielding it from air currents, the current starts to increase and continues to increase until the point *n* is reached. If the characteristic did not have the rising portion *de*, the current would continue to increase until it

destroyed the conductor by melting or by vaporizing it. The current and voltage readings for plotting the unstable or falling portion of the characteristic can only be obtained by connecting the conductor in series in a circuit having such characteristics that it tends to maintain a constant current.

If a pyroelectric conductor is to be operated on the unstable part of the characteristic from a constant voltage source, it must have a "ballasting" metallic resistance of such magnitude (preferably having a large positive temperature coefficient) connected in series with it that the voltage-current characteristic of the combination will have no unstable portion.

189. Leakage Conduction in Gases.—The gases under normal conditions are such poor conductors of electricity that the most sensitive galvanometers are unaffected by the current which flows through the air from one extended metal plate to a parallel plate maintained at a different potential. If, however, the air between the plates is continuously exposed to the action of some ionizing agent, such as a beam of X-rays, the current which flows between the plates is large enough to affect the more sensitive galvanometers, and the voltage-current characteristic of the layer of air between the plates is found to be as illustrated in Fig. 142. For a small range of voltage, the current is seen to increase with increase in voltage. Then for quite a range in voltage the current remains constant. At the voltage E_1, the current again starts to increase and then a further increase in voltage leads to a brush discharge or to a spark discharge between the plates. We conceive that the voltage-current characteristic of a gas not subject to the ionizing influence of X-rays is similar to this, but that the current is too feeble to be measured.

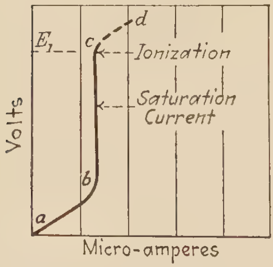


FIG. 142.—Voltage-current characteristic of leakage current in air.

The features of the characteristic shown in Fig. 142 are accounted for as follows: In a given gas at a given temperature ions are at all times being produced at a definite rate by the collisions between molecules, action of radioactive material, and, in the experiment, by the action of the X-rays. Starting

with a gas free of ions, these processes will not lead to an indefinite accumulation of ions in a given space, because recombination sets in, and the rate of recombination becomes greater and greater as the ions accumulate. Eventually, an equilibrium accumulation is reached at which the rate of recombination is just equal to the rate of production of ions. Imagine these processes to be going on in each cubic centimeter of air between the plates. When a very small voltage is applied between the plates, the ions existing between the plates are subject to forces which cause them to drift, the + ions toward the − plate and the − ions toward the + plate. If the voltage is doubled, the force on the ions and their velocity of drift are doubled. Consequently, the current will be doubled, provided the number of ions between the plates remains the same as at the lower voltage. But ions are now being lost from the region, not only by the natural processes of recombination, but by drifting to the plates and there discharging. At the higher voltage, ions are driven to the plates more rapidly, and consequently the equilibrium number of ions per cubic centimeter is lower, and the current will be slightly less than twice that at the lower voltage. It is evident that the current will keep on increasing with increasing voltage, in the manner indicated by the portion *ab* of the characteristic, until finally the ions are swept to the plates as rapidly as they are formed by the natural processes. Then the current should remain constant for further increases in voltage, as shown by the portion *bc*.

The current again increases for voltages higher than E_1 because the accelerating force acting on the ions in their drift toward the plates is now so great that those ions happening to have the longer free paths acquire such high velocities in these paths that, on colliding with neutral molecules, they readily knock off an electron. Thus the natural processes of **ionization by collision** are supplemented by a new method of ionization of collision. It will be readily seen that at slightly higher voltages this new method will be so prolific a producer of ions that a channel through the gas will become conducting, as evidenced either by sparking between the plates, or the playing of an arc between the plates.

Conduction between cold electrodes at voltages too low to cause ionization by collision may be called **leakage conduction.**

In addition we have to consider **disruptive, or spark conduction, arc conduction from an incandescent cathode,** and **electronic conduction in evacuated space.**

190. Disruptive Conduction in Gases.—The lightning discharge, spark discharge, brush discharge, and streamers from conductors at high potential, so-called arcs of 0.1 to 10 meters in length between the wires of high-tension systems, and finally the conduction in the partially evacuated Geissler tubes and in the tubes of the Moore lighting system, are all instances of **disruptive conduction** in gases. In all these cases the electric intensity along the **channel** in which conduction is occurring is high enough to produce, under the temperature and pressure conditions existing in the **conducting channel,** a copious supply of ions by collision. The temperature conditions obtaining within the channel are entirely different from those a few millimeters outside its boundary.

If the conducting channel is long, the material, shape, and temperature of the electrodes have little influence on the voltage-current characteristic of this type of conduction **after** the channel has been formed. But in the incipient stages of conduction (as in the brush discharge and in the spark discharge of a condenser) the shape and the temperature of the electrodes have a most important influence.

In disruptive conduction the spectrum of the light radiated from the conducting channel is the line spectrum of the gases in which the channel is formed, except that the electrodes may eventually get hot and metallic vapors may get into the channel. The voltage-current characteristic of disruptive conduction is similar in shape to the characteristic discussed below under Arc Conduction.

190a. Arc Conduction from an Incandescent Cathode.—In the carbon arc, the mercury vapor arc, and the flame arcs of commerce, an incandescent or hot spot on the cathode is the main source of the electrons which make the channel from cathode to anode conducting. The voltage between electrodes is much less than that necessary to maintain disruptive conduction between them, and the arc must be started either by bringing

the electrodes in contact and drawing them apart, or by applying a high voltage between stationary electrodes and maintaining it until the hot spot has formed. The anode may be either cold or hot, but to have **arc conduction**, as distinguished from disruptive conduction, the cathode **must** have a **hot** spot. While the carriers issuing from the hot spot seem to be largely electrons, still the vapors of the elements of which the cathode is composed are carried into the conducting channel, are ionized by collision, and give to the light from the flame arcs the characteristic line spectra of the elements in question. In the circuit shown in Fig. 143 a hot spot is maintained on a cathode C by causing a continuous current to pass between it and an anode A.

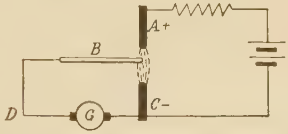


FIG. 143.—Hot-spot rectifier circuit.

In Fig. 143, a third water-cooled electrode B is shown connected to the cathode through an auxiliary circuit containing a source of e.m.f. It is found that if the electrode B is positive to C, a current will flow in the auxiliary circuit in the direction the generator G tends to cause it, but if the generator G is reversed so that B is negative to C no current will flow, unless, indeed, the voltage of the generator G' is high enough to give rise to disruptive conduction between B and C. It follows that if the source G is a source of alternating e.m.f., current will flow during those half cycles in which B is $+$ to C, but not during the other half cycles. This property of the arc is used to obtain rectified or pulsating unidirectional currents from alternating-current sources.

The voltage-current characteristics (with direct current) of arcs between various electrodes are shown in Fig. 144. It has been shown[3] that if the arc length is not less than 15 millimeters the relation between voltage, arc length L, and current can be expressed with fair exactitude by an empirical equation of the form

$$E = a + bL + \frac{c + dL}{i^n}, \tag{258}$$

[3] NOTTINGHAM, W. B.: *A New Equation for the Static Characteristic of the Normal Electric Arc.* Trans A.I.E.E., 1923, Vol. XLII, p. 302.

See also EDDY, W. N.: *Length—Voltage—Current—Pressure Characteristics of Normal Arcs for Different Electrode Materials*, Gen. Elec. Rev., March, 1922.

in which a, b, c, d, and n are constants whose values depend upon the electrode materials. The value of the exponent n seems to be dependent on the boiling point of the anode material, being 0.98 for carbon and 0.67 for copper. In all cases, as the current increases, the voltage between the arc electrodes actually decreases. The arc in itself is, therefore, unstable, and the current can only be held constant at any given value by operating the arc either from a constant current source or with a "ballasting" resistance connected in series with it.

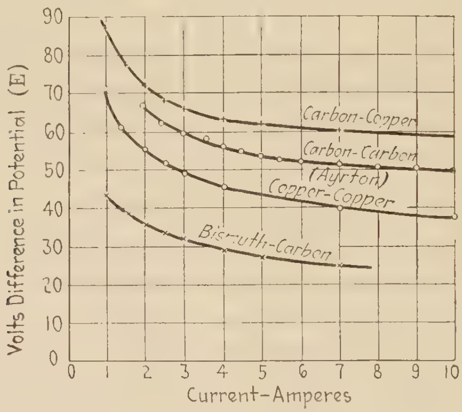


FIG. 144.—Volt-ampere characteristics with the arc length constant L = 3 mm.

The meaning of the term "resistance" can be profitably extended to cover more effectively the voltage-current relations in an arc by appending the following qualifying phrases:

For a constant current of value I_1 in the arc, the ratio of the voltage between the arc electrodes to the current I_1 may be termed the **static resistance** of the arc at the current I_1.

On the other hand, when an arc is already operating at the current I_1, a small increase in the current of amount ΔI is accompanied by a small increase in voltage of amount ΔE (a negative quantity). The ratio of $\Delta E/\Delta I$ is not only not equal to E/I, as it is in metallic conduction, but it is a negative quantity. To a small cyclic current which is superimposed on the steady current I_1, the arc has the properties of a **negative** resistance! The ratio of $\Delta E/\Delta I$ at the mean current I_1 may, accordingly, be

called the equivalent resistance of the arc under a small cyclic change, or, more briefly, the **cyclic resistance** at the mean current I_1.

In Fig. 145 is shown a circuit in which the instability of the arc, as expressed by a negative cyclic resistance, is used to sustain an oscillating current. An arc is a common branch to two circuits. Circuit *D* contains a direct-current generator and a high resistance, or, preferably, an inductance coil, which keeps the current delivered by the generator to the arc substantially constant. Circuit *A* contains a condenser and a generator which generates an alternating e.m.f. The condenser prevents the flow of a continuous current in circuit *A*, but permits the alternator to superimpose an alternating current on the direct current flowing in the arc. Because of the negative cyclic resistance of the arc, the alternating current which flows is greater than when the points *M* and *N* are connected by a jumper of zero resistance. If the ohmic resistance of the circuit *A* is less than the cyclic resistance of the arc, the alternator *G* may be omitted, and an oscillation started in the circuit *A* will be sustained and built up to some steady state magnitude.

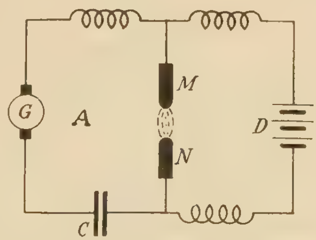


FIG. 145.—Arc circuit for sustaining oscillations.

191. Electronic Conduction in Evacuated Space.—The perfect insulating medium (aside from the difficulty of maintaining it) is highly evacuated space. It contains nothing to conduct, nothing to supply carriers by breaking down, and nothing to cause heating. Cold electrodes when separated by a centimeter of the best attainable evacuated space will sustain a higher voltage than when separated by a centimeter of the best insulating material known.

If a current is to flow between electrodes separated by evacuated space, it must be by reason of the emission of electrons from the negative electrode. The most effective way of causing a substance to emit electrons at a measurable rate is to raise its temperature. The rate of emission increases rapidly with

increase in the temperature of the substance, and the phenomenon is termed the **thermionic emission of electrons.**

The apparatus used in the study of the electronic currents made possible in space by the emission of electrons from hot metals is shown in Fig. 146. A wire W of the metal whose

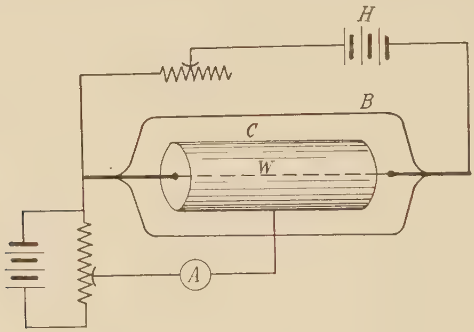


FIG. 146.—Circuit for measuring thermionic currents.

emissive properties are to be studied is mounted within a highly evacuated glass bulb B. The wire is maintained at any desired temperature by passing through it a heating current from the battery H. Coaxial with the wire is a cylinder C of platinum or molybdenum foil or gauze. By applying different voltages between the hot wire and the surrounding cylinder, voltage-current characteristics similar to those shown in Fig. 147 are obtained. The characteristic *abc* is for the filament at a temperature T_1, *ade* is for the same filament at a higher temperature T_2, and *afg* is for a still higher temperature T_3.

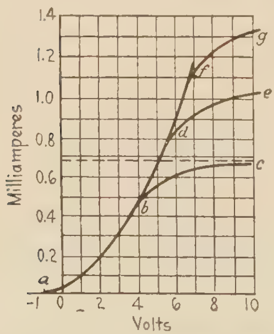


FIG. 147.—Voltage-current characteristics of thermionic currents.

To account for the features of these curves, imagine the incandescent filament, as an **isolated** body, to start to emit electrons at a constant rate. The emission of the electrons during the first instant will leave the filament positively charged and surrounded with a swarm of outwardly moving electrons. Consequently, the electrons emitted in the next instant will be sub-

ject to the attractive force of the positively charged filament. The electrons emitted during the third instant will be subject to the attraction of a still greater charge, and so on. Those electrons escaping through the surface with the lower velocities will shoot out to only a short distance before they are stopped and are then drawn back in, while those escaping with the higher velocities will shoot out to greater distances and remain out for longer periods.

It can be seen that a condition of equilibrium will be attained in which the filament has surrounded itself with an atmosphere of electrons which are "raining" back into the filament at a rate which is exactly equal to the rate of emission. At high temperatures this condition is attained in less than a millionth of a second. The atmosphere of electrons distributed in the space surrounding the filament is called a **space charge.** Now imagine an electrical connection to be made from the incandescent filament through a source of e.m.f. to any nearby conductor, as the cylinder C. When the source delivers zero e.m.f., the filament shares some of its + charge with the cylinder. Thereupon the outlying electrons, instead of falling back into the filament, are drawn to the positively charged cylinder and then return to the filament by way of the conducting circuit. Thus at zero impressed e.m.f. there will be a small current from the cylinder across the space to the filament, as shown by the characteristics.

The curves show that, upon charging the cylinder negatively with respect to the filament, this current decreases; it becomes zero if the cylinder is negative to the filament by -1 volt or more. Upon charging the cylinder positively with respect to the filament, the current is seen to increase with increasing voltage until $+7$ volts is reached. The characteristic *abc* shows that voltages in excess of this produce substantially no further increase in the current. The conclusion is that all of the electrons emitted by the filament at the temperature T_1 must be drawn to the cylinder at this voltage, and so no further increase in current with voltage is possible; the current is said to have attained its **saturation value** for the temperature T_1. However, if the temperature of the filament is raised, the rate of electron emission is also raised and the current keeps on increasing with the voltage to a new saturation value corresponding to the higher temperature T_2.

Thus each characteristic consists of four portions: (*a*) the horizontal portion, in which the value of the current is a function of the temperature, and is independent of the voltage, **provided** the voltage exceeds a certain value; (*b*) the rising portion, in which the current is a function of the voltage and is approximately independent of the temperature, **provided** the temperature exceeds a certain value; (*c*) the rounded knee connecting these two portions; and (*d*) the very small currents at negative voltages which are due to the initial velocities of the emitted electrons.

191a. Rate of Emission of Electrons.—Richardson, who published the first quantitive theoretical studies of thermionic emission (1902), assumed the existence of an electric force at the surface of bodies (see Sec. 145*g*), and postulated that the work done by the electrons in escaping against this force must be derived from the kinetic energy of thermal vibration of the more rapidly moving free electrons in the metal. By assumptions and arguments drawn from the kinetic theory of gases, Richardson has deduced the following equation for the number of coulombs of electrons per second, J, which reach a square centimeter of the surface with sufficient velocity to escape against the surface forces:[4]

$$J \text{ (coulombs per sq. cm. per sec.)} = Nq_o\sqrt{\frac{RT}{2\pi M}}\,\epsilon^{-\frac{\phi}{RT}} \quad (259)$$

in which, N represents the number of free electrons per cubic centimeter of metal.

q_o represents the electronic charge (1.591×10^{-19} coulombs).

T " the temperature of the metal in degrees Kelvin.

R " the gas constant in the formula $pv = RT$. $R = 8.62 \times 10^{-5}$ joules per degree for a coulomb of electrons.

ϕ represents the electron affinity of the metal (Sec. 157*a*), that is, the work done against the surface force when a coulomb of electrons escape.

M represents the mass of a coulomb of electrons, namely, 5.66×10^{-16} gram-sevens.

If the values of N and w for any metal are assumed to be independent of the temperature, Eq. (259) leads us to expect the following relation (generally known as Richardson's equation)

[4] RICHARDSON, O. W.: *The Emission of Electrons from Hot Bodies.*

between the temperature of the emitting wire and the current density of the outgoing electrons at the surface of the wire:

$$J \text{ (ampere per sq. cm.)} = a\sqrt{T}\epsilon^{-\frac{b}{T}}, \qquad (260)$$

in which a and b are constants, characteristic of each metal.

By an argument based upon thermodynamic considerations, Richardson[4] and others[5] have deduced another expression for the saturation current, namely,

$$J \text{ (ampere per sq. cm.)} = AT^2\epsilon^{-\frac{b_0}{T}}. \qquad (261)$$

From theoretical considerations, Dushman has assigned the value 60 to the coefficient A.

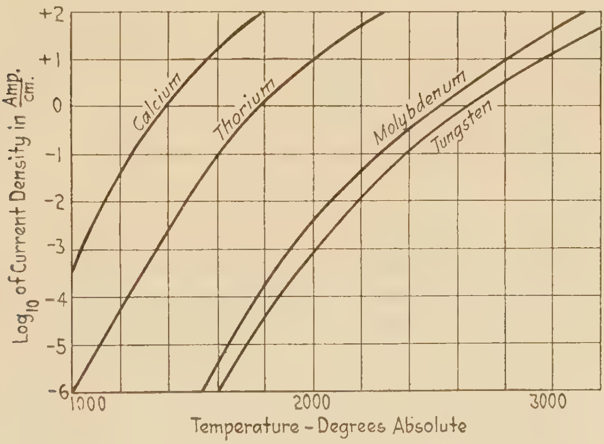


FIG. 148.—Relation between rate of electron emission and temperature.

By measuring the saturation currents from filaments at different temperatures, Richardson, Dushman, and others have shown that both Eqs. (260) and (261) will express the equation of the curve of observed values within the limits of experimental error. The variation of current density with temperature is so great that it is impossible to measure the temperature accurately enough to

[5] DUSHMAN, S.: *Electron Emission from Metals as a Function of Temperature*, Phys. Rev., June, 1923, February, 1924; DUSHMAN, S.: *Theory of Electron Emission*, Trans. Am. Electrochem. Soc., 1923, p. 101; WILSON, H. A.: *Theory of Thermionics*, Phys. Rev., 1924, p. 38.

decide between the two expressions. Figure 148 shows the relation between the rate of electron emission and the temperature for a number of metals. The curves are plotted from Eq. (261), using for b_0 the values determined by Dushman, as noted on the curves.

For the range of voltage in which the value of the current between a large cylinder and a fine emitting filament is limited by the effect of the space charge, the following equation for the current has been derived from theoretical considerations:[6]

$$I \text{ (amperes)} = \frac{8\pi p_0}{9}\sqrt{\frac{2q_0}{m}}\,\frac{l}{r}E^{3/2} \tag{262}$$

in which

p_0 represents the permittivity of space $= 8.85 \times 10^{-14}$.
l and r represent the length and radius of the filament.
E represents the voltage.

$$I \text{ (amperes)} = 14.7 \times 10^{-6}\frac{l}{r}E^{3/2}. \tag{262a}$$

192. Resistance of Insulators.—In the case of insulators made of the very high-resistance insulating materials, such as rubber and glass, the leakage current occurs mainly in a surface film of moisture and dirt rather than through the body of the insulator. The resistance of the surface film drops rapidly with increase in the humidity of the air. The following table lists the volume and the surface resistivities of various materials. The latter is the resistance in ohms between the opposite edges of a centimeter square.

[6] Child, C. D.: *Discharge from Hot CaO*, Phys. Rev., 1911, Vol. XXXII, p. 498; Langmuir, Irving: *Effect of Space Charge and Residual Gases on Thermionic Currents in a High Vacuum*, Phys. Rev., 1913, p. 450; Langmuir, Irving: *Fundamental Phenomena in Electron Tubes Having Tungsten Cathodes*, Gen. Elec. Rev., June, 1920.

Resistivities of Insulating Materials at 20°C.

Material	Volume resistivity, ohm-centimeters	Surface resistivity, ohm-centimeters, Humidity 50 per cent	Surface resistivity, ohm-centimeters, Humidity 90 per cent
Ice, −0.2°C.	2.8×10^8		
Transformer oils	10^{12}–10^{13}		
Paraffin oil	10^{15}		
Glass	10^{13}–10^{16}	10^{11}–10^{13}	10^6–10^9
Porcelain (glazed)	10^{14}–10^{15}	2×10^{12}	5×10^8
Hard rubber	10^{15}–10^{18}	3×10^{15}	2×10^9
Beeswax	8×10^{14}	6×10^{14}	5×10^{14}
Rosin	5×10^{16}	5×10^{14}	2×10^{14}
Amber	5×10^{16}	6×10^{14}	10^{11}
Mica	10^{13}–10^{15}	10^{12}	10^9
Sulphur	10^{16}–10^{17}	7×10^{15}	10^{14}
Sealing wax	10^{15}–10^{16}	2×10^{15}	10^{14}
Fused quartz	5×10^{18}	3×10^{12}	2×10^8
Paraffin (special)	5×10^{18}	10^{16}	6×10^{15}
Air	∞		

193. Resistance of Contacts between Metallic Conductors.—The resistance of the contact between metallic conductors depends upon the smoothness, cleanness, and hardness of the contact surface, the pressure of contact, etc. The resistance of the two series contacts from brass block to taper plug to block in plug boxes is of the order of 0.00005 ohm.

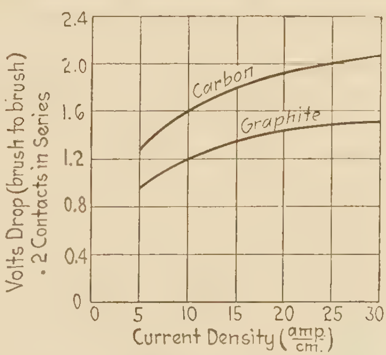


Fig. 149.—Voltage drop in commutator contacts.

In the sliding contact between a copper brush and the surface of the copper commutator, the resistance may be from 0.0045 to 0.018 ohm for a square centimeter of contact surface.

In the sliding contact between carbon brushes and the copper commutator, the resistance of the contact decreases as the current density increases. The voltage-current characteristic from + brush to − brush (two sliding contacts in series) is illustrated for the typical machine in Fig. 149. The voltage drop is roughly constant over the working range of current densities and is of the order of 2 volts.

194. Asymmetrical Conductors.—By an asymmetrical conductor is meant a conducting element or device which permits of the flow of a greater current through it in one direction than in the other at a given impressed voltage. The principal application of such devices is in obtaining from sources of alternating e.m.f., currents which are unidirectional, or which have a large unidirectional component. These devices include the following types:

Hot-spot arc conductors (mercury-arc rectifier).
Thermionic evacuated tubes (kenetron).
Crystal contacts (crystal radio detectors).
Polarization cells (electrolytic detector).

The hot-spot arc conductors and the thermionic evacuated tubes have this in common—that conduction is due to the emission of electrons from a hot cathode. In the arc, a spot on the cathode is kept at an emitting temperature by the continuous operation of an arc—sometimes an auxiliary arc, as illustrated in connection with Fig. 143. In the evacuated tube, the cathode is a filament which is kept at an emitting temperature by passing an auxiliary heating current through it. The rectifying tube circuit differs from Fig. 143 only in that the heating battery H is connected directly to the two terminals of the cathode C, and B is a molybdenum cylinder in the glass tube and coaxial with the filament. The arc rectifier is used to rectify currents at voltages of 3000 or lower, while an evacuated rectifier (trade name "kenetron") with a spacing of 1 centimeter between the cylinder and the filament is used on 100,000-r.m.s. volt circuits.

194a. Crystal Contacts.—A point contact between a metal and certain conducting crystalline substances, or between two of the conducting crystals, exhibits the remarkable asymmetrical

conducting properties illustrated in Fig. 150. Among the crystalline substances exhibiting these properties are carborundum (carbide of silicon), silicon, tellurium, zincite (red oxide of zinc, ZnO), chalcopyrite (iron-copper sulphide), and molybdenite (MoS_2). Although these crystal contacts have been extensively studied, the explanation of their asymmetrical properties is not clear.[7] Crystal rectifiers, or crystal detectors, as they are called in radio communication, have been widely used to supply to the telephone receiver partially rectified currents under the feeble alternating electromotive forces of high frequency which are induced in the receiving circuits.

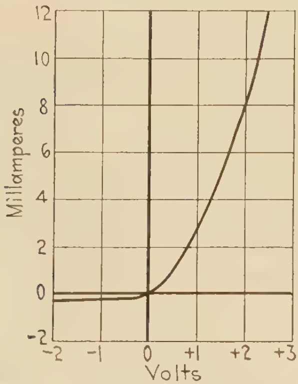


FIG. 150.—Voltage current characteristic of zincite-chalcopyrite detector.

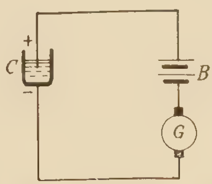


FIG. 151.—Polarization cell rectifier circuit.

194b. Polarization Cells.—The voltage-current characteristic of an electrolytic cell in which pronounced polarization effects take place is shown by curve C of Fig. 139. Suppose a battery B and a source of alternating e.m.f. G are connected in series with such a cell C, as in Fig. 151. If the cell is polarized by the current from the battery to a point just below the knee of curve C (Fig. 139), and if the peak value of the alternating e.m.f. is of the same order as the polarizing battery voltage, it will be seen that a current many times the polarizing current will flow during the half cycle in which the two e.m.fs. are in the same direction. During the other half cycle the current will be between zero and

[7] See PIERCE, G. W.: *Principles of Wireless Telegraphy*, Chaps. XVII and XVIII.

the polarizing value.[8] In other words, the superposition of the alternating e.m.f. on the continuous e.m.f. has caused a large increase in the average value of the unidirectional current in the circuit.

This principle was used in the **electrolytic detector** for obtaining rectified currents from the alternating e.m.fs. of high frequency induced in radio receiving circuits. In this detector, the anode was a platinum wire 0.0005 centimeter in diameter immersed for 0.01 centimeter in a 30 per cent solution of nitric acid. The cathode was the platinum cup holding the solution, or a much larger platinum wire dipping into the solution.

194c. Exercises.

1. State Ohm's law without mentioning resistance.

2. From the information given in Sec. 173, calculate the resistivity of mercury at 0°C.

3. After standing for many hours in a room at a constant temperature of 20°C., a coil of copper wire was found to have a resistance of 8.37 ohms. After having an electric current passed through it for several hours the resistance was found to be constant at 9.23 ohms. Calculate the apparent temperature of the coil. Is the coil actually at this temperature throughout?

4. From the data in Sec. 178, calculate the resistance at 20°C. of 1000 feet of No. 10 B. & S. copper wire (diameter 0.102 inch). Compare with the value given in Sec. 179.

5. The service wires from the street mains to a residence are generally of No. 8 B. & S. rubber-covered wire (diameter 0.128 inch). If the distance from the street pole to the meter is 125 feet, compute the resistance of the service wires.

6. Which would make the return of lower resistance for the current supplied to the incandescent lamps in a room, the No. 14 B. & S. copper wire which is generally used, or the ½ inch steel conduit or pipe which is frequently used to protect the wires? Assume that the resistivity of the steel relative to copper is 9.8. The internal and external diameter of the conduit are 0.62 and 0.84 inch, respectively.

7. The Keokuk-St. Louis power transmission system consists of two duplicate three-phase lines, each 144 miles long. Each three-phase line contains three conductors each of 300,000 circular mils cross-section. Compute the resistance of one conductor, the total weight of copper in the system, and the total investment in copper.

8. The National Electric Code specifies that in wiring for lighting service the greatest load that may be placed on one branch circuit is 1200 watts (at 110 volts). It is not considered good practice to allow the resis-

[8] For an oscillographic study of the detector see PIERCE, G. W.: *Principles of Wireless Telegraphy.*

tance voltage drop in the branch circuit to exceed 1½ per cent of the voltage supplied.

If all the load is concentrated at one outlet, what is the greatest length of No. 14 B. & S. conductor that should be used in connecting the outlet to the distributing panel?

9. A 15-horsepower motor for pumping is located 1000 feet away from the generating station where power is available at 550 volts and costs 1.5 cents per kilowatt-hour. What size conductor would you recommend:

a. If the motor is to furnish 15 horsepower output 10 hours per day, 365 days of the year?

b. If the motor is for emergency service and furnishes 15 horsepower output only 1 hour per week on the average?

It may be assumed that weatherproof wire at 15 cents per pound is to be used. The annual charges upon the investment may be taken at the rate of 10 per cent total. A motor efficiency of 87 per cent may be assumed. Overhead wires are to be used.

10. The resistance of a conductor measured at 15°C. was 1.64 ohms. At 80°C. the resistance measured 2.09 ohms. What is the resistance-temperature coefficient (at 20°C.) for the material composing this conductor?

11. Design a water rheostat to be used to load fully a 600-kilowatt 700-volt direct-current railway generator, the rheostat to be immersed in a river or tail race in which the resistivity of the water is 3000 ohm-centimeters.

12. The resistivity of a certain river water is 3000 ohm-centimeters at 18°C. From the data in Sec. 187*f*, make an approximate estimate of the number of grams of NaCl which it would be necessary to add to this water per liter in order to reduce the resistivity to one-half of the above value.

CHAPTER IX

CALCULATION OF THE ELECTRIC CURRENTS IN NETWORKS. KIRCHHOFF'S LAWS

195. The Common Starting Point for All Network Calculations. The relations which exist between the current and the electromotive force in a single conductor and the dimensions of the conductor have been presented in the preceding chapter. The object in the present chapter is to present the laws and the methods of calculation which enable us to predict the values of the currents which will flow in the branches of **networks** of many conductors, containing one or more sources of intrinsic

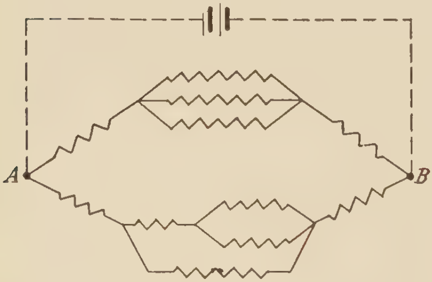


FIG. 152.—Series-parallel network.

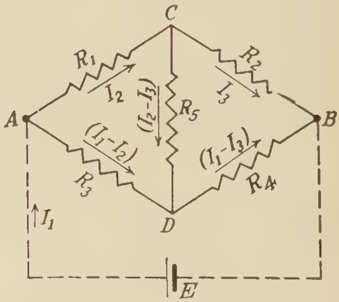


FIG. 153.—Bridge network.

electromotive force. Figures 152 to 154 are diagrams of the connections of typical networks. The problem is this:

Given the diagram of connections of the network, and the values of the e.m.f. of the generating sources, and the resistances and capacitances, etc. of the conductors, the condensers or the appliances in each branch: to calculate the value of the current which will flow in any branch of the network.

The first part of this chapter deals with the calculation of the **continuous** currents which flow in networks and non-cylindrical conductors under the driving forces of sources of continuous electromotive force.

If any constant of a circuit, or of a network, is suddenly changed, as by a switching operation, the currents and the electromotive forces in the parts of the circuit do not immediately assume their ultimate or **steady-state** relations to each other. The expressions for the currents which flow in the interval during which the circuit is settling into the steady state contain **transient terms** whose values rapidly reduce to zero. The latter part of the chapter deals with the calculation of these **transient** terms.

The calculation of the **alternating** currents which flow when the sources of unvarying e.m.f. are replaced by alternators delivering an e.m.f. which alternates in direction as a sinusoidal function of time is considered in a later chapter.

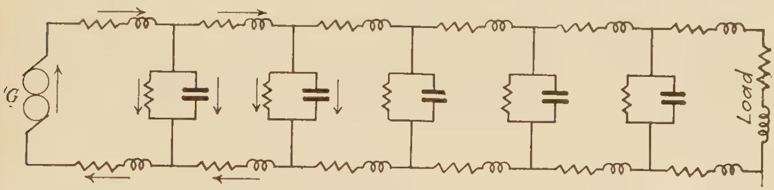


FIG. 154.—Long line network.

The starting point for all these network calculations is the same. It is to set up the equations which express in algebraic form the broad generalizations about the movements of electricity and the transformations and transfers of energy which accompany such movements. These generalizations are two in number:

1. The principle of the conservation of electricity.
2. The principle of the conservation of energy.

The first principle is applied to networks in the form known as Kirchhoff's Law of Currents, and the second in the form known as Kirchhoff's Law of Electromotive Forces.

196. Kirchhoff's Law of Currents.—The principle of the conservation of electricity has been stated and discussed in Secs. 21 to 24. Briefly summarized:

The two electricities are conceived to exist in invariable and equal quantities. Electrical phenomena are conceived to involve—not the creation or the destruction of electricity—but either the separation of

the positive from negative electricity or the circulation of one kind relative to the other.

Consider the application of this principle to the region bounded by the closed surface S of Fig. 155. The surface S is pierced by one or more conductors containing atmospheres of electrons which may drift into or out of the enclosed region.

If q represents the algebraic value of the net or unneutralized quantity of electricity within a surface S which is pierced at a single place by a conductor, then, by definition, the algebraic value of the conduction current through the surface in the **inward** direction is

$$i = \frac{dq}{dt} \text{ (defining } i\text{)}. \tag{187}$$

If the atmosphere of electrons is drifting inward, dq/dt is a negative quantity, and the current inward has a negative value, or the current is said to be flowing outward.

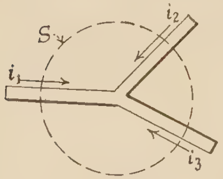


FIG. 155.—Currents toward a junction.

If the surface S is pierced by a number of conductors, and if the currents in the **inward** direction through the surface are represented by i_1, i_2, i_3, etc., then the rate at which **unneutralized** electricity is accumulating in the conductors within the region is equal to the sum of the currents.

$$\frac{dq}{dt} = i_1 + i_2 + i_3 + \cdot \tag{263}$$

For the case in which the currents i_1, i_2, i_3 are continuous currents (constant in value), the rate of accumulation of charge has a constant value. If this value is any value other than zero, unneutralized charge would accumulate indefinitely. The unlimited accumulation of charge is physically impossible. A very definite limit to the extent to which unneutralized charge may be accumulated in a given region is set by the fact that each additional increment of electricity must be forced into the region against the mounting repulsive forces of the accumulated charge, and it is impossible to devise electrical mechanisms which will exert unlimited electromotive forces. From this it follows that the following general statement, known as Kirchhoff's Law

of Currents, may be made about the values of the currents in a network of conductors.

196a. KIRCHHOFF'S LAW FOR CONTINUOUS CURRENTS (DEDUCTION).—At any junction point in a network of conductors carrying continuous currents, the sum of the algebraic values of the currents in the direction toward the junction point is zero.

$$\Sigma I = 0. \tag{264a}$$

In circuits in which the currents and the potential differences between portions of the circuits vary rapidly in time, there is a variable accumulation of electrons on some conductors and a variable deficit on others. For example, in the two-wire telephone line illustrated in Fig. 156, if the potential difference between wires A and B varies, there will be an accumulation of electrons on the short length MN of A and an equal withdrawal from the length OP of B, or vice versa. Accordingly, the portions MN and OP may be regarded as constituting a condenser

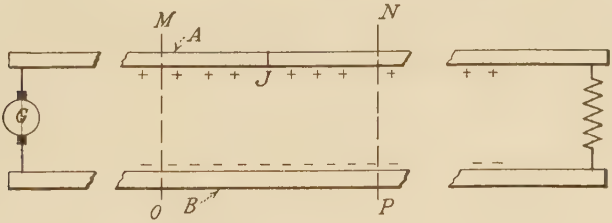


FIG. 156.—Long line.

of definite capacitance. The wire MN, then, has two functions. It serves as a conductor of electricity from M to N, and it serves as one electrode of a condenser of which the other electrode is OP. As a consequence, while electrons are accumulating or withdrawing from the portion MN, the sum of the instantaneous (algebraic) values of the currents at M and N in the direction toward the common junction point J is not zero. Such a circuit is said to have **distributed** capacity.

For the purpose of mathematically treating a circuit with distributed capacity (say a 1000-mile telephone line) it is customary to replace it by an equivalent artificial circuit made up of a great many branches each having **segregated** resistance and capacitance as in Fig. 154. Thus, if the section MN is short

enough, the artificial section shown in Fig. 157 is the equivalent of the section of the actual line in Fig. 156. In Fig. 157 the wires MN and OP are regarded as pure resistances having no capacitance relative to each other. The capacitance between wires in the actual circuit is, in Fig. 157, replaced by a condenser C connected through leads of zero resistance to the midpoints of the two telephone wires. In such an equivalent artificial circuit, **for either continuous or for variable currents,** the sum of the algebraic values of the currents toward any junction point is always zero.

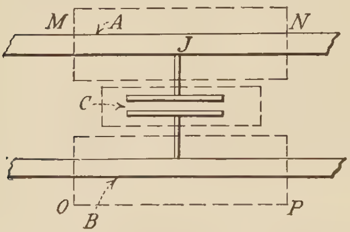


FIG. 157.—Equivalent artificial section.

Accordingly, the law of currents may be generalized to apply to either continuous or variable currents, as follows:

196b. LAW GENERALIZED FOR VARIABLE CURRENTS (DEDUCTION).—**At any junction point in an equivalent network of segregated elements, the sum of the instantaneous (algebraic) values of the currents in the direction toward the junction is at every instant equal to zero.**

$$\Sigma i = 0. \qquad (264)$$

197. Law of Electromotive Forces for Circuits.—The algebraic value of the electromotive force in the direction AB of a specified type of force in the part of a circuit included between any two terminals has been defined to be equal to the work which is done by the specified force per equivalent unit of positive electricity which flows from A to B.

The electromotive forces are so evaluated as to satisfy the principle that, **at every instant of time, the sum of the algebraic values of the electromotive forces in a specified direction around any closed circuit is zero.**

$$\Sigma e \text{ (around a closed circuit)} = 0 \qquad (207)$$

Consider any simple series circuit divided into parts in which the values of the electromotive forces are e_1, e_2, e_3, etc., in which the current has the same value i at all parts. Then

$$e_1 + e_2 + e_3 + e_4 = 0. \qquad (207a)$$

Upon multiplying both members of this equation by i, we have

$$e_1i + e_2i + e_3i + e_4i = 0. \qquad (265)$$

Each term[1] in the left member of the equation represents the time rate at which work is being done by the forces in question. If the value of any ei product is positive, it signifies that electricity is moving in the direction in which the force in question tends to move it. This is the case when the force in question represents the transformation of the non-electrical forms of energy (chemical, thermal, or mechanical) into an electrical form, or when it represents a decrease in the electropotential or electrokinetic energy associated (stored) with the region, or, finally, when the region in question is regarded as the "portal" through which energy is being transferred from another system to the circuit of which it is a part. On the other hand, a negative value for the ei product signifies the transformation of energy from an electrical form into the chemical, thermal, or mechanical form, or an increase in the electropotential or electrokinetic energy associated with the region, or the transfer of energy from the circuit of which the region is a part to another circuit.

Under these conditions it is seen from Eq. (265) that *the statement that "the sum of the electromotive forces around a closed circuit is zero" is equivalent to saying that in the transformations and exchanges of energy occurring in a closed system, the principle of the conservation of energy is satisfied.*

198. The Purpose of Algebraic Signs and Their Importance in Electric Circuit Theory.—In order to write the equations which express Kirchhoff's law of electromotive forces for the circuits of a network, we must know the physical laws which express the relations between the current and the electromotive force in the different types of conducting regions of which networks may be

[1] Upon multiplying both members of the power equation by any short interval of time dt, it becomes,

$$e_1\,i\,dt + e_2\,i\,dt + e_3\,i\,dt + \ . \ . \ . \ = 0 \qquad (266)$$

Since idt represents the quantity of electricity dq which passes through each part in the interval of time dt, this equation may be written in the form,

$$e_1\,dq + e_2\,dq + e_3\,dq + \ . \ . \ . \ = 0 \qquad (267)$$

Equations (266) and (267) are seen to be energy equations.

built up, and must write these laws in the form of equations. The most general and at the same time the most definite way of expressing these laws by equations is that in which the electrical quantities which appear in them, namely, current, electromotive force, quantity of electricity, electrostatic, and magnetic flux, are defined and treated as algebraic quantities. *The algebraic signs have the sole but vital purpose of specifying those directions which must be specified in order definitely to describe a current, a flux, or an electromotive force. Any method of describing or defining these terms which fails to specify directions is incomplete* and can be used only in cases which are so simple that no confusion as to directions can arise.

The conventions which are used throughout this text in relating the algebraic signs of the electrical quantities to direction in a circuit are as follows:

198a. Conventions Relating Algebraic Signs of Symbols to Directions.

1. For convenience in specifying directions, an arrow will be drawn in an arbitrarily selected direction along each branch of the network. The direction indicated by the arrow will be called the ARROW DIRECTION in THAT branch.[2]

2. Any symbol placed on the diagram to represent the current in a given branch will be understood to represent the algebraic value of the current in the ARROW direction along the branch. (This convention is nothing but a precise and rapid method of defining the symbol.)

3. Any symbol placed on the diagram to represent the electromotive force of specified forces in a given branch will be understood to represent the algebraic value of the electromotive force in the ARROW direction along the branch. (The known electromotive forces of batteries or generators are usually indicated by writing the numerical value on the diagram with separate polarity marks to indicate directions.)

[2] The use of arrows may be replaced by the use of equivalent systems, such as the letters, a, b, c, etc. at the junction points, and the symbols I_{ab}, etc. But some system of specifying directions and some convention relating the symbols used in the equations to these directions, are essential.

4. Any symbol (as q) placed on the diagram to represent the charge in a condenser will be understood to represent the algebraic value of the charge on that electrode which receives positive charge when the current in the arrow direction along the branch is positive. In other words, the symbol q is so defined that the following relation exists between q and i, where i represents the current in the arrow direction along the branch.

$$i = +\frac{dq}{dt} \text{ or } q = +\int i\,dt.$$

Other conventions relating to magnetic quantities will be added to this list in the chapters dealing with the magnetic effects of electric current.

199. The Constituent Elements of Electric Circuits and Their Electromotive Forces.—The constituent elements or regions

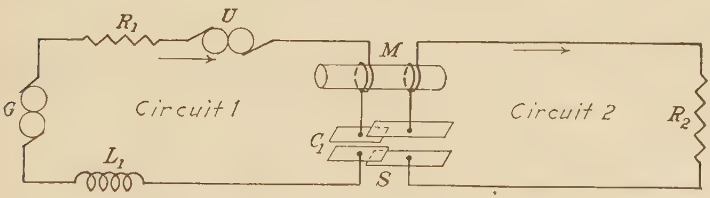


FIG. 160.—Constituent elements of a network.

which may enter into the make-up of the more involved networks are illustrated in the network shown in Fig. 160. Circuit 1 contains seven elements all connected in series. These elements are:

I. The region in which energy is transformed into the electrical form:
   1. A generating device G.

II. The regions receiving the transformed energy:
   1. A utilization device U (of the reversed generating type).
   2. A conductor or resistor element R.
   3. A condenser element C.
   4. An energy-transfer region of the elastance type S.
   5. An inductance or inertial element L.
   6. An energy-transfer region of the inductance type M.

199a. The Generating Device.—The generating device, or the source of intrinsic electromotive force—a battery, a thermocouple, or a dynamo—is a region in which, by the expenditure of chemical, thermal, or mechanical energy, the atmosphere of electrons is forced toward the negative terminal of the region. The intrinsic e.m.fs. of the dynamo and battery differ from the non-intrinsic e.m.fs. of resistance, elastance, etc. in that the latter are functions of the current in the circuit, while the former are not functions of the current, but are inherent or intrinsic characteristics of the source.[3]

In circuit studies, the e.m.f. of the source is either one of the known quantities, or it is an unknown whose value is to be determined not from specified conditions in the source, but so as to cause specified currents or electromotive forces in specified regions of the circuit. Therefore, we are not in this chapter concerned with the relations by which the e.m.f. of the source may be predicted from the specifications of the source.

When the source or seat of an intrinsic electromotive force is connected in a circuit, an electromotive force is said to be **impressed** in the circuit, and the intrinsic e.m.fs. in a circuit are frequently called the **impressed** electromotive forces.

Whether the electromotive force of the source connected in a given circuit in which the arrow directions have been adopted is to be designated as a + or a − quantity depends upon the direction in which the source is connected in its particular branch.

199b. Power Utilization Devices.—The utilization device U, such as a motor or a storage battery on charge, is a reversed generating device through which electrons are forced to move against the intrinsic electromotive force of the region, by reason of the greater electromotive force of other sources. In such a region electrical energy is reconverted into one of the other forms mentioned above. The utilization device is said to be the seat of an intrinsic e.m.f., **provided** the device, if acting independently of other sources of intrinsic e.m.f., can reverse the direction of flow of current through itself, and thereby transform energy of

[3] This statement is not strictly correct. When a source delivers current, there are always reactions which alter to some extent not only the **terminal** e.m.f., but also the **internal** or **intrinsic** e.m.f. of the source.

the other forms into the electrical form. Thus an electric motor, or a storage battery while charging, are seats of intrinsic electromotive forces, since each acting alone can convert energy into the electrical form, the former when driven as a dynamo, and the latter as a voltaic cell. Resistor heating elements are not utilization devices having intrinsic electromotive forces, since such heating elements cannot convert thermal energy into electric.

The statements made about the intrinsic electromotive forces of the generating devices apply to the utilization devices of the reversed generator type.

199c. Conductor or Resistor Elements.—The conductors and the resistors (such as lamps and heating elements, shown at R in Fig. 160) appearing in the networks will be assumed to be metallic conductors. For other classes of conductors with their non-linear relations the calculations would be much more difficult. In metallic conductors, the relation between the current and the **electromotive force of resistance**[4] is the straight-line relation expressed by Ohm's law (Sec. 170), namely,

$$E_R \text{ (e.m.f. of resistance in volts)} = -RI. \qquad (230)$$

[4] Some writers express Kirchhoff's law in the following form: "At every instant of time, the sum of the intrinsic e.m.fs. around any circuit is equal to the sum of (the portions of the intrinsic) e.m.fs. **consumed** in the different regions of the circuit, or is equal to the sum of the RI drops in potential."

When the law is expressed in this form, the e.m.fs. expressing the distribution of the transformed energy among the different regions are called "the e.m.f. **consumed by** resistance, inductance, or elastance, or the e.m.f. **expended in** resistance, inductance, or elastance."

The two forms of expression, namely, "the e.m.f. of . . . " and "the e.m.f. **consumed by** . . . " must not be confounded. One is the negative of the other. The expression for the "e.m.f. consumed by resistance" is

$$e_{R'} = +Ri.$$

In complex circuits there is no possibility of getting correct results unless this distinction is realized, and one form or the other is consistently adhered to. In a number of cases the expression "the e.m.f. consumed by . . . " is an awkward form to use. For example, in the discussion of the discharge of a condenser which has been charged to some known voltage and then connected in a discharge circuit, it leads to awkward situations to give the voltage to which the condenser is charged in the form "the voltage consumed by the condenser is 100 volts."

The e.m.f. of resistance is a non-intrinsic e.m.f. It represents the work done per unit quantity of electricity which passes through the conductor by the impeding force of the atomic impacts upon the electrons which are driven through the conductor by some source of intrinsic e.m.f. The work done by the forces of impeding impacts in the region of resistance R in the interval of time dt is

$$dW \text{ (joules)} = E_R I\,dt = -RI^2 dt. \tag{269}$$

That is to say, the processes going on in a conductor or a resistor never deliver but always absorb energy. From Joule's experiments, it is known that this energy is all converted into the form of heat energy in the body of the conductor.

199d. Condenser Elements.—The condenser C of Fig. 160, is a region (as two extended conducting sheets or wires, insulated from each other and one connected to each terminal of the region), through which a continuous (unidirectional) drift of electrons cannot occur. The passage of electrons into the region by one terminal and out by the other results in a separation of charge in the region; one conducting system acquires an excess of electrons and the other a deficit. After the passage of a small charge into such a region, the entry of further charge is against the forces arising from the separated charge. This requires the delivery of energy to the region. The energy so delivered is not lost, since it all becomes available when the charge is allowed to flow out of the region.

From the experimentally established straight-line relation between the quantity of electricity in a condenser and the potential difference between its terminals (see Sec. 72) the expression for the **electromotive force of elastance** (or **of condensance**) e_c may be written.[4]

$$e_c = -Sq = -\frac{q}{C} \tag{50a}$$

$$e_c = -S\int i\,dt. \tag{270}$$

The elastance S, or the capacitance C, are constants whose values depend upon the dimensions and the material of the condenser.

The work done by the forces of elastance in the interval of time dt is

$$dW \text{ (joules)} = e_c i dt = - Sq\, i\, dt \qquad (271)$$

This may be positive or negative, depending upon whether the signs of q and i are unlike or alike; that is, under some conditions the condenser receives energy and under others it delivers it again.

199e. Energy Transfer Regions of the Elastance Type.—The region S of Fig. 160 is a region in which a transfer of energy in the electrical form may take place from circuits 1 to 2, or vice versa. In this region an electromotive force—**the electromotive force of mutual elastance**—is **induced** in circuit 1 by reason of a separation of charges which has taken place in circuit 2, or vice versa. The two circuits are said to be **elastively coupled** (or **capacitively coupled,** or **electrostatically coupled**).

The expression for the **induced e.m.f. of mutual elastance** e_1 is the experimentally established relation of Sec. 109, namely,

$$e_1 = -S_m q_2. \qquad (151)$$

The energy delivered to the primary by the forces of mutual elastance in the interval of time dt is

$$dW \text{ (joules)} = e_1 i_1 dt = -S_m q_2 i_1 dt. \qquad (272)$$

199f. Inductance or Inertial Elements.—The inductance element L of Fig. 160 is a region, such as a wire wound in the form of a coil, in which the inertial effects are very pronounced. Energy must be delivered to this region during any interval in which the current (or the velocity of the electrons) is increasing. The delivered energy is not expended, since it becomes available, while the current is decreasing to zero. The inertial forces called into play when the velocity is changing are said to give rise to an **electromotive force of (self-) inductance.** The detailed study of this effect and of the following inductance effect will be taken up in Chap. XIII, in which it will be shown that the **e.m.f. of inductance** e_L is given by the expression

$$e_L = -L\frac{di}{dt} \qquad \text{(from a later chapter)}$$

in which L is a constant for any coil.

199g. Energy Transfer Regions of the Inductance Type.—The region M of Fig. 160 is a region in which a transfer of energy may take place from circuits 1 to 2, or vice versa. In this region an electromotive force—**the electromotive force of mutual inductance**—is induced in circuit 1 by any **variation** in the current in circuit 2, or vice versa. The two circuits are said to be magnetically coupled. In Chap. XIII the following expression for the **e.m.f. of mutual inductance** is derived:

$$e_1 = -M\frac{di_2}{dt}. \qquad \text{(from a later chapter)}$$

200. Calculations for a Simple Battery Circuit.—Imagine a simple circuit (Fig. 161) made up of one conductor of resistance R_2 joining the two terminals of a voltaic or a storage cell. In this circuit, the current is the same through any cross-section (from Kirchhoff's current law), so there is only one current to deal with. For convenience in specifying directions around the circuit, let an arrow be placed pointing in that direction in which the battery tends to force current. Then, arbitrarily, let the current in the direction of the arrow be represented by $+I$, and that in the opposite direction by $-I$. The symbol representing the **current in the direction of the arrow** is placed on the diagram. In this simple circuit Kirchhoff's e.m.f. law will give the one equation necessary to fix the value of I. There are three e.m.fs. acting in the closed circuit:

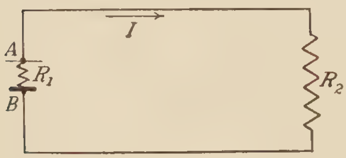


FIG. 161.—Simple circuit.

a. The intrinsic or the internal e.m.f. of the battery. The value of this e.m.f. in the direction of the arrow will be represented by E. E is assumed to be constant, although its value does change in time if the current flows long enough to produce marked changes in the concentration of the solution near the electrodes.

b. The e.m.f. of resistance in the conductor, namely, $-R_2I$.

c. The e.m.f. of resistance in the conducting path from terminal to terminal inside the cell, namly, $-R_1I$, in which R_1 is the internal resistance of the cell.

The sum of these three e.m.fs. is zero.

$$E - R_2I - R_1I = 0 \tag{273}$$

Solving,
$$I = \frac{E}{R_1 + R_2}. \tag{274}$$

The manner in which the current varies with the resistance R_2 of the conductor is shown by the curve marked "current" of Fig. 162. The current has the maximum possible value E/R_1 when R_2 is zero, and it decreases to zero as R_2 is increased without limit.

If an electrostatic voltmeter is connected across the cell terminals while the cell is supplying current, the voltmeter will read not the internal e.m.f., but the terminal e.m.f. e_t of the cell.

$$e_t = E - R_1I = R_2I.$$

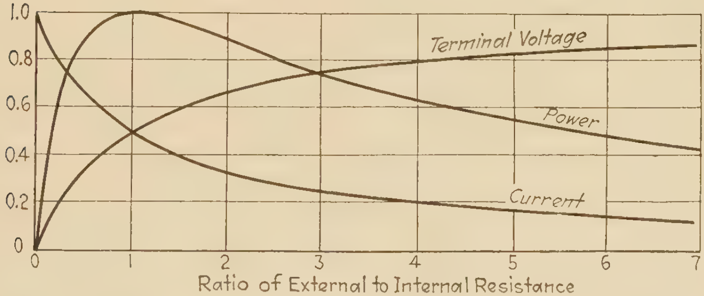


FIG. 162.—Variation of current, terminal voltage and power with external resistance.

Substituting the value of I from Eq. 274 in this expression, we obtain

$$e_t = E - \frac{R_1E}{R_1 + R_2} = \frac{ER_2.}{R_1 + R_2}. \tag{275}$$

To determine the internal resistance of a cell by the only possible method, namely, the method of experimental measurement, we make use of Eq. (275) in the following manner: We first measure the terminal voltage of the cell when R_2 is either infinite or is very great in comparison with the unknown internal resistance R_1. Equation (275) shows that the internal voltage E is equal to the terminal voltage thus measured. We then connect across the cell a resistance R_2 of known value, and measure either

the current I which flows, or the terminal voltage of the cell e_t while the current is flowing. Upon substituting the value of E, I, and R_2 in (274) or E, e_t and R_2 in (275), the unknown internal resistance R_1 may be computed.

The manner in which the terminal e.m.f. varies with the resistance R_2 of the conductor is shown by the curve marked "terminal voltage" of Fig. 162. The terminal e.m.f. rises from zero at $R_2 = 0$ to E as R_2 becomes great in comparison with R_1.

The useful power delivered by the cell is that expended in the external circuit of resistance R_2 namely,

$$P = e_t I = \frac{E^2 R_2}{(R_1 + R_2)^2}. \tag{276}$$

The manner in which the power delivered by the cell varies with R_2 is shown by the curve marked "power" in Fig. 162. The power delivered is zero when R_2 is zero, or when the cell is short circuited. It rises to a maximum as R_2 is increased and then again falls to zero as R_2 is made infinitely large.

To find the value of R_2 which results in the maximum power delivery to it through a fixed resistance R_1, we take the derivative of P with respect to R_2, equate it to zero, and solve for R_2. Upon doing this, it is found that the delivery is a maximum when the external resistance R_2 is made equal to the internal resistance R_1. In this case the power delivered outside and the power wasted inside the cell have the same value, $E^2/(4R_1)$, or the efficiency of utilization of the electrical energy of the cell is 50 per cent.

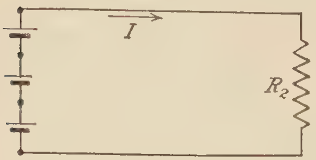


Fig. 163.—Cells in series.

201. Cells in Series and in Parallel.—Suppose that n cells all similar to the cell used above are connected in series as shown in Fig. 163. Writing Kirchhoff's law of e.m.fs. for the circuit,

$$-R_2 I + (E - R_1 I) + (E - R_1 I) + \text{to } n \text{ terms} = 0$$

or

$$-R_2 I + n(E - R_1 I) = 0.$$

Solving for I, we obtain

$$I = \frac{nE}{nR_1 + R_2}. \tag{277}$$

Suppose that the n cells are all connected in **parallel** to two heavy copper terminals of negligible resistance, as shown in Fig. 164. Writing Kirchhoff's law of currents for the junction point A,

$$-I + (I_1 + I_1 + I_1 + \text{to } n \text{ terms}) = 0$$

or

$$I = nI_1. \tag{278}$$

Writing the law of electromotive forces for a complete circuit comprising the conductor of resistance R_2,

$$E - R_1I_1 - R_2I = 0.$$

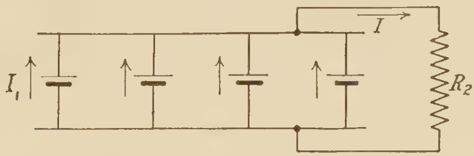


FIG. 164.—Cells in parallel.

Substituting the value of I_1 from Eq. (278) in this equation and solving for I,

$$I = \frac{E}{R_2 + \frac{R_1}{n}}. \tag{279}$$

A comparison of Eqs. 274, 277, and 279 shows that a battery consisting of n identical cells connected in series is the **equivalent** of a single cell having an internal e.m.f. and an internal resistance each n times as great as the e.m.f. and resistance of one of the constituent cells. On the other hand, a battery consisting of n identical cells all connected in parallel is the **equivalent** of a single cell having an internal e.m.f. equal to that of one of the constituent cells and an internal resistance only $1/n$th as great as the internal resistance of one of the constituent cells.

202. Equivalent Resistance and Conductance of a Network Connecting Two Terminals.—A problem which frequently occurs is that of calculating the resistance or the conductance between two terminals which are connected—not by a single wire of known dimensions—but by a number of wires of known resistance or of known conductance. In general terms, the problem is that of computing the **equivalent resistance or con-**

ductance of the network of conductors between two specified terminals.

The various combinations of the conductors connecting two terminals may be divided into the following four types:

1. Series combinations, as in Fig. 165.
2. Parallel combinations, as in Fig. 166.
3. Series-parallel, as in Fig. 152.
4. Bridge networks, as in Fig. 153.

Simple formula may be derived by which the equivalent resistance and conductance of combinations falling under any of the first three types may be very easily calculated. The equivalent resistances of bridge networks are rarely computed, since any derived formula is of very limited application, and is not easy to apply.

By the equivalent conductance of a network connecting two terminals A and B, we mean the conductance of a single conductor which, for a given difference of potential between A and B, will permit the same current to flow from A to B as does the network. This definition suggests the general method of deriving the formula for the equivalent conductance of any network, namely to imagine a battery of zero internal resistance and of electromotive force E to have its terminals connected to A and B. By applying Kirchhoff's laws to the network, we obtain equations from which we may derive an expression for the current I delivered by the battery. Then the equivalent conductance G of the network is equal to I/E. We proceed to apply this method.

203. Conductors in Series.—When two or more conductors are connected end to end between A and B, as in Fig. 165, they are said to be connected in **series.** From Kirchhoff's current law, the current has the same value in every part of this circuit.

Let E represent the electromotive force of the battery which we imagine to be connected across AB, and let R_1, R_2, R_3, etc. represent the known resistances of the separate conductors. By writing Kirchhoff's e.m.f. law for this circuit, we obtain the following equation:

$$E - R_1I - R_2I - R_3I - \cdots = 0, \tag{280}$$

whence

$$\frac{E}{I} = R_1 + R_2 + R_3 + \cdots$$

But by definition, the equivalent resistance R_t from the terminal A to B is the ratio of E to I. Therefore,

$$R_t = R_1 + R_2 + R_3 + . \tag{281}$$

The equivalent conductance is the reciprocal of R_t.

Equations (281) and (280) may be stated in the following form:

203a. Resistance of Conductors in Series.—*The equivalent resistance of a series combination of conductors is equal to the sum of the individual resistances.*

The total voltage across a series combination of conductors is divided among them in the proportion of their resistances.

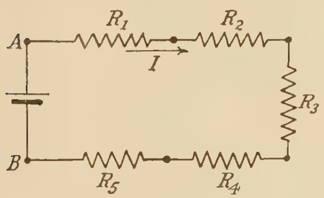


FIG. 165.—Conductors in series.

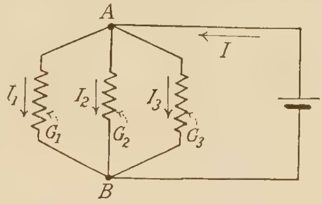


FIG. 166.—Conductors in parallel.

204. Conductors in Parallel.—When two or more conductors are connected to two common terminals A and B, as in Fig. 166, they are said to be connected in **parallel.**

Let E represent the electromotive force of the battery connected across AB; let G_1, G_2, G_3, etc. represent the conductances of the separate conductors, and let I_1, I_2, I_3, etc. represent the currents in the same conductors and I the current in the battery in the direction of the arrows.

By writing Kirchhoff's law of currents for the junction A, we obtain the equation

$$I - I_1 - I_2 - I_3 = 0. \tag{282}$$

By writing the electromotive force law for circuits through the battery and the respective conductors, we obtain the equations

$$\left.\begin{aligned} E - \frac{I_1}{G_1} &= 0 \quad \text{or} \quad I_1 = G_1E \\ E - \frac{I_2}{G_2} &= 0 \quad \text{or} \quad I_2 = G_2E \\ E - \frac{I_3}{G_3} &= 0 \quad \text{or} \quad I_3 = G_3E \end{aligned}\right\} \tag{283}$$

Substituting these values of I_1, I_2, I_3, etc. in Eq. (282), we have

$$I = G_1E + G_2E + G_3E,$$

whence
$$\frac{I}{E} = G_1 + G_2 + G_3.$$

But, by definition, the equivalent conductance G_t from the terminal A to B is the ratio of I to E. Therefore,

$$G_t = G_1 + G_2 + G_3 + . \qquad (284)$$

Equations (284) and (283) may be thus stated:

204a. Conductance of Conductors in Parallel.—*The equivalent conductance of a parallel combination of conductors is equal to the sum of the conductances of the branches.*

The total current delivered to a parallel combination of conductors divides among the branches in proportion to their conductances.

Since G_t, in Eq. (284), is the reciprocal of the equivalent resistance, we may rewrite Eq. (284) in the form

$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Whence the formula for the equivalent resistance is

$$R_t = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}. \qquad (285)$$

A comparison of Eq. (285) with the simpler Eq. (284) shows why calculations pertaining to parallel combinations should be carried on in terms of the conductances of the parts rather than in terms of the resistances.

205. Series-parallel Networks.—A series combination of conductors may be put in parallel with one or more like combinations, the resulting parallel combination may be put in series with another, etc., until very many conductors have been combined in one **series-parallel** network, such as that illustrated in Fig. 152. The series-parallel combination is, in reality, not a new type. It is a combination of the first two types. By using the formulas derived for these two types, the equivalent resistance

or conductance of the network may be easily determined. This is accomplished **by starting with the simple combinations** and finding the equivalent constants of these, then applying the same laws again to the new simple combinations which appear, and so on, until the equivalent resistance of the whole network has been obtained, provided all combinations are series or parallel arrangements. This process of calculation cannot be carried completely through if the network contains any bridge connections of the kind pointed out below.

The distinction between the series-parallel network (for which the equivalent resistance may be easily computed) and the bridge network discussed below (for which the calculation is very laborious) is this: *A network is not of the series-parallel type unless the currents I_1, I_2, I_3, etc. leaving a junction point can be identified as currents of the same value all meeting again at a second junction point.*

206. Bridge Networks.—There are numberless bridge networks which may be connected between two points. One of these, known as the Wheatstone bridge, is shown by the full lines of Fig. 153.

It is evident that for any type of networks such as that shown in Fig. 153 we could apply Kirchhoff's laws, to determine all the currents and to determine the equivalent resistance of the networks by the general method outlined in Sec. 202. This is not done, however, in the bridge networks, for two reasons: (*a*) any one of these networks appears less frequently than the series or parallel arrangements; (*b*) a much more important reason is that the resulting formulas are too complicated to be remembered, or even to be conveniently used. As an illustration, the formula for the equivalent resistance from A to B in Fig. 153, which is the simplest type of bridge network, has been derived in Sec. 207*b*.

Formulas are not depended on for bridge networks, but Kirchhoff's laws are applied directly to the solution of each numerical problem, using the methods described in Sec. 207.

207. Calculation of the Currents in Any Electric Network.—In the preceding sections we have applied Kirchhoff's laws to

series and parallel combinations and have derived general formulas which relieve us of the need of directly applying the laws to the solution of particular circuits of these types. But for the general class of networks containing bridge networks or containing batteries or generators at various points throughout the network, the calculations must be carried out by the direct application of Kirchhoff's laws. In practically all cases, the problem is to calculate the currents in the various branches of a network in which we know the resistance of each branch and the intrinsic e.m.f. and polarity of each battery or generator.

The application of Kirchhoff's laws to such a network yields a number of algebraic equations between the unknown currents and the constants of the network, and the values of the currents are determined by solving these simultaneous equations. In such a process, the correct use of algebraic signs is **absolutely essential.** This requires a rigid adherence to some system of conventions or agreements concerning the meaning and use of signs. A feature of this text is the careful attention paid to the statement of a systematic set of conventions. The following statement of the steps in the solution of network problems emphasizes these conventions, and should assist the student to apply Kirchhoff's laws in a systematic way.

207a. Steps in the Systematic Solution of Network Problems. **Step 1.**—Draw a diagram of the network, showing the total resistance of each branch and the location of each battery or generator. (An essential first step in the attack of any physical problem is the picturing of the given physical conditions. Usually this is best accomplished by means of a diagram.)

Step 2 (Optional).—Search the diagram for parallel combinations of branches which are free of battery or generator e.m.fs. Redraw the diagram, replacing the parallel combination by an equivalent conductor. Again search the diagram for new parallel combinations which may appear, and redraw. Repeat until the diagram has been reduced to the simplest possible form. The application of Kirchhoff's laws to the simplified diagram will give the currents in the equivalent conductors, and additional calculations (see step 9) will be necessary to determine the currents in the original branches.

Step 3.—Draw an arrow along each branch for convenience in specifying directions (see Sec. 198a for the conventions as to directions and algebraic signs).

Step 4.—On the diagram, along each branch, place a symbol, such as I_1, I_2, I_3, etc., to represent the algebraic value of the **current in the direction of the arrow.** (The value of the current against the arrow will then be represented by the same symbol with the opposite sign.)

Likewise, beside each generator or battery write the known algebraic value of the generated **e.m.f. in the direction of the arrow** along that branch. From the conventions (Sec. 198a) this value is positive when the generator tends to send current in the direction of the arrow, and the value is negative when it tends to force current in the other direction.

Step 5.—To obtain the equation between the currents meeting at junction points, write the equation expressing Kirchhoff's current law, namely, "The sum of the algebraic values of the currents in the direction **toward a** junction point is zero." In any particular branch, the arrow along that branch may be toward the junction point, in which case the current in that branch should be represented **in the equation** by the symbol written on the diagram. Or the arrow in any branch may point away from the junction point, in which case that current should be represented **in the equation** by the negative of the symbol on the diagram. It will be found by experience that independent equations may be written for one less than the total number of junction points, that is. seven junction points will yield six independent equations.

Steps 4 and 5.—A straightforward elimination of unnecessary current symbols by the combination of steps 4 and 5. In step 4, when writing symbols on the diagram to represent the currents along the branches in the direction of the arrows, one may reduce the number of separate symbols required by applying Kirchhoff's law of currents at each junction. By so doing, the current in one branch at each junction point may be represented by a combination of the symbols representing the currents in the other branches meeting at that junction. The number of separate symbols for unknown currents may be reduced in this way to the

number of independent equations which can be obtained by applying the e.m.f. law, as in the next step. If this plan is followed, any further attempt to use the current law will result in useless identities, such as 0 = 0.

Step 6.—For any chosen circuit in the network, write the equation expressing Kirchhoff's electromotive force law, namely, "around any closed circuit, the sum of the algebraic values of the e.m.fs. in a specified direction is zero." The specified direction around a closed circuit may be arbitrarily chosen as the clockwise direction or the counterclockwise direction on the diagram. The chosen direction around any circuit probably will be in the direction of the arrows along some branches composing the circuit. In such cases, the battery or generator e.m.fs. should be written in the equations just as they appear on the diagram, since, from the agreement in step 4, the values on the diagram are the values in the directions of the arrows. The e.m.fs. of resistance in these branches should be written as the product of $-R$ and the symbol on the diagram which represents the current in the direction of the arrow. If the chosen direction around any circuit is against the arrow in a given branch, the e.m.fs. in that branch should be written with signs opposite to those indicated above.

Apply the law to one closed loop after another until the total number of independent equations is equal to the number of unknown symbols appearing in the equations. Choose closed circuits which comprise few branches and which, therefore, yield short equations. Choose circuits so that each new circuit contains one or more new branches, thus insuring that this equation is independent of the previous equations; remember that every branch must be traced through at least once before the system of equations can be complete. These hints usually enable one to obtain the requisite number of independent equations without algebraic tests to determine whether or not the equations are independent. Any equation which is not independent results in an identity, such as 0 = 0, in the course of the attempt to solve simultaneously.

Step 7.—By the usual algebraic methods solve the independent equations to determine the currents. The algebraic work must be systematic in order to avoid useless work. For example, if at

first there are five unknowns and five equations, the work should be definitely planned to eliminate one particular unknown, leaving four unknowns and four equations. Then as the next step, eliminate one more, leaving three unknowns and three equations, and so on. The **substitution method** of elimination is recommended, since in the hands of students who have forgotten the details of different methods it leads to a more orderly elimination and to less aimless juggling of equations.

The substitution method of elimination consists in solving any one of the five equations for an expression representing one of the currents, say I_3 in terms of the others. This expression for I_3 is then substituted wherever I_3 appears in each of the other four equations. The result **must** be four equations from which I_3 has been eliminated. Now from these four equations another unknown is eliminated by a similar substitution, and so on.

Step 8.—Interpret the results. If any current symbol, such as I_3, is found to be equal to a negative number, say -8, it means that in that branch the current in the direction of the arrow is -8 amperes, or the current is 8 amperes flowing in the counterarrow direction.

Step 9.—If the network has been simplified by replacing a parallel combination by an equivalent conductor, the currents in the original branches may be found from the current in the equivalent conductor by applying the principle that the current in the equivalent conductor is to be divided among the branches in proportion to their conductances.

207b. Example.—As an illustration of the use of this method let us compute the resistance from A to B of the network shown by the full lines in Fig. 153.

Imagine a battery of voltage E to be connected through leads of negligible resistance to A and B.

Step 3.—Let the arrow directions along the branches be as indicated.

Steps 4 and 5.—At junction A, let I_1 and I_2 stand for the current in the arrow directions in the branches BA and AC respectively. Then the current in the branch AD is $I_1 - I_2$.

At the junction C, let I_3 represent the current in the branch CB. Then the current in the branch CD is $I_2 - I_3$.

At the junction D, the current in the branch DB is $I_1 - I_3$.

At the last junction D, the three currents toward the junction sum up to zero. Failure to do this would be evidence of a blunder in the preceding work.

Step 6.—Writing the law of electromotive forces for the circuit $BEACB$,

$$E - R_1 I_2 - R_2 I_3 = 0. \tag{286}$$

For the circuit $ACDA$ it is

$$-R_1 I_2 - R_5(I_2 - I_3) + R_3(I_1 - I_2) = 0. \tag{287}$$

For the circuit $BDCB$ it is

$$+R_4(I_1 - I_3) + R_5(I_2 - I_3) - R_2 I_3 = 0. \tag{288}$$

From Eq. (286)

$$I_3 = \frac{E}{R_2} - \frac{R_1 I_2}{R_2}.$$

Substituting for this value I_3 in Eq. (287) and (288) we obtain two equations containing two unknowns, I_1 and I_2. Solving either of these equations for I_2 and then substituting this value for I_2 in the other equation, we obtain the following:

$$\frac{E}{I_1} = \frac{R_1R_3(R_2 + R_4) + R_2R_4(R_1 + R_3) + R_5(R_1 + R_2)(R_3 + R_4)}{(R_1 + R_3)(R_2 + R_4) + R_5(R_1 + R_2 + R_3 + R_4)}. \tag{289}$$

But E is the voltage between the terminals, and I_1 is the current taken by the network; therefore,the quotient E/I_1 gives the expression for the resistance of the network between A and B.

208. Wheatstone Bridge Network for the Measurement of Resistance.—The most common method of measuring an unknown resistance is a comparison method carried out with the apparatus known as the Wheatstone bridge. The circuit of the bridge is illustrated in Fig. 167. The unknown resistance R_3 is connected in a network with three other known resistances, R_1, R_2, and R_4. The four resistances are called the four **arms** of the bridge. The three known resistances are in the form of coils of resistance wire so connected to the contacts of dial switches that they can be readily set at any desired values.

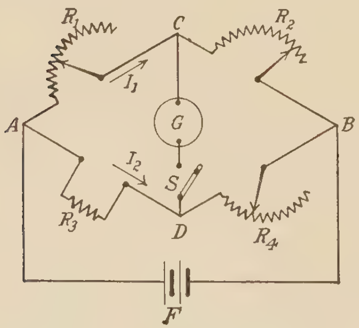


FIG. 167.—Wheatstone bridge.

A battery F, usually of one or more dry cells, is connected to the vertices A and B, and a sensitive detector of current, G, such as a galvanometer, is connected through a switching key S across the vertices C and D.

To measure the value of the unknown resistance R_3, the variable resistances are adjusted until upon closing the key S no

deflection of the galvanometer occurs. The bridge is then said to be **balanced.** In this balanced condition, the current in the branch CD is zero, the e.m.f. of resistance across it is zero, and the current in the two upper arms has the same value I_1 in each arm; likewise, the current in the lower arms has the same value I_2 in each arm.

Since the voltage across CD is zero, the e.m.f. law when written for the circuit $ACDA$ yields the equation,

$$R_1I_1 = R_3I_2,$$

and when written for $BDCB$ it yields

$$R_2I_1 = R_4I_2.$$

Whence, dividing equals by equals, and equating

$$\frac{R_3}{R_4} = \frac{R_1}{R_2}. \tag{290}$$

From the known values of R_1, R_2, and R_4 which cause the bridge to balance, the value of the unknown resistance R_3 may be readily computed. If the resistances are all non-inductive (see Chap. XIII) the battery F is frequently replaced by a source of alternating e.m.f. having a frequency within the audible range, and the galvanometer G is replaced by a telephone receiver.

209. The Use of the Mesh Currents in Network Calculations.—An alternative way (frequently more convenient) of carrying on network calculations is to regard the network as made up of circuits, each of which has some portion of itself in common with one or more of the other circuits. The calculations are then carried on in terms of the currents in these circuits, the **mesh currents,** instead of the current in the branches.

Thus we may solve the problem in Sec. 207*b* as follows:

Steps 3 and 4.—Let I_1, I_2, and I_3 represent the currents flowing in the arrow direction in the meshes indicated in Fig. 168.

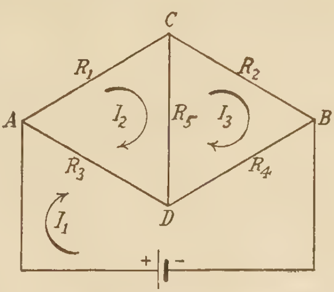


FIG. 168.—Mesh currents.

Steps 5 and 6.—When we come to write the e.m.f. law for any one of these meshes, we note that we must take into account the e.m.fs. from three effects:

a. The intrinsic e.m.fs. of the sources in the mesh.

b. The e.m.f. of resistance due to the mesh current.

c. The e.m.f. of **mutual resistance** arising from the fact that some portion of a circuit which is part of a network is necessarily common to some other mesh and is also traversed by the current of the other mesh.

Now the e.m.f. law when written for a simple series circuit is of the form

$$E_1 + E_2 + \ . \ . \ . \ +(-R_1I_1) + (-R_2I_1) + (R_3I_1) + = 0, \qquad (207)$$

in which E_1, E_2, etc. represent the intrinsic e.m.fs. in the circuit; R_1, R_2, etc. represent the resistances of parts of the circuit; and I_1 represents the value of the current in the arrow direction. If the circuit has portions in common with other circuits, a third group of terms must be added to express the e.m.fs. in these portions due to the currents of the other meshes.

For the sake of mathematical similarity let us agree to use the same form for the e.m.f. of **mutual resistance** as for the customary e.m.f. of resistance. Thus

$$E_1 = (-RI_1) \text{ (e.m.f. of resistance)} \qquad (230)$$

$$E_1 = (\ -\ R_{1,2}I_2) \text{ (e.m.f. of mutual resistance)} \qquad (291)$$

in which E_1 is the e.m.f. in the arrow direction in mesh 1 due to the fact that the current I_2 in the arrow direction in mesh 2 flows through the resistance $R_{1,2}$ which is common to meshes 1 and 2. In other words, the e.m.f. equation for mesh n will be of the form

$$E_n + (-R_nI_n) + (-R_{n,p}I_p) = 0. \qquad (292)$$

If this convention is adopted, and if the mutul resistance between two meshes is always represented by the symbol $+R_{n,p}$, *then a positive numerical value will have to be assigned to* $R_{n,p}$ *if a current in the arrow direction in mesh n flows through the common branch* $R_{n,p}$, *in the same direction as would a current in the arrow direction in mesh p, otherwise a negative value must be assigned to it.*

Writing the law of electromotive forces for mesh *BEADB*,

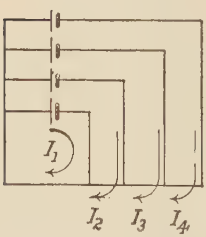


FIG. 169.—Mesh currents in general circuit.

$$E - I_1(R_3 + R_4) - R_{1,2}I_2 - R_{1,3}I_3 = 0$$

or

$$E - I_1(R_3 + R_4) + R_3I_2 + R_4I_3 = 0.$$

For meshes *ACDA* and *BDCB*, it is

$$-I_2(R_1 + R_5 + R_3) + R_5I_3 + R_3I_1 = 0$$

and

$$-I_3(R_4 + R_5 + R_2) + R_5I_2 + R_4I_1 = 0.$$

Upon solving these equations, and making the proper comparison with the previous solution, the results will be found to be identical.

210. Theorems Relating to Networks.—In the most general case, each mesh of an n-mesh network may contain a source of e.m.f. and may have parts in common with every other mesh of the network. Such a network is illustrated in Fig. 169. By writing the law of electro-

motive forces for each circuit, the following simultaneous linear equations are obtained:

$$\left.\begin{aligned} R_{1,1}I_1 + R_{1,2}I_2 + R_{1,3}I_3 + R_{1,4}I_4 &= E_1 \\ R_{2,1}I_1 + R_{2,2}I_2 + R_{2,3}I_3 + R_{2,4}I_4 &= E_2 \\ R_{3,1}I_1 + R_{3,2}I_2 + R_{3,3}I_3 + R_{3,4}I_4 &= E_3 \\ R_{4,1}I_1 + R_{4,2}I_2 + R_{4,3}I_3 + R_{4,4}I_4 &= E_4 \end{aligned}\right\} \tag{293}$$

In which $R_{k,k}$ represents the resistance of the kth circuit, and $R_{p,k}$ represents the resistance which must be mutiplied by the current in the kth circuit to get the e.m.f. of mutual resistance in the pth circuit.

These equations may be solved by the method of determinants. Thus in the notation of determinants, the expression for I_2 is

$$I_2\begin{vmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} \end{vmatrix} = \begin{vmatrix} R_{1,1} & E_1 & R_{1,3} & R_{1,4} \\ R_{2,1} & E_2 & R_{2,3} & R_{2,4} \\ R_{3,1} & E_3 & R_{3,3} & R_{3,4} \\ R_{4,1} & E_4 & R_{4,3} & R_{4,4} \end{vmatrix}. \tag{294}$$

Using the symbols,

$$D = \begin{vmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} \end{vmatrix} \tag{295}$$

and

$$M_{2,3} = \begin{vmatrix} R_{1,1} & R_{1,2} & R_{1,4} \\ R_{3,1} & R_{3,2} & R_{3,4} \\ R_{4,1} & R_{4,2} & R_{4,4} \end{vmatrix}$$

(called the minor of the second row and third column).

The expression for the current in mesh 2 may be written

$$I_2 = -\frac{E_1M_{1,2}}{D} + \frac{E_2M_{2,2}}{D} - \frac{E_3M_{3,2}}{D} + \frac{E_4M_{4,2}}{D}. \tag{297}$$

From this result, the following theorem may be deduced.

210a. Superposition Theorem.—*If a network contains two or more generators located in different meshes, the current flowing in any mesh is the algebraic sum of the currents which would flow in the same mesh if each generator in turn were the only one acting in network. All other generators are assumed for the time being to have zero e.m.f.*

Suppose there is a driving e.m.f. (source of intrinsic e.m.f.) in the mesh 4 only. The current it causes in mesh 2 is given by

$$DI_2 = E_4 \begin{vmatrix} R_{1,1} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,3} & R_{2,4} \\ R_{3,1} & R_{3,3} & R_{3,4} \end{vmatrix}. \tag{298}$$

On the other hand, suppose mesh 2 above contains a driving e.m.f. The current it causes in mesh 4 is given by

$$DI_4 = E_2 \begin{vmatrix} R_{1,1} & R_{1,2} & R_{1,3} \\ R_{3,1} & R_{3,2} & R_{3,3} \\ R_{4,1} & R_{4,2} & R_{4,3} \end{vmatrix}. \tag{299}$$

Now since $R_{p,k} = R_{k,p}$, these two minors are identical save that the rows in one are the columns of the other. Therefore, they are equal in value. Therefore, if the driving e.m.fs. are equal, the currents are equal. From this the following proposition, known as the reciprocity theorem, may be advanced.

210b. Reciprocity Theorem.—*The current in the kth mesh due to a given driving e.m.f. in the pth mesh is equal to the current caused in the pth mesh by an equal driving e.m.f. in the kth mesh.*

In both of the above theorems, the word "branch" may be substituted for "mesh" whenever the latter occurs.

211. Theorems Used in Mapping the Field in Conducting Materials.—By a **line of (electric) flow** in a conductor is meant a line whose direction at each point coincides with the direction in which positive electricity tends to move.

By a **tube of (electric) flow** is meant the tubular surface formed by all the lines of flow which can be passed through points in the boundary of any small area in the conductor.

In an isotropic conductor the lines and tubes of flow coincide with the lines and tubes of electric intensity.

211a. Relation between the Electric Intensity and the Current Density Vectors.—Imagine a small right *cylinder* of conducting material of length l and cross-sectional area a, enclosing a point P. Let the elements of the cylinder coincide with a

tube of flow in which the current is I. Then the difference in potential between the end faces is

$$E = RI = \rho \frac{l}{a} I$$

or

$$\frac{E}{l} = \rho \frac{I}{a},$$

but E/l is the electric intensity F at P, and I/a is the current density J at P.

Whence

$$\mathsf{F} = \rho \mathsf{J} \tag{300}$$

and

$$\mathsf{J} = \gamma \mathsf{F}. \tag{300a}$$

211b. Law of Currents Expressed by Surface-integrals.—The current over any small plane surface of area a at a point in a conductor at which the current density has the value J is

$$dI = J \cos (J,n) da, \tag{301}$$

in which the angle (J,n) is the angle between the J vector and the normal n to the patch of area da.

If the surface is large and the current density is variable over the surface, the current over the surface is found by taking the surface-integral (or the flux) of the current density over the entire surface (see Sec. 84).

$$I = \int J \cos (J,n) da. \tag{302}$$

Now the current over a **closed** surface in the outward direction is the time rate at which + electricity is leaving the volume enclosed by the surface. Therefore we may write: Therefore

$$\int^{\text{closed surface}} J \cos (J,n) da = -\frac{dq}{dt}. \tag{303}$$

In the case of unvarying currents, the surface-integral over a **closed** surface must be zero. Equation (303) shows that if it were not zero, the quantity of electricity in the volume would increase without limit. This we know to be impossible. Therefore

$$\int^{\text{closed surface}} J \cos (J,n) da = 0 \text{ (for unvarying flow)}. \tag{304}$$

211c. Law of Currents Expressed in Vector Notation.—Dividing both members of Eq. (303) by the volume v enclosed by the surface,

$$\int^{\text{closed surface}} \frac{J\cos(J,n)da}{v} = -\frac{d}{dt}\frac{q}{v}.$$

If the volume v becomes infinitesimal, the left member is the **divergence** of the current density and q/v is the volume density of charge ρ. Hence in vector notation:

$$\operatorname{div} \mathbf{J} = -\frac{d\rho}{dt} \tag{305}$$

and

$$\operatorname{div} \mathbf{J} = 0 \text{ (for unvarying flow).} \tag{306}$$

211d. The Impossibility of a Volume Distribution of Charge in a Homogeneous Conductor with Unvarying Flow.—Since $J = \gamma F$, and since **within** any **homogeneous** conductor γ is a constant, it follows from Eq. (304) that the electric intensity over any closed surface lying wholly within a homogeneous conductor must satisfy the relation

$$\int^{\text{closed surface}} F\cos(F,n)\,da = 0. \tag{307}$$

It follows from Gauss's theorem that the quantity of electricity enclosed by the surface must be zero.

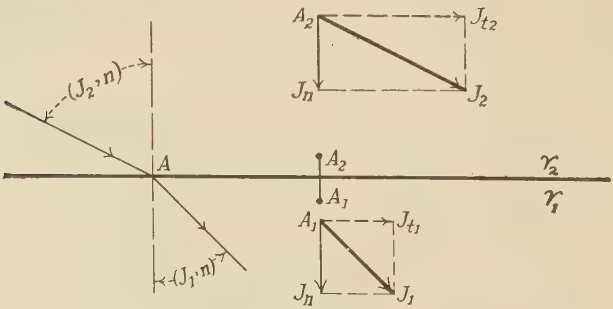


Fig. 170.—Refraction of lines of flow at interface.

211e. The Refraction of the Lines of Flow at an Interface.—Let us consider the conditions which must be satisfied at an interface between two conducting materials having not only different conductivities but different permittivities. Consider the two points in Fig. 170. A_1 is in the conducting material of lower conductivity γ_1, having the permittivity p_1; A_2 is in the material

of higher conductivity γ_2, and of permittivity p_2. Let these points be adjacent to each other at infinitesimal distances from the interface. The portion of the interface near the points may be regarded as a small plane surface, as illustrated in the figure.

Letting symbols with the subscripts $_1$ and $_2$ stand for the values of the quantities at the points A_1 and A_2, respectively, we have:

In conductor 1, $\quad J_1 = \gamma_1 F_1$

and in conductor 2, $\quad J_2 = \gamma_2 F_2. \qquad (308)$

At the points A_1 and A_2, the components of the electric intensity tangential to the interface must be substantially equal.

$$F_{t1} = F_{t2}. \qquad (309)$$

Therefore, from Eqs. (308) and (309),

$$\frac{J_{t1}}{J_{t2}} = \frac{\gamma_1}{\gamma_2}. \qquad (310)$$

At the points A_1 and A_2 **with unvarying currents,** the components of the current density normal to the interface must be equal, otherwise obvious charge would accumulate without limit at the interface. Whence

$$J_{n1} = J_{n2}. \qquad (311)$$

Therefore, from Eqs. (308) and (311)

$$\frac{F_{n1}}{F_{n2}} = \frac{\gamma_2}{\gamma_1}. \qquad (312)$$

At any point A (Fig. 170) on the interface, let (J_1, n) and (J_2, n) represent the angles between either the electric intensity vectors or the current density vectors and the normals to the surface in the dielectrics of small and large conductivity, respectively.

$$\tan\,(J_1, n) = \frac{F_t}{F_{n1}} = \frac{J_{t1}}{J_n}$$

$$\tan\,(J_2, n) = \frac{F_t}{F_{n2}} = \frac{J_{t2}}{J_n}.$$

Therefore, $$\frac{\tan\,(J_1, n)}{\tan\,(J_2, n)} = \frac{F_{n2}}{F_{n1}} = \frac{J_{t1}}{J_{t2}} = \frac{\gamma_1}{\gamma_2}. \qquad (313)$$

This relation may be stated as follows:

211f. (Deduction).—*In the case of an* **unvarying electric current,** *the lines of electric intensity and the lines of flow in passing from one conducting medium to another undergo an abrupt change in direction, being* **refracted** *at the interface in such a way that*

a. The incident and refracted lines lie in the same plane, which is perpendicular to the interface at the point of incidence.

b. The angle (J, n) *between the line and the normal to the surface is always greater in the conductor of greater conductivity, the ratio of the tangents of these angles in the two conductors being a constant which is equal to the ratio of the two conductivities* (*see Fig.* 170).

If the conductivity of one material is many times that of the other (as in the case of an iron electrode dipping into river water in which γ_2/γ_1 is of the order of 10^8), Eq. (313) cannot possibly be satisfied if (J_1, n) (the angle between the line of flow and the normal in the material of lower conductivity) differs from zero by more than a fractional part of a second of arc. In other words, the lines of flow start out into a poor conductor in directions which are normal to the surface of an electrode of high conductivity.

211g. Analogy between Electric Fields in Dielectrics and Conducting Materials.—A comparison of the equations and deductions in this section for conducting regions with the equations and deductions for non-conducting dielectrics (Secs. 98 to 102) will show that the laws for the two regions are of identically the same mathematical form, with the following analogies between the different physical quantities.

In the conducting field		*In the non-conducting field*
Electric intensity F	corresponds to	electric intensity
Current density J	" "	Electrostatic flux density D
Conductivity γ	" "	permittivity p
Resistivity ρ	" "	elastivity s
Tube of flow	" "	tube of electrostratic flux

This means that if an electric field is set up in a region containing any number of non-conducting dielectrics by maintaining two or more of the metallic bodies at fixed potentials, no alteration whatsoever will occur in the distribution of the electric intensities if the dielectrics become conducting, provided only that

the ratio of the conductivity to the permittivity is the same for all the materials in the field. That is,

$$\frac{\gamma_1}{p_1} = \frac{\gamma_2}{p_2} = \frac{\gamma_3}{p_3} = \text{etc.} \tag{314}$$

211h. Accumulation of Obvious Charge at an Interface.—We have seen that at two adjacent points on opposite sides of the interface between two **non-conducting** dielectrics, the normal components of the electric intensities are inversely proportional to the permittivities.

$$\frac{F_{n1}}{F_{n2}} = \frac{p_2}{p_1} \text{ (zero conductivity).} \tag{108}$$

On the other hand, if the materials conduct at all, no matter how poorly, the normal components of the intensities become inversely proportional to the conductivities when the steady state of unvarying flow is reached.

$$\frac{F_{n1}}{F_{n2}} = \frac{\gamma_2}{\gamma_1} \text{ (unvarying flow).} \tag{312}$$

As long as the latter condition is not satisfied, the current densities toward and from the interface in the two materials are not equal and **obvious** charge accumulates on the interface. Upon closing a switch which impresses the full voltages between the metal electrodes of a conducting system of dielectrics, the relations expressed in Eq. (108) are instantaneously assumed. If at any interface p_2/p_1 is not equal to γ_2/γ_1, charge of such a sign accumulates at the interface as to cause the intensities to drift from the relation expressed in Eq. (108) to that expressed in Eq. (312) (see Sec. 214 for the equations applying to the transient conditions).

212. Method of Computing Resistances of Non-cylindrical Conductors.—The resistance between end faces of a right cylindrical conductor of a material whose resistivity is known may readily be computed by the experimentally determined relation of Sec. 177*a*, namely, $R = \rho l/a$. If the conductor is of some other shape, as in Fig. 171, its resistance can be **computed** from its dimensions and from the known resistivity of the material, provided it is possible to map out the equipotential surfaces and

the lines of electric intensity within the conductor. The lines of electric flow coincide in direction with the lines of intensity, and if the distance between the equipotential surfaces is taken small enough, the tubes of flow divide the conducting material between two adjacent equipotential surfaces into cells which are for our purpose right cylinders, all in parallel. Consequently, the conductance of the material lying between two equipotential surfaces will be the sum of the conductances of the elementary cells.

The equipotential surfaces, in turn, divide the conductor into elementary conductors all arranged in series between the terminals. Consequently, the resistance of the conductor from terminal to terminal will be the sum of the resistances of these elementary slices.

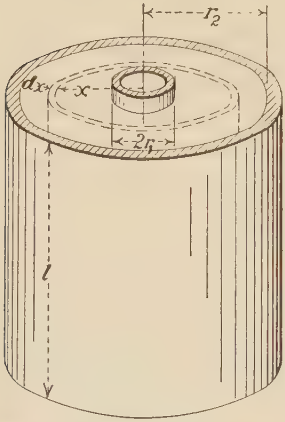


FIG. 171.—Water rheostat: co-axial pipe electrodes.

212a. Example.—Let us compute the resistance of a water resistor in which the metal electrodes are two coaxial iron pipes, and the resistor material is the river water with which the space between the pipes is filled (see Fig. 171). Let l represent the axial length of the conducting shell of water, and r_1 and r_2 represent its inside and outside radii. Since the conductivity of the steel is at least 10^6 times that of the water, the surfaces of the pipes are equipotential surfaces. From conditions of cylindrical symmetry the lines of flow are radial lines, and the equipotential surfaces are cylindrical surfaces coaxial with the pipes. These equipotential surfaces divide the conductor into elementary shells all connected in series from one pipe to the other.

Consider any shell of radius x and of thickness dx. Its resistance dR from equipotential surface to equipotential surface is that of a cylinder of length dx and of cross-sectional area $2\pi lx$. Therefore, if ρ represents the resistivity of the water,

$$dR = \frac{\rho dx}{2\pi lx}.$$

The resistance of the water from pipe to pipe is the sum of the resistances of all shells lying between them.

$$R \text{ (ohms)} = \int dR = \int_{r_1}^{r_2} \frac{\rho}{2\pi l} \frac{dx}{x}$$

$$R \text{ (ohms)} = \frac{\rho}{2\pi l} \log \frac{r_2}{r_1}. \tag{315}$$

212b. Example.—The case in which the conductance of a non-cylindrical conductor can readily be computed by resolving it into simple tubes of flow all in parallel is shown in Fig. 172. The resistor is water held in an annular trough of non-conducting material of wood or clay tile. The electrodes are the metal plates P, P. The lines of flow are circular arcs and the elementary tubes of flow are of constant cross-sectional area throughout their length. By taking the sum of the conductances of the tubes of flow it may be readily shown that the conductance from plate to plate is,

$$G \text{ (mhos)} = \frac{\gamma h}{\theta} \log \frac{r_2}{r_1} \frac{\text{(mho-centimeters)}}{\text{(radians)}}, \tag{316}$$

in which r_1 and r_2 represent the inside and outside radii of the trough.
h " " depth of the water.
θ " " length of the arc of flow in radians.
γ " " conductivity of the water.

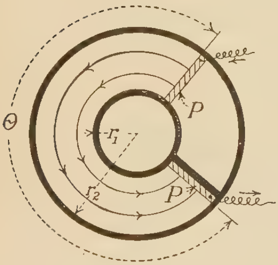


Fig. 172.—Water rheostat: annular trough.

213. Charge and Discharge of a Condenser.—The connections of a circuit for studying the phenomena during the charge or the discharge of

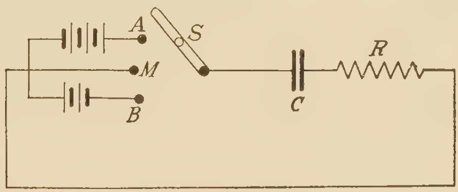


Fig. 173.—Circuit for charging and discharging a condenser.

a condenser through a resistance are illustrated in Fig. 173. A condenser of capacitance C is connected in series with a non-inductive resistance of R ohms. The switch S may be thrown to any one of three positions. If closed at A, the condenser is charged by the A battery through the resistance R. Upon opening the switch after charging, the condenser is left charged to the voltage of the A battery except as its charge gradually leaks through the dielectric. If, after charging, the switch is opened and is then thrown to the M position, the condenser discharges through the resistance R. If, after charging, the switch is thrown to the B position, the condenser may be charged to a higher or a lower voltage, or in the reverse direction, depending upon the connections and the relative voltages of the two batteries. Let us derive the equations which express the way in which the current and the voltage of the condenser vary in time during charge and discharge.

Let i represent the instantaneous value of the current.
e_c " " instantaneous value of the e.m.f. of the condenser.
q " " instantaneous value of the charge in the condenser.
E " " constant voltage of the battery which is connected in series with the condenser. (With the switch S on M, E is zero.)

Assuming an arrow direction, and writing the e.m.f. law for this circuit, we obtain

$$E - Ri - \frac{q}{C} = 0. \tag{317}$$

This is an equation with two dependent variables which are related by the equation

$$i = \frac{dq}{dt}. \tag{187}$$

Upon substituting this value of i in Eq. (317) we obtain

$$E - R\frac{dq}{dt} - \frac{q}{C} = 0,$$

a linear differential equation in which the variables may be separated by transposition, yielding

$$\frac{dq}{q - CE} = -\frac{dt}{CR}.$$

The solution of this equation is

$$\log (q - CE) = -\frac{t}{CR} + K_1$$

$$q - CE = \epsilon^{-\frac{t}{CR} + K_1}$$

$$q = CE + K\epsilon^{-\frac{t}{CR}}. \tag{318}$$

The value of the integration constant K is obtained by the following argument:

Let time be measured from the moment of the switching operation.

Let Q_0 represent the initial charge in the condenser—that is, the charge at the moment of the switching operation.

Let Q_u represent the **ultimate** charge of the condenser—namely, EC.

Let Q_d represent $Q_0 - Q_u$. (319)

Or Q_d represent the initial value minus the ultimate value of the charge, a quantity which we will call the **discrepancy** in the charge in the condenser.

For the instant $t = 0$, Eq. (318) reduces to

$$q_{(t=0)} = Q_0 = EC + K$$

$$Q_0 = Q_u + K.$$

Whence K must equal $Q_0 - Q_u = Q_d$ (320)

and
$$q = Q_u + (Q_0 - Q_u)\epsilon^{-\frac{t}{CR}} \tag{321}$$

or
$$q = Q_u + Q_d\epsilon^{-\frac{t}{CR}} \tag{321}$$

Since
$$e_c = -\frac{q}{C},$$

$$e_c = -\frac{q}{C} = -\frac{Q_u}{C} - \frac{Q_d}{C}\epsilon^{-\frac{t}{CR}}$$

$$e_c = E_{cu} + E_{cd}\epsilon^{-\frac{t}{CR}} \tag{322}$$

$$= -E + (E_{co} + E)\epsilon^{-\frac{t}{CR}}, \tag{322}$$

and, since

$$i = \frac{dq}{dt}, \tag{187}$$

$$i = -\frac{E_{cd}}{R}\epsilon^{-\frac{t}{CR}} \tag{323}$$

$$i = \frac{E + E_{co}}{R}\epsilon^{-\frac{t}{CR}}. \tag{323}$$

Figure 174 shows the current and the voltage of the condenser during the charge of an ordinary Leyden jar (C = 0.004 microfarad) through a 250-ohm resistance from a 500-volt battery.

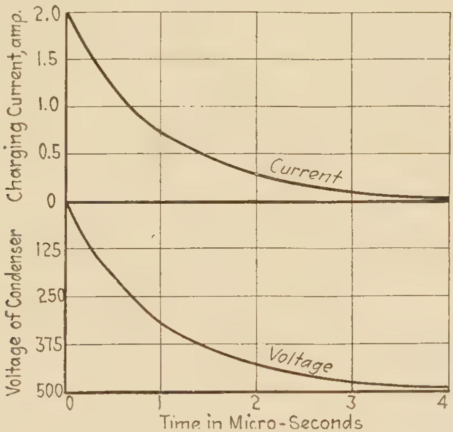


FIG. 174.—Current and voltage of a condenser during charge.

The time in which the exponential term decreases to $1/\epsilon$ *th* (36.8 *per cent*) *of its initial value is called the* **time constant** *of the circuit.*

For a circuit containing a condenser of capacity C in series with a resistance of value R, the expression for the time constant T_c is

$$T_c\text{(seconds)} = CR \text{ (farads, ohms)}. \tag{324}$$

For Fig. 174. the time constant is 10^{-6} seconds.

214. Residual Charge and Discharge.—If a condenser in which the dielectric is glass, mica, or paraffined paper is connected to a source of voltage for 5 or 10 minutes, and is then discharged by momentarily connecting the terminals by a wire, it appears at first to be completely discharged. If the condenser is allowed to stand for a minute with the two electrodes insulated, a new charge of the same sign as the original gathers on the plates, and a second much smaller spark may be obtained. This process may be repeated five or six times with some dielectrics, the spark becoming fainter each time. These charges which gradually appear after a condenser has been discharged are called **residual charges,** and the subsequent discharges are called **residual**

discharges. This effect can be accounted for if it be assumed that the dielectric is not homogeneous but consists of layers or strata of different resistivity. Let us investigate the properties of condenser which, for simplicity, we consider to be made up of only the two strata shown at *a* Fig. 175. The equivalent circuit is shown at *b*.

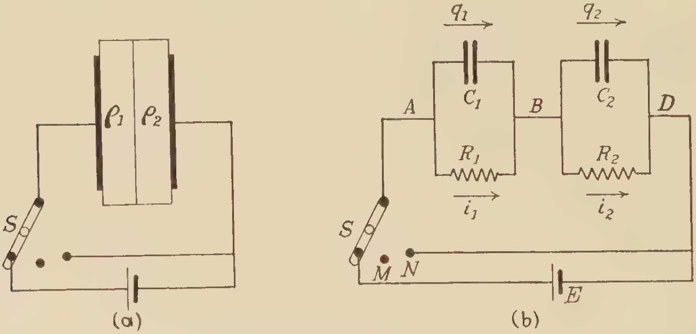


FIG. 175.—Equivalent circuit of a stratified condenser.

Let the arrow directions be as indicated. The e.m.f. law for the circuit $AC_1\,BC_2\,DA$ yields

$$E - \frac{q_1}{C_1} - \frac{q_2}{C_2} = 0. \tag{325}$$

For the circuits AC_1BA and BC_2DB it yields

$$-\frac{q_1}{C_1} + R_1i_1 = 0$$

$$-\frac{q_2}{C_2} + R_2i_2 = 0. \tag{326}$$

The current law for the junction B yields

$$\frac{dq_1}{dt} + i_1 - \frac{dq_2}{dt} - i_2 = 0. \tag{327}$$

Substituting the values of i_1 and i_2 from Eq. (326) in Eq. (327),

$$\frac{dq_1}{dt} + \frac{q_1}{C_1R_1} - \frac{dq_2}{dt} - \frac{q_2}{C_2R_2} = 0. \tag{328}$$

Substituting the value of q_2 and $\frac{dq_2}{dt}$ from Eq. (325) in Eq. (328) and collecting terms,

$$\frac{C_1 + C_2}{C_1}\frac{dq_1}{dt} = -\left[q_1 - \frac{EC_1R_1}{R_1 + R_2}\right]\frac{R_1 + R_2}{C_1R_1R_2}$$

or

$$\frac{dq}{q_1 - \dfrac{EC_1R_1}{R_1 + R_2}} = -\frac{R_1 + R_2}{R_1R_2(C_1 + C_2)}dt.$$

Whence

$$\log\left(q_1 - \frac{EC_1R_1}{R_1 + R_2}\right) = -\frac{t}{\dfrac{R_1R_2(C_1 + C_2)}{R_1 + R_2}} + K_1.$$

Writing C for $(C_1 + C_2)$

and R for $\dfrac{R_1R_2}{R_1 + R_2}$, (329)

$$q_1 = \frac{EC_1R_1}{R_1 + R_2} + K\epsilon^{-\frac{t}{CR}} \tag{330}$$

and

$$e_{c1} = -\frac{ER_1}{R_1 + R_2} - \frac{K}{C_1}\epsilon^{-\frac{t}{CR}} \tag{331}$$

and

$$i_1 = \frac{E}{R_1 + R_2} + \frac{K}{C_1R_1}\epsilon^{-\frac{t}{CR}}. \tag{332}$$

214a. Example.—For a glass-plate condenser 60 by 60 by 0.6 centimeters thick having two strata each 0.3 centimeters thick, each having a relative permittivity of 8 but having resistivities of 2×10^{14} and 8×10^{14} ohm-centimeters, the constants are:

$$C_1 = C_2 = 8.5 \times 10^{-9} \text{ farads}$$
$$R_2 = 4R_1 = 8.3 \times 10^{10} \text{ ohms}$$
$$CR = 282 \text{ seconds.}$$

From the equations derived above, the curves plotted in Fig. 176 have been drawn. These curves show: (a) the e.m.f. of condenser 1, (b) the e.m.f. of

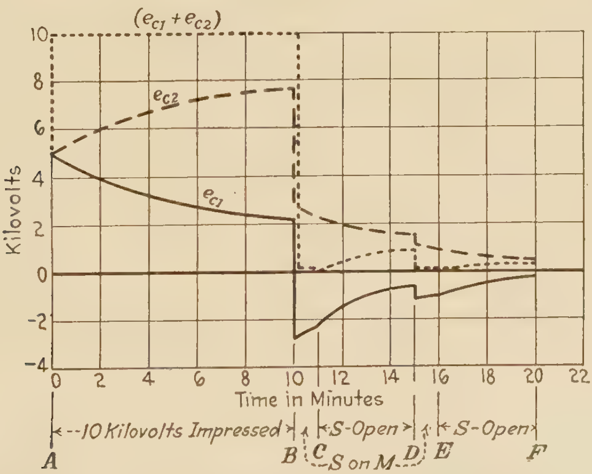


FIG. 176.—Voltage of stratified condenser during discharge.

condenser 2, and (c) the sum of these two e.m.fs. during the following sequence of events: (1) At the instant A, the switch S is closed impressing $-10{,}000$ volts on the condensers; this is kept on for 10 minutes. (2) At the instant B, the source E is disconnected and the condenser is discharged by holding the switch on the N contact for 1 minute. (3) At C the jumper between A and D is opened by moving the switch to the M contact and the

net charge left on the interface between the two strata is allowed to leak through the strata for 4 minutes. During this interval, Eq. (322) applies to each of the condensers. At D the condensers are again discharged by moving the switch to the N contact for 1 minute, and so on.

215. Exercises.

1. A storage battery whose e.m.f. is 20 volts and internal resistance 0.5 ohm is connected to a conductor whose resistance is 5 ohms. What current does the battery supply? What is the voltage impressed on the conductor?

2. Compute the resistance between the terminals A and B of Fig. 177.

3. If resistances are to be measured by taking simultaneous voltmeter and ammeter readings, and if it is proposed to make no corrections in the readings for the current taken by the voltmeter or for the voltage drop in the ammeter, sketch the manner in which the instruments should be connected:

a. For the measurement of extremely high resistances.

b. For the measurement of extremely low resistances.

Why? Discuss.

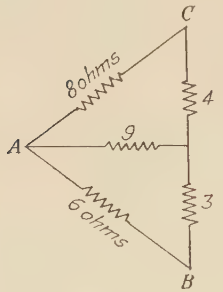


FIG. 177.

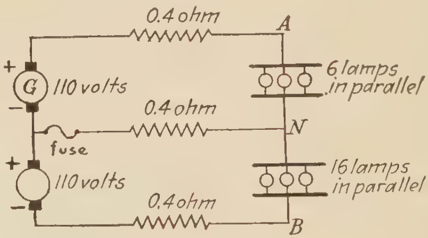


FIG. 178.

4. Two voltmeters having full-scale deflections of 150 and 50 volts and resistances of 17,936 and 5864 ohms respectively are connected in series with a coil of 8500 ohms resistance across 220-volt mains. What will each instrument read?

5. A 150-scale voltmeter whose resistance is 17,527 ohms is connected in series with an unknown high resistance across 220-volt mains. If the voltmeter reads 141.5 volts, what is the value of the unknown resistance?

6. Each cell of an n-cell battery has an e.m.f. of 2.1 volts and an internal resistance of 0.12 ohm.

a. If the battery is to supply current to a circuit whose resistance (exclusive of the battery) is 0.09 ohm, will the larger current be supplied when the cells are all in parallel or all in series?

b. Same question as *a*, save that the circuit resistance is 0.2 ohm. For what resistance of the external circuit will the battery supply equal currents for the parallel and the series connection of the cells?

7. To increase the range of a 50-scale ammeter whose resistance is 0.0005 it is shunted with a resistance of 0.0003 ohm. If the instrument reads 47.3 amperes, what is the total current in the circuit?

8. In Fig. 167, Sec. 208, let the battery and the galvanometer with its key be interchanged in position. Deduce the relation which must now exist between the resistances of the branches to reduce the galvanometer current to zero?

9. The circuit of a 220-volt three-wire distribution system is shown in Fig. 178. Each lamp may be considered as having a resistance of 200 ohms. Find the voltage across each set of lamps.

10. Assume that a fuse had been placed in the neutral wire as shown and that this fuse has burned out. Calculate the voltage across each set of lamps under this condition. What is the objection to the use of fuses in the neutral wire of a three-wire system?

11. Calculate the resistance between A and B for the arrangements shown in Figs. 179*a* and 179*b*.

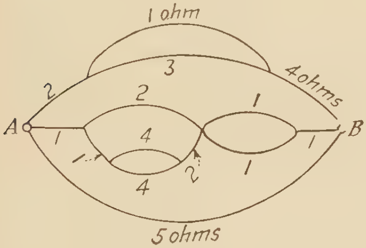


FIG. 179*a*.

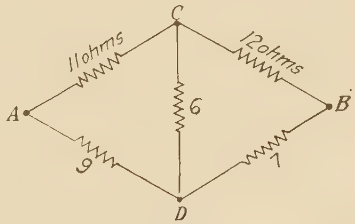


FIG. 179*b*.

12. *a.* Calculate the currents in the branches of Fig. 180 for the case in which $E_1 = 0$, $E_2 = 8$.

b. Calculate the currents in the same circuit for the case in which $E_1 = 6$, $E_2 = 0$.

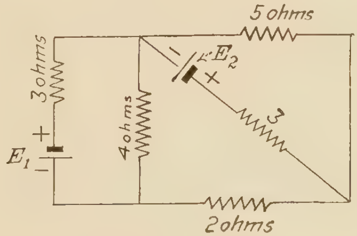


FIG. 180.

c. Calculate the currents for the case in which $E_1 = 6$, $E_2 = 8$. Does the principle of superposition apply in general to electric e.m.f. and currents in circuits consisting of metallic conductors? Does it apply to circuits containing gaseous conductors?

13. The greatest lighting load which may be placed on any branch circuit (under the regulations of the National Electric Code) is 1200 watts. It is customary to use No. 14 B. & S. rubber-covered wire for the branch circuits. In good practice the voltage drop to the farthest lamp of a branch circuit is not allowed to exceed 1.5 per cent of the impressed voltage.

If the lamps are all concentrated at the far end of the circuit, what is the greatest length of circuit (distance from distributing panel to lamps) which may be used?

14. *a.* Two heavy "mains" each of length l and having a resistance of r ohms per unit length are used to supply power to lamps which are connected (in parallel) across the mains. The lamps are uniformly spaced along the entire length of the mains. Suppose that the current taken by the lamps is c amperes per unit length of circuit. Compute the drop in voltage to the farthest point of the circuit.

b. Analyze the result, and formulate a simple rule for computing the drop to the farthest point of a pair of mains which supply a uniformly distributed load.

15. Referring to exercise 13, if the lamps are uniformly distributed along the length of the branch circuit, what is the greatest length of circuit which may be used?

16. A hemispherical metal basin of radius r_2 is filled to the brim with water of resistivity ρ. A metal sphere of radius r_1 is suspended at the center of this basin so that the sphere is half immersed in the water.

a. Derive the expression for the resistance from the sphere to the hemisphere.

b. What simple form does this expression assume if the radius of the sphere is very small in comparison with the radius of the basin?

c. What form does the expression assume when the difference between the radii, r_2 and r_1, is small in comparison with either radius?

17. *a.* A metal wash tub about 24 inches in diameter and 15 inches in depth is filled with water (resistivity 3000 ohm-centimeters) and a metal ball 2 centimeters in radius is half submerged near the axis of the tub. Calculate approximately the resistance from the ball to the tub. Assign upper and lower limits between which the true value certainly lies? How close together can you bring these limits?

b. Compute the resistance between two metal spheres 4 centimeters in diameter placed a mile apart and each half immersed in the surface of a lake, the resistivity of the water being 3000 ohm-centimeters.

18. A ground for a radio receiving set is made by driving a pipe into the earth. The pipe is 2 inches in diameter and is driven 6 feet into the earth. Estimate the resistance of this ground if the resistivity of the earth is 200 ohm-cms.

19. Calculate the currents in the branches, exercise 12, using the mesh current notation.

CHAPTER X

FORCES IN THE MAGNETIC FIELD UPON CONDUCTORS CARRYING CURRENT

THEME: The quantities introduced to specify the magnetic field are defined in terms of **mechanical force** and current.

221. The Magnetic Field.—In the description of the effects of the electric current in Chap. VI, it has been stated that two wires each carrying electric currents exert mechanical forces upon each other by reason of the currents. Likewise, it has been stated that mechanical forces exist between a wire carrying a current and any magnets or pieces of iron in the vicinity of the wire. It has also been stated that if a circuit in the neighborhood of a wire which carries a current is moved about relatively to the wire, or if the circuit and the wire are held stationary but the current in the wire is caused to vary in magnitude, an electromotive force is induced in the circuit. These effects are utilized in ammeters for the measurement of current, and in electromagnetic generators for the generation of electromotive forces.

Any region in which a wire is observed to be subject to a force by reason of a current which it is carrying, or in which an electromotive force is induced in a circuit by reason of its motion, or in which a magnet is subject to mechanical forces is called a **magnetic field.** It is called a magnetic field because such regions were first set up by magnets and were first studied by observation of the forces upon magnets. All the effects observed in the magnetic field are now attributed to, and described in terms of, the **differential motions** of electricity, and so the magnetic field may also be called the **electrokinetic field** in distinction from the electrostatic field. The electrostatic field results from a separation of negative electricity from positive; the electrokinetic field results from the motion of negative electricity relative to positive—or from the differential motion of the

two kinds. Magnetic effects do not result from the motion of equal amounts of positive and negative electricity with equal velocities in the same direction (as in the motion of an uncharged body), but only when there is a **difference** in the rates (quantities per second) at which the two kinds are moving across any surface in a given direction.

Our previous observations of the electrokinetic effects described above have been mainly of a qualitative nature. They were intended to indicate only in a qualitative way the general principles utilized in the ammeters, voltmeters, and dynamos which were to be used in the study of the properties of conductors. We now propose to study in a quantitative way these diverse effects which result from the differential motion of electricity, in order to derive the laws and principles relating to electrokinetic phenomena.

222. Choice of Effects to Be Used in Defining and Measuring Magnetic Quantities.—The first question which arises is this: Which one of the effects cited in the history of the discoveries of the magnetic effects in Sec. 127 shall be chosen, by means of which to define and measure the quantities which must be introduced to describe the properties of the field. The effects are:

a. The force upon a permanent magnet in a magnetic field.

b. The force upon a coil or a conductor carrying a current in a magnetic field.

c. The electromotive force induced in a coil while it is being moved in a magnetic field.

d. The momentary electromotive forces induced in a stationary coil when the magnetic field is suddenly established or suddenly vanishes.

It will be found that these effects are all closely interrelated, and that it is possible to choose any one of the effects as a means for investigating, defining, and measuring magnetic fields.

If the historical development were to be followed, the magnetic field would be investigated and the magnetic quantities would be defined by means of the force on a magnet. In the light of present knowledge, we unhesitatingly reject this plan, because **magnetism** is no longer regarded as a fundamental entity. The

fundamental experiments described in the next section lead to a conclusion of fundamental importance, namely, that the field around a permanent magnet has the same properties as the field around a conductor carrying a current. This fact led Ampere about 1820 to argue that the causes of the two fields were probably the same. He assumed the existence of electric currents in the atoms or molecules of permanent magnets and other magnetic materials, and showed that the properties of the magnetic materials could be accounted for in this way. Other physicists had assumed the existence of **magnetic charge** of some kind located at the poles of magnets and were able to account for the actions of magnets in this way. With the wide acceptance of electron theories, Ampere's theory of molecular currents has been generally accepted as the more reasonable explanation. It is obvious that, under these conditions, the fundamental definitions of magnetic quantities should be in terms of electric charge, rather than in terms of apparent forces between imaginary magnetic charges located at the poles of magnets. Another reason for rejecting the plan of developing magnetic theory in terms of the forces on magnets is that the electrical engineer has little occasion to make use of theory dealing with the force on magnets. On the other hand, the force on a conductor carrying a current in a field and the electromotive force induced in a moving circuit are matters of everyday concern to the engineer. There is a great advantage in having the fundamental definitions and experimental laws stated in terms of these effects of everyday use, because the definitions themselves then grow to have more precise meanings, and all calculations can more readily be traced back to the fundamentals.

Since permanent magnets, and the ferromagnetic materials, iron, nickel, and cobalt, owe their peculiar magnetic properties to the motions of the electrons in their atomic structures, it is evident that the presence in a magnetic field of ferromagnetic materials with their **concealed** electronic motions and constraints will add to the things which must be accounted for, and will, therefore, add to the complications in building up electrokinetic theory from the fundamental thing, namely, the motion of electricity. Accordingly, this chapter treats only of magnetic fields from which permanent magnets and all ferromagnetic

materials have been excluded. Chapter XIV will show the application of the fundamental principles and definitions to fields containing magnets and ferromagnetic materials.

The choice of the effect to be used to define magnetic quantities now lies between the mechanical force exerted upon a coil or a conductor carrying a current, and the electromotive force induced in a coil which moves in the magnetic field. Both of these are important effects and little more can be said in favor of one than of the other. We have chosen, however, to define the magnetic quantities primarily in terms of the mechanical force acting upon a coil or a conductor carrying a current in the magnetic field.

223. Fundamental Magnetic Experiments.—Before considering the apparatus by means of which the mechanical forces on conductors may be measured and the properties of the magnetic field defined, it is well to review the following historical experiments which served in a striking way to illustrate the features of the forces.

223a. Experiment 1. Force upon Magnets and Soft Iron (Oersted, 1820).—*a*. Let a straight wire be held in a north and south line over a magnetic needle (Fig. 181). If a continuous current is passed through the wire from south to north, the north end of the needle deflects toward the west.

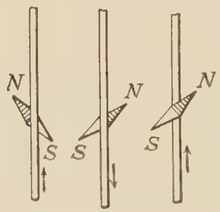


Fig. 181.—Force on a magnetic needle.

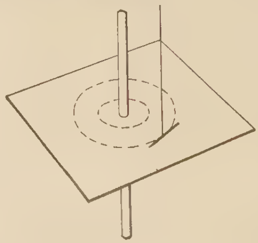


Fig. 182.—Force on a magnetic needle.

The direction of the deflection will be reversed (1) by reversing the direction of the current, or (2) by holding the wire under, instead of over, the needle. The deflection of the needle is very slight if the wire is at a considerable distance from the needle, and it approaches 90 degrees if a wire with a large current is approached close to the needle. Plates of wood, glass, copper, or of any material save iron, nickel, and cobalt, may be interposed between the needle and the wire without affecting the result.

b. Let a balanced needle of soft iron wire be suspended by a fine fiber in the vicinity of a long, straight section of wire carrying a current. The needle will be found to assume a direction tangent to a circle which lies in

a plane perpendicular to the wire and is concentric with the wire (see Fig. 182). Needles of glass, copper, etc. are not so affected.

These experiments show that the magnetized steel needles and needles of soft iron, when placed in the field of a long, straight conductor, are subject to turning moments, which tend to make the needles assume directions which are **tangent to circles concentric with the conductor.**

c. Let the conductor be now bent into a loop and let either the magnetized or the soft iron needle be suspended at the center of the loop, with its length lying in the plane of the loop. The turning force upon the needle, tending

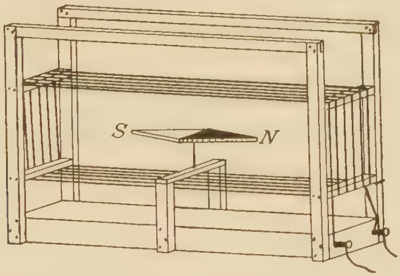


FIG. 183.—Schweigger's coil for multiplying the force on a needle (1821).

to make it assume a position with its length perpendicular to the plane of the loop, is greater than the force due to the straight wire, because both the wire above and the wire below now tend to make the needle deflect in the same direction. If, now, the single loop is replaced by a coil having a number of loops or turns (Fig. 183), the force exerted upon the needle by a given current is greatly multiplied.

223b. Experiment 2. Forces between Straight Wires (Ampere, 1820).—Let two flexible wires be loosely suspended side by side (about 1 centimeter

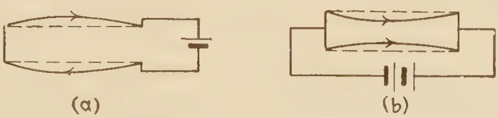


FIG. 184.—Force between parallel wires.

apart) between two supports, as indicated by the dotted lines in Fig. 184. If a current is passed through the wires in opposite directions, the wires repel and draw apart to the positions shown by the full lines of Fig. 184*a*. If current flows through the two wires in the same direction, they attract and draw together, as shown in Fig. 184*b*. The greater the distance between the parallel wires the less is the force between them.

223c. Experiment 3. Forces between Coils.—*a*. From the reactions between the parallel wires in experiment 2, one would predict that two wire

loops placed face to face, as in Fig. 185, should attract each other if (as indicated by the arrows) current flows around both loops in the same direction. On the other hand, if the current flows around the loops in opposite directions, they should repel. These predictions are readily confirmed by experiment.

b. If one loop is smaller than the other and is suspended at the center of the larger but with the planes not parallel, the loops exert a turning moment on each other. If one of the loops is free to turn, it turns until the planes coincide, with the current flowing around both loops in the same direction.

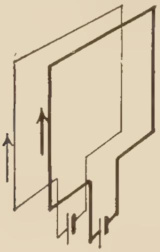


Fig. 185.—Force between two coils.

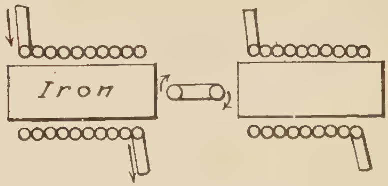


Fig. 186.—Coils with iron core.

c. If one or both of the loops is replaced by a coil having a number of turns, the forces due to a given current are greatly multiplied. We shall find that each turn acts on every other turn just as it would if the two turns alone carried current.

d. If a small coil is mounted in the field of two large tubular coils with its plane at right angles to the planes of the loops of the large coils, it is subject to a turning moment. If, now, the large tubular coils are provided with iron or steel cores (as illustrated in Fig. 186), the turning moment is multiplied many fold.

e. If the long sides of the small coil of *d* take the form of extremely flexible metal ribbons strung under tension between fixed supports, then the flexible ribbons deflect in the direction the coil tends to turn.

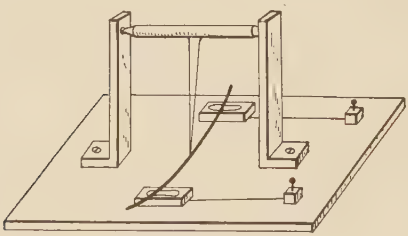


Fig. 187.—Force is exerted at right angle with the wire.

223d. Experiment 4. The Force on a Short, Straight Length of a Conductor Carrying a Current is at Right Angles to the Length of the Conductor (Ampere, 1820).—Let a copper wire in the form of a short horizontal circular arc be mounted on a vertical central axis, so that it is perfectly free to move in the direction of its length (see Fig. 187). The wire makes contact with the convex surface of the mercury in the two troughs near its ends. Through these contacts, current may be sent along the wire, and yet the wire is quite free to move in the direction of its length, save for the slight surface-tension

restraints exercised by the mercury. Let this arc be mounted in the strong magnetic fields of magnets or of large coils. The magnets and the coils setting up the field may be moved in any manner, and the currents in the field coils and through the movable wire arc may be started and stopped or varied in any manner, without obtaining any evidence that the forces on the wire have the slightest tendency to move it in the direction of its length. From this we conclude that the force on any short, straight length of a conductor carrying a current in a magnetic field is at right angles to the length of the wire.

223e. Experiment 5. The Magnetic Field of the Doubled-back Conductor (Ampere, 1820).—*a.* Let a paper-insulated or a rubber-insulated wire carrying a current be doubled back upon itself, so that the current flows in opposite directions along two parallel wires which are very close together, as in Fig. 188*a*. By approaching such a long, narrow loop to a delicately

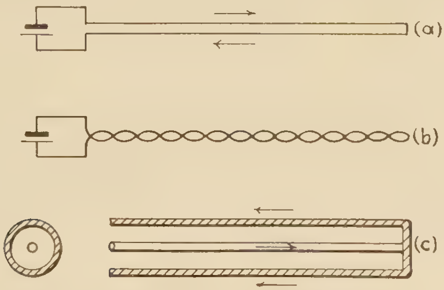


FIG. 188.—Doubled back conductors.

suspended magnetic needle or to a suspended coil carrying a current, it may be demonstrated that the magnetic field set up by the loop is extremely feeble except in the immediate neighborhood of the conductors.

b. Let the loop of Fig. 188*a* be now twisted, as in Fig. 188*b*, or let one conductor remain straight and let the return conductor crook back and forth in close proximity to it in the sawtooth fashion shown in Fig. 205. A repetition of the tests will show that the field set up by the current in the twisted wires is far weaker than the weak field set up by the narrow loop of Fig. 188*a*.

c. Let the current be now carried in opposite directions by the concentric rod and tube shown in Fig. 188*c*. A repetition of the tests will show that the current causes no magnetic field whatsoever external to the tube.

We conclude that equal currents flowing in opposite directions in filaments that are very close together substantially neutralize each other's magnetic effects. This fact is of great importance in the construction and use of electrical apparatus, since it makes it possible to convey current to and from any instrument in such a way that no disturbing magnetic field is set up by the current in the wires leading to the instrument.

From *b* we conclude that the magnetic effect of a straight elementary length at points at a great distance from it is substantially the same as that

of any two elementary lengths which, joined together, have the same termini as the short, straight element. That is, an elementary length may, for analytical purposes, be resolved in any manner into two or more compound lengths.

223f. Experiment 6. Magnetic Equivalence of a System of Current Meshes and a Current Loop Having the Same Boundary (Ampere).—Consider a current flowing in a circuit of any shape (as in Fig. 189). Imagine any surface whatsoever of which this circuit is the contour or boundary. Imagine this surface to be divided into meshes of any size by two systems of lines. The previous experiment would lead us to predict that a current of strength I flowing around the boundary will be the magnetic equivalent of currents of strength I flowing around each mesh in the same direction as that around the boundary, for this hypothetical system of mesh currents will give rise to two equal currents in opposite directions, along every line, such as AB, which lies in the interior. The magnetic effect of the currents in these portions of the mesh is, therefore, zero. On the other hand, the currents in those parts of the meshes which coincide with the boundary combine to reproduce a current of strength I around the boundary. This conclusion may be experimentally confirmed by tests on a loop circuit and a mesh circuit having the same contour as the loop.

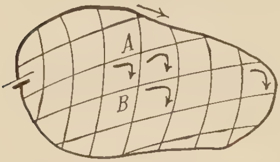


Fig. 189.—Loop and equivalent mesh.

224. Direction-finding Coil for Studying the Magnetic Field.—For the purpose of studying the properties of the magnetic field, let the coil shown in Fig. 190 be constructed. This coil, which we shall call the **direction-finding** coil, consists of about 400 turns of 0.25 millimeter insulated copper wire wound around the periphery of a thin wooden disk. The disk may be circular or square, or may have an odd-shaped contour, and may have an area of about 5 square centimeters. The disk with the coil around its periphery is supported in a gimbal mounting containing two axes at right angles to each other. The coil is thus free to turn so that its **normal** axis will point in any direction whatsoever. By the **normal axis** of the coil is meant not the axis on which the

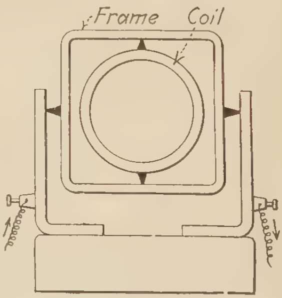


Fig. 190.—Direction-finding coil.

disk is mounted, but the line perpendicular (normal) to the plane of the coil at its center. A test current from a battery or other source of continuous current is conveyed to and passed through the coil through the bearings. The two bearings and the two halves of each of the axes must be insulated from each other.

Let this direction-finding coil, through which the small continuous test current is flowing, be placed with its center at any specified point P in a magnetic field. The coil will be found to be subject to forces which cause it to turn so that its **normal** axis always assumes the same direction as long as the center is located at P. Let the direction assumed by the axis be noted, and then, by means of a reversing switch, let the direction of the current in the coil winding be reversed. It will be found that the coil will swing around so that its normal axis turns through an angle of 180 degrees.

It is evident that the position assumed by the coil depends upon the direction in which the test current flows around the coil. From this it follows that if we wish to study and definitely specify certain features of magnetic fields by describing the directions assumed by the axis of an exploring direction-finding coil, we must agree to specify which one of the two directions along the normal axis is to be called **the** direction of the axis. Furthermore, we must adopt some convention which relates **the** (specified) direction along the axis to the **direction of the current** around the coil (that is, around the axis). The best way to indicate on coils and on coil diagrams the direction along the axis which has been selected as **the** specified direction is to place an arrow along the axis in the specified direction. Accordingly, in the subsequent discussion we will always refer to **the** direction along the axis either as the **arrow** direction or as the **specified** direction. In order to have uniform practice, all investigators should relate the arrow direction along the normal axis (or through the coil) to the direction of the test current around the axis (or around the coil) by the same convention. The convention in general use for this purpose is as follows:

224a. CONVENTION SPECIFYING THE ARROW DIRECTION ALONG THE AXIS OF (OR THROUGH) A COIL.—The arrow direction along the axis of (or through) a coil is defined to bear to the arrow direction for current around the coil the same relation that the direction of advance-

ment of a right-hand screw bears to the direction of rotation. This is called the "right-hand screw convention" (see Fig. 191).

In some cases the following equivalent statement of the convention is easier to apply: If a portion of the circuit be grasped in the right hand with the thumb pointing along the circuit in the arrow direction for the current, as in Fig. 192, then the direction in which the fingers point in looping through the circuit is defined to be the arrow direction along the axis or through the coil.

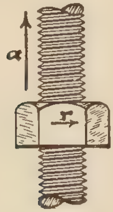


FIG. 191.—Right-hand screw relation.

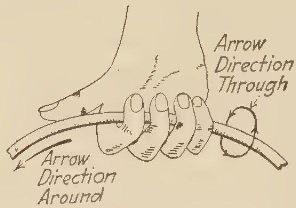


FIG. 192.—Right-hand rule for arrow directions around and through a coil.

225. Right-hand Screw Convention.—In the following discussion of relations in the magnetic field, we will have frequent occasion to make use of the right-hand screw convention, which is in general use in mathematical work, namely,

When a given direction around a loop bears the same relation to a given direction through the loop that the direction of rotation of a right-hand screw bears to its direction of advancement, the two given directions are said to be related to one another by the right-hand screw convention (see Fig. 191).

226. Lines of Magnetic Flux Density, or Lines of Force (DEFINITION).—Let the magnetic field surrounding one or more conductors carrying a continuous current be explored in the following manner by means of the direction-finding coil. Place the coil, through which the small test current is flowing, with its center at any point in the field and note the direction in which the normal axis of the coil points. Move the coil so that its center travels a short distance in the direction of this axis and note the new direction in which the axis now points. Move the coil a short distance in this direction and so on. If, starting at any point P in the field, a path or line is traced out by this step-by-step process, it will be found that the path eventually returns to the starting point P. That is, **all lines traced in this manner form closed loops.**

A second striking feature of these lines is that **the loops are always found to be linked with one or more loops of the conductor carrying the current giving rise to the magnetic field.** That is to say, if a string is stretched along any line (loop) traced out by the direction-finding coil, and if the two ends of the string are tied together, it is found that the conducting circuit carrying the current loops through the string circuit one or more times in such a manner that it is impossible to separate these two closed circuits without cutting one or the other. Some of the loops which may be traced out in the field set up by a current in

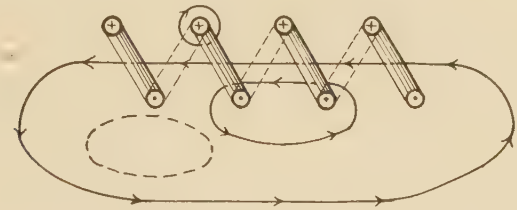


FIG. 193.—Closed loops of magnetic flux density.

the coil of Fig. 193 are indicated by the full lines. The direction-finder never traces out a loop (such as the dotted loop) which is not linked with the conductor carrying the current.

The loops whose features have been partly described above are called **lines of magnetic flux density.**[1] The explanation of why this somewhat cumbersome name, "magnetic flux density," has been applied to these lines will be found in those relations of

[1] Many writers call these lines **lines of magnetic force,** and others call them lines of **magnetic intensity.** The early writers called them lines of force because they were first traced out by a study of the forces experienced by compass needles when placed in a magnetic field. A balanced compass needle, if free to take any position whatsoever, will be found to set itself along the line of magnetic flux density passing through its center, with its north-seeking pole pointing in the positive direction along the line of magnetic flux density. That is, the compass takes up a position which can be accounted for upon the supposition that the compass is subject to a torque, one force being directed toward the north and being applied at a point called the **north (seeking) pole** not far from one end of the compass, and the other force being directed toward the south and being applied at a point called the **south (seeking) pole** not far from the other end. These notions will be found to be misleading if accepted as physical facts.

a mathematical nature between the magnetic quantities which will be discussed in the next chapter.

A third feature of the closed lines of magnetic flux density, in the magnetic field set up by the current in any circuit or any number of circuits, is this. Let that direction along any loop in which the arrow direction of the axis of the direction-finding coil points be (somewhat arbitrarily) called the (positive) direction along the line of flux density. It is found by experiment that the net or resultant current flowing through any one of these closed lines always flows through the loop in the same direction as does the test current in a direction-finder whose center lies on the line. This experimentally determined relation between the arbitrarily defined positive direction along the (closed) lines of magnetic flux density and the direction of the field current through the loop may be best remembered by the following modification of a rule first given by Ampere.

226a. AMPERE'S RULE (MODIFIED) FOR PREDICTING THE POSITIVE DIRECTION ALONG THE LINES OF MAGNETIC FLUX DENSITY ENCIRCLING A CURRENT (EXP. DET. REL.).—**Imagine the conductor to be grasped in the right hand, with the thumb pointing along the conductor in the direction of flow of current; then the fingers encircling the conductor point in the positive direction along the lines of magnetic flux density (see Fig. 192).**

We may now frame the following definition of the term, "line of magnetic flux density."

226b. A LINE OF MAGNETIC FLUX DENSITY is a (closed) curve in a magnetic field such that the tangent to it at each and every point P coincides in direction with the direction of the normal axis of a small direction-finding coil whose center lies at P. The POSITIVE DIRECTION along a line of magnetic flux density is defined to be the arrow direction of the normal axis of the direction-finding coil whose center lies on the line.

It should be borne in mind that these lines of magnetic flux density are imaginary lines. That is, they are imaginary in the sense of existing only in the imagination of the investigator, or imaginary in somewhat the same sense that the circles of latitude and longitude of the geographers are imaginary. The tendency which must be guarded against in the use of the notion of these lines of flux density is the universal tendency to take it for granted that any construct of the imagination to which a name has been

given has a physical existence. One must go slowly in thus "idolizing" lines of flux density. The effect of such a course is to divert the thought from the fundamental things, the things which can be actually measured, namely, the forces on an actual coil, to a scheme which was invented for convenience in describing the fundamental thing. If such a diversion occurs, thought tends to take, and frequently does take, the form of unbridled speculation having a very remote connection with reality.

Another convenient method of tracing out the shapes of the lines of flux density in the vicinity of the conductors carrying the current is to place a sheet of cardboard in the plane of some of the line of flux density loops and to sprinkle iron filings upon

FIG. 194.—Field around a long straight wire.

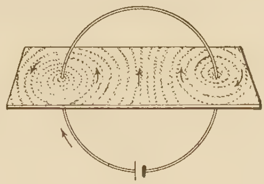


FIG. 195.—Field about a circle of wire.

the cardboard. If the board is gently tapped while a large current is flowing through the wires, the iron filings will arrange themselves in closed chains of particles which are linked with the conductors. These chains mark out the shape of the lines of flux density. Figures 194 to 197 are photographs or sketches of the lines which have been mapped out by means of iron filings. The positive directions along the lines of flux density have been indicated by arrows. The direction of the current in the conductors piercing the plane of the paper has been indicated by the usual conventions. The cross (+), the tail of an arrow, indicates that the current flows vertically down through the plane of the paper, while the dot (·), the point of an arrow, indicates that the current flows up. Figure 194 is for a long, straight conductor perpendicular to the plane of the paper.

The lines of flux density are seen to be circles having the conductor at the center. Figure 195 shows the lines of flux density in the vicinity of a circle of wire whose plane is perpendicular to

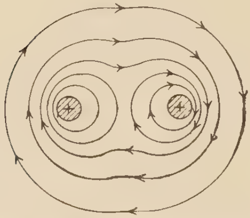


Fig. 196.—Field about two parallel wires.

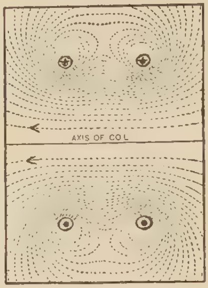


Fig. 197.—Field about a two turn coil.

the plane of the paper. It is also fairly representative of the conditions around two long, straight wires perpendicular to the plane of the paper and carrying currents in opposite directions. Figure 196 is for two long, straight wires carrying currents in the same direction. Figure 197 is for a coil having two turns widely spaced.

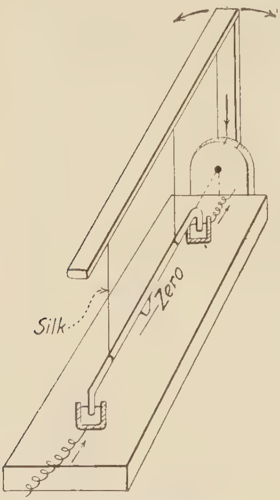


Fig. 198.—Force-finder.

227. The Force-finder and Torque-finder for Studying the Magnetic Field. We propose to make our first quantitative study of the properties of the magnetic field by determining the forces which act upon a short straight length of a current-carrying conductor in different positions in the field. For this purpose, let a short, straight wire be either suspended, as in Fig. 198, or mounted on delicate calibrated springs. Let a continuous test current be conveyed to and passed through this wire, either by mercury cups into which the ends of the wire dip, or by very flexible connections. The conductors leading the current to and from the suspended wire should be carried away from the vicinity of the wire along the straight line determined

by the test wire. Ampere's experiment with a conductor free to move in the direction of its length (Sec. 223*d*) has shown that the force on such a conductor never has a component along the length of the wire. The force is always perpendicular to the length of the wire, that is, perpendicular to the direction of flow of the test current. Therefore, any apparatus designed to measure the force needs only to be arranged to measure forces which are perpendicular to the wire. The instrument just described will be called the **force-finder.**

For the purpose of measuring the **turning moment** or the **torque** of the forces acting upon a small coil carrying a current at a given position in a magnetic field, let a coil similar to that shown in Fig. 93 be constructed. This coil, which we shall call the **torque-finder,** may be identical with the Siemens' dynamometer of Fig. 93 (Sec. 134*a*), save that the fixed coil is omitted and the movable coil alone is used. The coil is not free to assume any position, but by turning a knurled head h it may be constrained to take a position differing from the position it would assume under the turning forces of the field alone. To this head is attached one end of a helical spring S. the other end of which is attached to the coil. By turning the head, the spring is twisted, and the coil is constrained to move from the position which it would assume under the forces of the field to a new position N. A pointer moving over a scale indicates the amount by which the spring must be twisted to hold the coil in the new position N against the turning forces of the field. The scale may be graduated to read directly the torque exerted by the spring on the coil.

228. Mechanical Force Acting upon a Conductor Which Carries a Current in a Magnetic Field (Exp. Det. Rel.).—By means of the **force-finder** the forces acting upon a short, straight conductor which is located at any point P in the magnetic field may be studied. By mounting the wire with its center at a fixed point P and measuring the force on the wire when it is carrying a given test current and is successively pointed in many different directions, the following laws are arrived at:

a. The force on the wire is always perpendicular to the length of the wire.

b. The force varies with the direction in which the wire points. One and only one direction can be found for the wire in which the wire is subject to no force. This direction is found to coincide with the direction of the lines of magnetic density as mapped out by the direction-finding coil. Any number of directions can be found in which the wire experiences a definite force f_1, but these directions are all found to make the same angle θ with the lines of magnetic flux density at the point P. A comparison of the forces measured for various values of θ gives the following relation:

c. The force on a short conductor of a given length carrying a given current at a given point P in a magnetic field is directly proportional to the sine of the angle (B, l) between the conductor and the lines of magnetic flux density at P.

d. The fact that the force is always perpendicular to the length of the conductor is not sufficient to determine its direction, since there are any number of perpendiculars to a wire at a given point. A further study brings out the additional fact that the force is always perpendicular to the lines of magnetic flux density. That is, the force is exerted along a normal to the plane determined by the straight test conductor and the line of magnetic flux density through P. Further observations show that the rule known as the general right-hand force rule (see below) correctly states the direction along this normal in which the force acts.

In the above study, the length of the conductor and the magnitude of the test current I flowing through it were kept constant. Now, by varying the length and the current while keeping the test conductor in a fixed position in the field, the following laws are determined:

e. The force on the conductor is directly proportional to its length l.

f. The force is directly proportional to the value of the test current I flowing in the conductor.

The above statements are not rigorously exact unless the test conductor is made very short.

These laws may all be embodied in the following complete equation, by means of which the force on any short conductor at any point P may be calculated.

$$f \text{ (dyne-sevens)} = BIl \sin (B, l) \tag{341}$$

in which,

f is the force acting on a short, straight wire at any point in any magnetic field.

I is the current in amperes flowing in the wire.

l is the length of the wire in centimeters.

(B, l) is the angle between the wire and the line of magnetic flux density through P.

B is a quantity which characterizes and describes the magnetic field in the neighborhood of the point P. The nature of this quantity is discussed, and it is named in the following section.

229. Magnetic Flux Density.—The quantity B which was experimentally arrived at in the above investigation is a quantity whose value is independent of the proportions of the particular force-finder which may have been used to study the properties of the field. As stated, it is a quantity which characterizes and describes certain properties of the field in the neighborhood of the point P. If a similar experimental study is made at any other point in the field, the same relations are found to hold, except that, in general, the coefficient B is found to have a different value, and the line of magnetic flux density is found to have a different direction.

If, then, we think of B as the quantity by means of which certain conditions at one point in a field may be compared with those at another point in the same field or in a different field, it is evident that B must be regarded as having direction as well as magnitude, since both are necessary completely to specify the conditions at a point. B is, therefore, regarded as a vector quantity pointing in the one direction at P which has special properties, namely, in the positive direction along the lines of magnetic flux density. Various names have been assigned to this vector quantity. It was originally called the **magnetic induction,** but the Standardization Committees have adopted the name **magnetic flux density.** By transposing Eq. (341), we obtain the following equation defining the magnetic flux density B:

$$\mathsf{B}\ (\text{dyne-sevens per amp.-cm.}) = \frac{f}{Il \sin\ (B,\ l)}\ \frac{(\text{dyne-sevens}).}{(\text{amperes, cms.})} \qquad (342)$$

229a. MAGNETIC FLUX DENSITY (Definition).—**The magnetic flux density B at a point P in a magnetic field is defined to be a vector quantity whose direction is that singular direction through the point in which a short, straight test wire carrying a current must be placed if the force on the wire is to be zero, and whose magnitude is equal to the force upon the short, straight test wire per centimeter of length and per ampere of test current when the direction of the test wire makes a right angle with the direction for zero force. The (positive) direction of the B vector is defined to be that one of the two directions (along the singular line) which is related to the direction in which the current flows around a direction-finder in stable equilibrium at P by the right-hand screw relation.**

Briefly, the magnetic flux density is equal in magnitude to the force in dyne-sevens, per ampere-centimeter upon a test conductor placed at right angles to the lines of magnetic flux density.

229b. Unit of Magnetic Flux Density (Definition).—*The magnetic flux density at a point is unity if a wire* 1 *centimeter in length, carrying a current of* 1 *ampere, in a direction at right angles to the lines of magnetic flux density, is acted upon with a force of* 1 *dyne-seven. The descriptive name of this unit is the* **dyne-seven per ampere-centimeter.** *However, the derived name which has been applied to this unit is the* **weber per square centimeter**[2]

The words "per square centimeter" in the derived name of the unit (namely, the weber per sq. cm.) can at this stage have no significance to the student. On the other hand, the descriptive name of the unit (namely, the dyne-seven per amp.-cm. has a simple and all-important physical significance. It is recommended, therefore, that the student call the unit by its descriptive name until, in the next chapter, he encounters the relations of a mathematical nature from which the derived name has been obtained. He may then drop the physically significant name for the name of mathematical significance.

[2] A smaller unit, which is a decimal submultiple of the practical unit of flux density, is in very common use. This smaller unit is the **eighth-weber per sq. cm.** This unit is more commonly used under the name of the **maxwell per sq. cm.** or the **line per sq. cm.**

$$\left.\begin{array}{l}\text{One eighth-weber per sq. cm.}\\ \text{One maxwell per sq. cm.}\\ \text{One line per sq. cm.}\end{array}\right\} = 10^{-8} \text{ webers per sq. cm.}$$

The **maxwell per sq. cm.** is not a unit of the Practical System of Units, but it is the unit of flux density in the Electromagnetic System of Units.

This practice of at first using a descriptive name of physical significance will parallel that followed in dealing with electric intensity in the electrostatic field. It will be recalled that the first name applied to the unit of electric intensity was the descriptive, physically significant **dyne-seven per coulomb.** Later this was superseded by the derived name, the **volt per centimeter.**

In much of the above discussion, as in the definition of the unit of flux density, it has been assumed that the properties of the magnetic field are substantially uniform in the vicinity of the point P. If the field is non-uniform, as it is likely to be in the immediate vicinity of the currents giving rise to the field, then in order rigorously to define the conditions at a point, we must imagine the wire of the force-finder to be infinitesimally short and must define the flux density to be the force acting upon the wire per centimeter of length.

230. The Force on a Conductor Carrying a Current.—Now that we have come to regard the factor B which occurs in the experimentally derived formula for the force on a wire carrying a current as a quantity which specifies certain properties of the field at the point P, and have proceeded to name this quantity the magnetic flux density at the point, we may embody this name in the following more complete statement of the law relating to the force which acts upon a conductor when it carries a current in a magnetic field.

230a. LAW FOR THE FORCE ON A CONDUCTOR CARRYING A CURRENT (EXP. DET. REL.).—**If a short length l of a conductor carrying a current I lies in a region of a magnetic field at all points of which (region) the magnetic flux density has the same direction and the same value B, then the force exerted upon this short length of the conductor is given by the formula**

$$f \text{ (dyne-sevens)} = BIl \sin (B, l) \begin{cases} \text{dyne-sevens per amp.-cm.,} \\ \qquad \text{or} \\ \text{webers per sq. cm.,} \end{cases} \qquad (341)$$

In this formula, (B, l) represents the angle between the direction of the short length of wire and the B vectors in the region.

The force is exerted along a normal to the plane which is determined by the short (straight) length of wire and the B

vector for any point occupied by the wire. The direction along this normal in which the force acts may be determined by the rule known as the "right-hand rule for the force," namely, *point the first finger of the right hand in the direction of the B vector, and the thumb in the direction of the current. The second finger will point along the normal in the direction of the force on the wire (see Fig.* 200).

This law purports to be a law for computing the force on a conductor, but in reality it is a definition of magnetic flux density in terms of measured forces. Therefore, we cannot hope to use this law to determine the forces on a conductor by computation alone, unless we can devise a method of determining the flux densities in a region without actually measuring the forces exerted on a force-finder. The problem in the subsequent chapter will be to devise methods and rules for predicting or calculating the value of the flux density at any point in the field which is set up by currents of known value flowing in conductors and coils of known configuration.

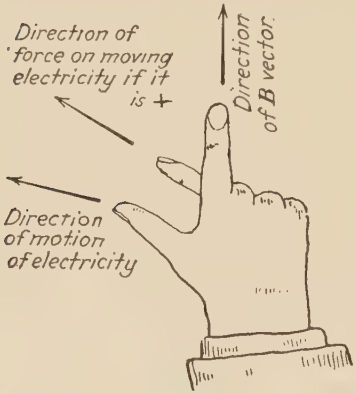


Fig. 200.—Right-hand rule for force.

231. Calculation of Torque on a Coil from the Equation for the Force on a Short Wire (Deduction).—Because of the difficulty of conveying current to the force-finder, it is not so convenient or satisfactory to use as the torque-finder. We therefore propose to use Eq. (341) to deduce the torque upon a small test coil, and to confirm the deduced results by experiments with the torque-finder. This is mainly for the purpose of indicating the use of the torque-finder to measure flux densities.

Let C (Fig. 201), represent the boundary of any plane coil, and AA any fixed axis in the plane of the coil. Let us use the equation for the force on a short, straight conductor carrying a current in a magnetic field to compute the torque on this coil. Let the torque be first computed for the case in which the coil has been placed in a **uniform** field in the position of maximum torque. That is, for the case in which the coil has been so placed that the vectors representing the flux density lie in the plane of the coil, and make a right angle with the axis AA.

For convenience, take the line AA as the X axis of a system of rectangular coordinates. Let us consider the torque which results from the forces on the

two elementary portions of the boundary EF and GH included between the planes at x_1 and $x_1 + dx$.

The force on the elementary portion EF of length dl is

$$df = BI(dl) \sin (B, l). \tag{341}$$

The torque of this force about the axis AA is

$$\text{torque of } EF = BIy(dl) \sin (B, l).$$

But $$(dl) \sin (B, l) = dx.$$

and $$y(dx) \text{ represents the area } EFF'E'.$$

Therefore, $$\text{torque of } EF = BI \text{ (area of } EFF'E').$$
$$\text{torque of } GH = BI \text{ (area of } GHE'F').$$

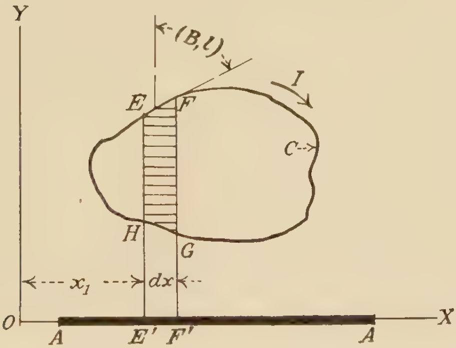


FIG. 201.—Torque on a coil.

But the application of the right-hand force rule shows that these two torques are oppositely directed; the former is normal to the plane of the paper in an upward direction, and the latter in a downward direction. Therefore, the net or resultant torque of the two forces is proportional to the horizontally cross-hatched area $EFGH$.

That is,

$$\text{torque of } (EF + GH) = BI \text{ (area of } EFGH).$$

From this it is readily seen that the total torque resulting from the forces on all portions of the coil is,

$$\text{torque on coil} = BI \text{ (area of coil)}.$$

That is, from computations based upon the equation for the force on a short, straight wire (Eq. (341)), we draw the conclusion that the torque on a plane coil in a uniform field is directly proportional to the area bounded by the coil, and is independent of

a. The shape of the coil.

b. The location of the turning axis relative to the coil, provided only that the axis lies in the plane of the coil.

The above calculations are for the simple case in which the coil lies in the position of maximum torque. By the use of spherical trigonometry in any case in which the turning axis and the B vector are at right angles, the

deduction may be drawn that the torque is directly proportional to the coil area and is independent of the shape.

232. Torque upon a Coil in a Magnetic Field (Exp. Det. Rel.).—Let the torque-finder be set up with its center at some fixed point P in a magnetic field, and with its normal axis pointing in the positive direction along the lines of magnetic flux density passing through P. In this position, the magnetic field exerts no torque on the coil. By turning the head H, and thus exerting a torque upon the coil through the spring, the coil may be constrained to take up and remain stationary in a new position with its normal axis making some angle (B, n) with the lines of magnetic flux density. Under these conditions the torque exerted by the magnetic field upon the coil is exactly equal and opposite to the torque exerted by the spring upon the coil. The torque exerted by the spring may be computed from the reading of the pointer on the scale. By measuring in this manner the torque exerted by the field upon the coil for a great many values of (B,n) between 0 and 180 degrees and comparing the values, the following law is discovered:

The torque τ upon a small plane coil carrying a given test current and located with its center at a given point P in a magnetic field is directly proportional to the sine of the angle (B,n) between the normal axis of the coil and the line of magnetic flux density passing through P. The torque tends to turn the coil so that its normal axis coincides with the B vector, and so that the direction of the current around the coil is related to the B vector by the right-hand screw relation.

$$\tau \text{ varies as } \sin (B,n). \tag{343}$$

That is to say, if the normal axis of the coil is deflected by a given angle (B, n) which may be measured in **any direction whatsoever** from the line of flux density passing through the coil center, the torque has a definite numerical value which depends only upon the value of (B,n) and not upon the direction in which (B,n) is measured from the line of flux density. The normal axis of the coil may be made to describe the surface of a cone making an angle (B,n) with a line of flux density as an axis without affecting the numerical value of the torque exerted by the field. The torque is a maximum when the axis of the torque-finding coil makes an angle of 90 degrees with the lines of flux density. This is, accordingly, a convenient position to use in making subsequent torque measurements.

We have determined the manner in which the torque upon a given torque-finding coil located with its center at a given point P in a magnetic field varies with the direction in which the normal axis of the coil points. Let us now set about to determine how the torque is affected by the proportions of the torque-finder. The proportions of the torque-finder which may be varied are:

a. The value of the test current, I.
b. The number of turns N in the coil.
c. The amount of area a bounded by the small plane coil.
d. The shape or contour of the coil.
e. The location or the mounting axis relative to the coil.

By varying the above proportions of the torque-finder, one at a time, and measuring the torque experienced by the coil when placed at a given point in a given field with its normal axis at right angles to the lines of flux density, the following laws are discovered:

a. The torque is directly proportional to the test current I.

b. The torque is directly proportional to the number of turns N in the torque-finding coil.

c. The torque is directly proportional to the plane area (a) bounded by the coil.

d. The torque is independent of the contour of the coil. The coil may bound a circle, a square, a rectangle, or an irregular shaped area and the torque will be the same, **provided** all these coils enclose or bound the same area.

e. The torque is independent of the location of the mounting (torque) axis of the coil.

If the field in the vicinity of P is non-uniform in its properties, the above statements are not rigorously exact unless the area bounded by the torque-finder coil is made very small.

The complete equation expressing the magnitude of the torque upon a coil at a given point in the field, therefore, takes the form

$$\tau \text{ (dyne-sevens, centimeters)} = KNIa \sin (B,n). \qquad (344a)$$

Upon experimentally determining the values of the coefficient K for a number of points in the field and comparing these values with the values of the flux density B at these points as determined by the methods outlined in Sec. 228, it is found that K in the above equation is identical in value with the flux density as previously defined in Eqs. (341) and (342) in terms of force.[3]

Therefore, Eq. (344a) may be written

$$\tau \text{ (dyne-seven-cms.)} = BNIa \sin (B,n). \qquad (344)$$

The torque on a plane coil carrying a current I is seen to be directly proportional to the value of the product (NIa). In Sec. 243 it is shown that the strength of the magnetic field which a small plane coil sets up at distant points is directly proportional to the value of this same product NIa. This product is, therefore, a measure of an important "electrical dimension" of the coil; it is found convenient to have a name for the product. It is called the **magnetic moment** of the coil. We will represent it by the symbol M.

[3] If we had chosen to define magnetic quantities not in terms of the force upon a wire, but in terms of the torque upon a coil, then magnetic flux density would have been defined by Eq. (344). Rearranging (344), it becomes

$$B = \frac{\tau}{NIa \sin (B,n)}. \qquad (347)$$

That is, the flux density at a point would have been defined as the maximum torque on a small test coil per ampere-turn per square centimeter of coil area.

232a. MAGNETIC MOMENT (Definition).—**By the magnetic moment M of a small plane coil of N turns carrying a current I and bounding an area of a square centimeters is meant the product NIa.**

$$M(\text{ampere-turn, centimeters}) = NIa \qquad (345)$$

Equation (344) may now be written in the form

$$\tau \text{ (dyne-seven-cms.)} = BM \sin (B,n). \qquad (346)$$

233. Force on a Moving Charge as Deduced from the Force on a Wire Carrying a Current (Deduction).—We have seen that the torque upon a plane coil in a magnetic field may be accounted for (in the sense that it may be computed) by means of the expression, Eq. (341), which gives the force acting upon an elementary length of wire carrying a current. Let us now tentatively adopt the hypothesis that the force which acts upon an elementary length of wire through which electrons are moving is, in turn, to be accounted for in terms of the force acting upon individual electrons when in motion in a magnetic field. That is, we tentatively advance the following view as to the mechanism of the phenomena.

We assume that a charged body, moving at a point P in a magnetic field with a velocity V (a vector quantity), is subject to a force which is in a direction normal to the plane determined by the V and the B vectors at the point P, and in the direction given by the general right-hand force rule. Accordingly, the electrons of the atmosphere of free electrons drifting through a wire must be conceived to experience a side thrust which causes a non-uniformity in the distribution of the electrons over the cross-section of the wire. There is a slight shift of the free electrons toward one side of the wire, leaving unneutralized positive nuclei on the other side. The force experienced by the moving electron atmosphere may be conceived to be transmitted to the more rigid structure of the wire either by the electrostatic attraction of the excess electrons along one side of the wire for the unneutralized positive nuclei on the other side, or by the higher velocity of the electron bombardment on one side of the wire than on the other.

If we assume that the wire carrying the current has an atmosphere of free electrons of Q coulombs per centimeter length of wire, which is drifting through the wire with an average velocity

of V centimeters per second, then the current in the wire in the direction of drift has the value

$$I \text{ (amperes)} = QV \text{ (coulombs per cm., cms. per sec.)}. \quad (348)$$

(Note: Q stands for the algebraic value of the moving charge; if the movement is of electrons, Q is a negative quantity.)

But from Eq. (341), the force experienced by the wire per centimeter of length is

$$f \text{ (dyne-sevens)} = BI \sin (B,l) \text{ (webers per sq. cm., amperes)}.$$

Upon substituting the value of I from Eq. (348) in the above, the following expression for the force on a moving charge of Q coulombs is deduced:

$$f\text{(dyne-sevens)} = QVB \sin(V,B) \text{ (coulombs, cms. per sec. webers per sq. cms.)}. \quad (349)$$

These results are summarized in the following law:

233a. MECHANICAL FORCE ON A CHARGED BODY MOVING IN A MAGNETIC FIELD (Exp. Det. Rel.).—**The mechanical force f exerted by the magnetic field upon a body carrying the charge Q and moving with the velocity V at a point in the field where the magnetic flux density has the value B is equal to the product of the charge times the magnetic flux density times the component of the velocity normal to the direction of the magnetic flux density.**

The force is directed along the normal to the plane which is determined by the V and the B vectors, and acts in that direction (along the normal) in which a right-hand screw would advance if it were rotated in the direction in which the V vector must be turned to make it point in the same direction as the B vector.

A second and very useful form of the rule for determining the direction of the force on a moving charge is as follows:

233b. The Right-hand Rule for the Force on a Moving Charge. *The mechanical force experienced by an electric charge which is moving in a magnetic field is along the normal to the plane which is determined by the V and B vectors. The direction along the normal in which the force acts may be determined as follows:*

a. Point the first finger of the right hand in the direction of the B vector (the positive direction along the lines of magnetic flux density) (see Fig. 200).

b. Point the thumb in the direction of motion of the charge.

c. Then the middle finger will point along the normal in the direction of the force which the moving charge will experience if it is a positive charge. If it is a negative charge, it will experience a force in the opposite direction.

It should be remembered that, from the mode of derivation, Eq. (349) gives the force on a **cloud** of electrons drifting through a wire with a net directed velocity of V centimeters per second. We have reason to believe that this net velocity is a very, very small fractional part of the actual velocities of the individual electrons in their zigzag paths. **Under these conditions,** it is not safe to conclude (without further experimental evidence) that Eq. (349) expresses the law for the force on an individual electron. As a matter of fact, experiments upon electrons moving through evacuated spaces indicate that Eq. (349) gives the force on individual electrons. It is the necessity for this confirming experimental evidence which leads us to class Eq. (349) as an experimentally determined relation rather than a deduced relation.

234. The Force on a Moving Charge Expressed in Vector Notation.—In vector analysis, an expression of the form $(\mathsf{V} \times \mathsf{B})$, in which V and B are vector quantities, is called the **vector product** of V times B. Such a product is defined as follows:

234a. THE VECTOR PRODUCT OF TWO VECTORS (DEFINITION).—The vector product of two vectors—written $(\mathsf{V} \times \mathsf{B})$—is defined to be a third vector whose magnitude is equal to the product of V times B times the sine of the angle (V, B) between the V and B vectors. The third vector is along the normal to the plane determined by the V and B vectors in that direction in which a right-hand screw would advance if rotated in the direction in which the V vector must be turned to make it point in the same direction as the B vector.

A comparison of this definition of a vector product with the statement for the force on a moving charge will show that in vector notation the force on a moving charge is expressed by the following vector equation:

$$\mathsf{f} = Q(\mathsf{V} \times \mathsf{B}) \text{ (vector product).} \tag{350}$$

235. Motion of a Charge in a Magnetic Field in a Vacuum.—Suppose a charged particle is by some agency shot with a velocity V into a magnetic field in an evacuated vessel or into the earth's magnetic field beyond the limits of the earth's atmosphere. Since the force on the moving charge is always at right angles to the direction of motion, the forces of the field have no influence whatsoever upon the speed with which the particle continues

to move, but simply upon its direction of motion. For simplicity, suppose the particle is shot into an extended uniform field. If the direction of the motion is parallel to the lines of magnetic flux density, the charge experiences no forces whatsoever and continues to move in a straight line with unchanging velocity. If the charge is shot into the field in a direction at right angles to the lines of magnetic flux density, it experiences in a uniform field a constant force which is always at right angles to the velocity. The particle therefore moves in a circle which lies in a plane perpendicular to the B vectors. If the charged particle is shot into the field at any angle other than zero or 90 degrees with the B vectors, its velocity may be resolved into a

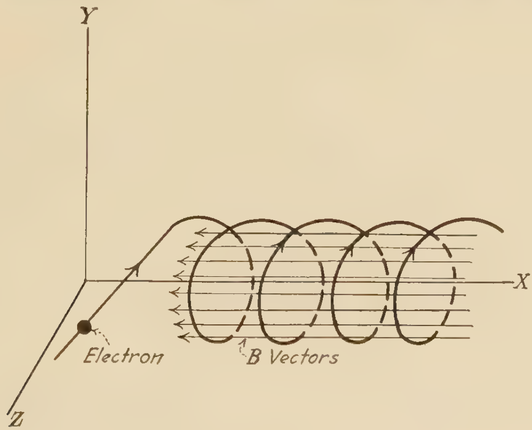


Fig. 202.—Path of an electron in a magnetic field.

component V_1 parallel to the B vector and a component V at right angles to the B vectors. The component V_1 is unaffected by the magnetic field and so the moving charge continues to have a component velocity V_1 along the lines of magnetic flux density. The component V would lead the charge to describe a circle about a line of magnetic flux density. As a result, the charge describes a helix having a line of magnetic flux density as its axis, as illustrated in Fig. 202.

If a charged body of mass m gram-sevens and charge Q has a component velocity V at right angles to the B vector, the centripetal force on the body is

$$f(\text{dyne-sevens}) = QVB$$

and the centripetal acceleration a is

$$a(\text{cms. per sec. per sec.}) = \frac{QVB}{m}.$$

But if a body moves with a velocity V in a circle of radius r, the acceleration toward the center is

$$a = \frac{V^2}{r}.$$

Equating the two expressions and solving, the following expression is obtained for the radius of the helix which the charged body describes in a uniform field about a line of magnetic intensity.

$$r \text{ (cms.)} = \frac{mV}{QB} \quad \frac{\text{(gram-sevens, cms. per sec.)}}{\text{(coulombs, webers per sq. cms.).}} \tag{351}$$

Those electrons shot off from the sun which come under the influence of the earth's magnetic field are diverted toward the polar regions from their straight path. The magnetic flux density B in the earth's field averages 0.5×10^{-8} webers per square centimeter, for the electrons $q = 1.59 \times 10^{-19}$ coulombs. $m = 9.0 \times 10^{-35}$ gram-sevens. Substituting these values in Eq. (351), the radius of the helix described by the electrons as they travel toward the polar regions is

$$r \text{ (cms.)} = 1.1 \times 10^{-7} \times V \text{ (cms. per sec.).}$$

236. Physical Significance of Magnetic Flux Density.—The preceding description of the forces acting on a charged body moving in a magnetic field gives the simplest and most fundamental idea we have obtained of the significance of the quantity which we call the magnetic flux density. We may make the statement that at a given point the B vector points vertically up and has the value of 10 microwebers per square centimeter without implying the existence of a medium. If we postulate a medium, the statement conveys to us no definite picture or conception of the peculiar state of the medium at the point. The most fundamental thing it does convey is the experimental fact that a force is exerted upon a moving charge which is proportional to the magnitude of B and **is in a direction perpendicular to** B. This last statement is important. No force (caused by a magnetic field) has ever been observed in the direction of the B vector. The conceptions which are founded upon the idea of magnetic poles and which picture them as acted upon by forces in the direction of B are misleading. To repeat, the fundamental physical phenomenon associated with the B vector is the **side push** upon moving charges—a side push which is perpendicular both to the direction of motion of the charges and to the B vector.

237. Engineering Applications.—The applications of the phenomena described above are so well known as to require little comment. The mechanical forces upon coils and conductors carrying current in a magnetic field are utilized in electric motors to cause the rotation of the armature group of coils in the magnetic field of the field group.

In the moving coil meters, these turning forces are balanced by the opposing forces of twisted springs or of wire suspensions. A pointer which moves over a scale indicates the amount of the twist, and therefore the force of the spring. The scale is usually graduated to indicate directly the current rather than the force. In the string galvanometer and the oscillograph, these forces make it possible to obtain a photographic record showing the variations in time of a rapidly varying current. The varying current is passed through a fine wire stretched in a magnetic field. The force on the wire is proportional to the current. By arranging the wire so that its deflection is proportional to the force, a record of the variations in the current is obtained by photographing on a rapidly moving film either the position of the shadow of the vibrating wire or the position of a spot of light which is reflected from a small mirror mounted on the wire.

In the "magnetic blowout circuit breaker," the circuit is broken in the magnetic field set up by coils designed for that purpose. In this field, the arc which is formed between the separating contacts of the switch is rapidly extinguished because the side forces on the moving charges in the arc force the arc to one side and very rapidly draw it out in length.

Under the enormous currents which are obtained when modern high-power equipment is accidentally short circuited, the mechanical forces between conductors rise to values which at times wreck generators, transformers, bus structures, and cable layouts by tearing the conductors from their moorings.

The design of equipment in which the above effects are utilized may be arbitrarily divided into two parts: First, the design of the current-carrying coils which are to move in some desired way under the forces acting on them in a given magnetic field; second, the design of other current-carrying coils which will produce the magnetic field. Thus far we have derived the laws by which the forces on any coil in a given magnetic field may be computed. We proceed in the next chapter to derive the laws which will make possible the design of magnetic fields, that is, the laws which will permit of the computation of the flux density at any point in the field which is set up by conductors of known configuration carrying known currents.

238. Exercises.

1. A direction-finding coil is placed in a magnetic field with its center at a point P, and the coil comes to rest with its normal axis pointing to the north. A force finder is then centered at the same point. The straight movable conductor on which the force is measured is 10 centimeters long, the current through it is 15 amperes from west to east, and the force on it is 4 grams. What is the value and direction of the flux-density vector at the point?

What should be the direction of the force on the conductor?

2. In a given force-finder, the straight conductor on which the force is measured is 10 centimeters long, and the current through it is 12 amperes.

When the conductor is carrying current from east to west at a point P in a magnetic field, the force on it is found to be zero. When the instrument is swung around 90 degrees so that the direction of the current is from north to south, the conductor is acted on by an upward force of 5×10^{-4} dyne-sevens. What is the value and direction of the flux-density vector at this point?

3. In exercise 2, what would be the force on the conductor if the instrument were swung only 45 degrees?

4. When a given direction-finding coil is placed at a certain point in a magnetic field and left free to turn, it comes to rest with the normal axis straight up. If a spring is attached to it and the coil is rotated and held in a position with the normal axis horizontal, the torque necessary to so hold the coil is found to be 2×10^{-3} dyne-sevens-centimeters. The coil has 900 turns, each carrying 0.1 ampere and each bounding a plane area of 2 square centimeters. What is the value and direction of the flux-density vector at this point?

5. What torque would be necessary to hold the coil of exercise 4 in a position with the normal axis making an angle of 60 degrees with the vertical?

6. An electron is shot into a uniform field at an angle of 60 degrees to the B vectors, and with a velocity of 10^8 centimeters per second. The flux density has the value of 10^{-4} webers per square centimeter. Find the diameter and the pitch of the helical path traversed by the electron.

CHAPTER XI

THE CALCULATION OF MAGNETIC FLUX DENSITIES DUE TO KNOWN DISTRIBUTIONS OF CURRENT

PART I—AMPERE'S FORMULA

239. Choice of Forms for the Expression of Magnetic Laws.—The experimental study, in the preceding chapter, of the forces upon a conductor carrying a current has given us a method of determining and specifying an important measurable property of magnetic fields which varies from point to point. The method is to introduce a vector quantity, B, which we call the **magnetic flux density at the point.** This vector quantity has been defined in terms of the force upon a test conductor; and direction-finders, force-finders, and torque-finders have been devised for finding the direction and measuring the value of the flux density at any point in the air portion of a magnetic field. If the flux densities at points on a line, which is to be occupied by a wire, have been measured, we can compute the forces on the wire when it is carrying a current of any given value. Thus far in the present study, we are dependent upon experimental methods for determining the values of the flux densities in the field set up by a current in a circuit even of the simplest configuration.

The object of this chapter is to determine the laws which will make it possible to compute the flux density at any point in the field which is set up by conductors of known configuration carrying known currents. The relations we seek must be obtained by measuring the values of the flux densities at selected points in the fields of circuits of different configurations carrying known currents. From an analysis of the data thus obtained, the laws expressing the relation between the flux densities and the dimensions of the current-carrying circuits must be deduced.

Before continuing this study of the relations in the magnetic field it may be helpful to review the features of the two alternative

forms in which it has been found possible to put the laws and carry on calculations relating to the electrostatic field; namely, the **action-of-element-upon-element** form, and the **line-, surface-, and volume-integral** form.

In the **action-of-element-upon-element** type of treatment, it is postulated that the action of one charged body upon another is to be explained and calculated in terms of the action of each element of charge upon every other element of charge. This leads to the devising of experiments in which the aim is to determine the law expressing the force between the elementary charges. These experiments (together with subsequent experiments) lead to the conception of electrons and protons which, when stationary, act upon each other with forces whose magnitudes are expressed by Coulomb's inverse square law. In this treatment, no question is raised as to a medium or mechanism by means of which distant elements act upon each other. The connecting link between distant elementary charges (and also between adjacent elements) is a formula which tells **how much** these elements influence each other.

In the **line-, surface-, and volume-integral** type of treatment, a study is made of the mode of variation in space of the vector quantities by which the properties of the electrostatic field are specified. It is found that the mode of space variation of these vectors may be expressed very simply by means of the following line-, surface-, and volume-integrals.

1. *The circuitation (line-integral around a closed line) of the electric intensity is zero for any path whatsoever.*

2. *The flux (surface-integral) of the electrostatic flux density over any closed surface whatsoever is equal to the quantity of electricity enclosed within the surface.*

The plan followed in developing electrostatic theory in this text has been to deduce these relations from the inverse square law of force. In pursuance of this plan, the first relation has been shown to be true of all vector fields in which the elementary vectors are directed **radially** to or from centers which have **spherically symmetrical properties**, and the second relation has been shown to be characteristic of all radially directed vector fields, provided the magnitude of the elementary vectors varies inversely as the **square** of the distance from the centers.

In presenting the laws relating to the magnetic field we propose to follow the sequence given below—a sequence quite similar to that followed in presenting electrostatic theory.

1. We postulate that the action of one current-carrying circuit upon another is to be accounted for in terms of the forces between the elementary lengths of the circuits, and proceed to deduce the laws expressing the action of one elementary length upon another.

2. By the aid of these laws the mode of space variation of the flux density vector will be studied; from this study, laws will be deduced which express the magnetic relations in terms of **line- and surface-integrals** involving the B vector.

240. Fundamental Magnetic Experiments.—The following two experiments, together with the six experiments of Sec. 223 and the experiments with the direction-finder and the force-finder described in the previous chapter, furnish the necessary data for the derivation of the laws by which the magnetic flux densities may be predicted.

Experiment 7. Relation between the Magnitudes of the Flux Density and of the Current Setting Up the Field (Exp. Det. Rel.).—Let a magnetic field be set up by passing a current through a conductor of any configuration. The magnetic flux density at any fixed point P in this field may be measured by the torque-finder. By measuring the flux density corresponding to a number of different values of the current I, the following law is arrived at.

240a. The flux density B at any given point due to a current I in a given coil is directly proportional to the current.[1]

$$B \text{ is proportional to } I. \tag{352}$$

Experiment 8. The Flux Densities in the Field of a Long, Straight Conductor (Exp. Det. Rel.).—The lines of magnetic intensity in the vicinity of a long, straight conductor of circular cross-section (with the return conductor in a remote region) carrying a current have been found to be circles centered about the wire, and lying in planes at right angles to the length of the wire. If the flux densities at points in such a field are measured by means of the torque-finder, the following relation is discovered. This relation was determined in the year of Oersted's discovery (1820) by Biot and Savart from measurements of the relative magnitudes of the forces experienced by a compass placed at various distances from the long, straight wire. It is frequently called the **Biot-Savart Relation.**

[1] These statements are true only if the field is free from the ferromagnetic substances, iron, nickel, and cobalt. The ferromagnetic substances exhibit saturation effects.

240b. *Biot-Savart Relation* (1820).—*The flux density B at a point P at a distance r from the axis of a long, straight wire carrying a current I is found to be directly proportional to the current I and inversely proportional to the distance r. The law may be written in the form*[2]

$$B = \frac{\mu I}{2\pi r} \text{ (for a long, straight wire)} \tag{353}$$

in which, μ is a proportionality constant whose value is to be experimentally determined. Its value depends upon the units in which B, I, and r are expressed, and upon the medium surrounding the wire, whether air, evacuated space, oil, etc. This constant μ we shall subsequently call the **permeability** of the medium. μ is found to have substantially the same value for all materials save the ferromagnetic substances. Moreover, in the ferromagnetic materials, B is not proportional to I. All the discussion which follows applies to space free of ferromagnetic materials, save the discussion of the properties of ferromagnetic materials in Chap. XIV.

241. Ampere's Formula for Computing Magnetic Flux Densities (Generalization) (1820–1823).—By analyzing the experimental data contained in the six experiments of Sec. 243 and in the Biot-Savart relation, Ampere was able to deduce the following rules for computing the flux density at any point in the magnetic field of coils of known configuration carrying known currents.

The magnetic flux density at a point P due to the current I in an electric circuit in an infinitely extended homogeneous medium may be calculated by the following procedure.

Step 1.—*Divide the circuit into short lengths, each so short that it may be regarded as a straight conductor.*

Step 2.—*Calculate the differential component dB, which is contributed to the magnetic flux density at P by the current in each differential length dl by the formula*

$$dB \text{ (webers per sq. cm.)} = \frac{\mu I \sin (r,l) dl}{4\pi r^2}, \tag{354}$$

[2] We write the Biot-Savart relation in the form $B = \left(\frac{\mu}{2\pi}\right)\frac{I}{r}$, rather than in the form $B = (k)\frac{I}{r}$, because it has been found that the placing of the factor $1/2\pi$ in this equation avoids the occurrence of π as a factor in subsequent equations of more frequent use.

in which (r,l) is the angle between the short length dl and the line r to the point P (see Fig. 203).

The differential vector dB is directed along the normal to the plane determined by the line r and the length dl, in that direction along the normal in which the fingers point if the elementary length is grasped in the right hand with the thumb pointing in the direction of the current in the elementary length. (This is Ampere's rule as stated in Sec. 226a.)

Step 3.—*The resultant (determined by the polygon construction) of all the infinitesimal vectors is the magnetic flux density at P.*

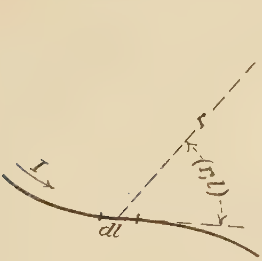


FIG. 203.

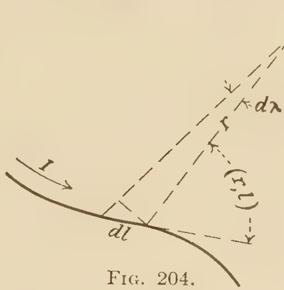


FIG. 204.

FIGS. 203 and 204.—Ampere's formula.

An examination of Fig. 204 will show that the following alternative form of step 2 may be readily derived from Formula 354. In some cases the alternative form of the second step is more convenient to apply than the first form.

Step 2 (ALTERNATIVE FORM).—*Calculate the differential component dB, which is contributed to the magnetic flux density at P by the current in each differential length dl by the formula*

$$dB \text{ (webers per sq. cm.)} = \frac{\mu I \, d\lambda}{4\pi r}, \qquad (355)$$

in which, $d\lambda$ is the angle subtended at P by the length dl (see Fig. 204). The differential vector dB is directed along the normal to the plane determined by the line r and the length dl, in that direction along the normal in which the fingers point if the elementary length is grasped in the right hand with the thumb pointing in the direction of the current in the elementary length.

242. The Manner of the Derivation of Ampere's Formula.—In 1820, when Oersted announced the discovery of the forces acting between perma-

nent magnets and electric currents, the French physicist, Ampere, at once became intensely interested in magnetic phenomena. One week after the account of Oersted's discovery reached Paris, Ampere announced the discovery of the forces between one conductor carrying current and another, and he set himself the task of discovering the fundamental laws relating to these forces. He took the position that all magnetic forces are simply the action of one current element upon another, and that permanent magnets are acted upon by such forces only because there are electric currents within the atoms or molecules of the magnet. Ampere decided, therefore, that the fundamental law of electrokinetics, or magnetism, is the law of force between the current in one very short element of a conductor and the current in another element, and he sought the equation which expresses this law. Since in this text we have defined the magnetic flux density at a point in terms of the force on a short element of wire, Ampere's search for an equation expressing the law of force between two elementary lengths of wire is equivalent to a search for the equation by which we may compute for any point the flux density which is to be attributed to the current in a given element of circuit.

It is possible, by means of the force-finder, to measure the force exerted upon an elementary length of an electric circuit in a magnetic field. Since, however, continuous currents do not exist except in complete circuits, it is not experimentally possible to isolate an elementary length of the circuit which is the cause of the magnetic field, and thus directly to measure the force this elementary length exerts upon another elementary length. The force-finder always measures the resultant force exerted upon it by all the elementary lengths of the circuit or circuits causing the field. Therefore, the formulas for computing the force with which one elementary length acts upon another cannot be based upon the direct measurement of forces between elements, but must be deduced from an analysis of the force with which one complete circuit acts upon another circuit or upon a force-finder. Ampere at first thought that the "guess and trial" method must be used to deduce the formula for the flux density due to an elementary length of circuit. The "guess and trial" method would be to assume an equation for magnetic fields corresponding somewhat in form to the equations which were known to hold in electrostatic and gravitational fields. Then by applying this equation to different arrangements of conductors, and comparing the calculated results with measured results, this equation could be modified so that finally it would give correct results. In later years he decided that a rigorous deduction of the formula could be made, and proceeded to work out his arguments, some of which are very ingenious. We are not concerned here with the question as to whether his arguments are conclusive or not, since the law does not rest upon these arguments today. We shall indicate some of them, nevertheless, merely to show the manner in which the form of Eq. (354), or its equivalent, may have been suggested to Ampere.

First of all, in postulating that the effect of the current in one element dl could be calculated independently of all other elements and then added to the effect of the current in other elements, Ampere was assuming that the

law of linear superposition of effects applies to the magnetic field. Now, as illustrated in Sec. 30, the law of linear superposition does apply when, and only when, the magnitude of the effect is directly proportional to the (first power) of the magnitude of the cause. It is evident, therefore, that when Ampere assumed the existence of a relation of the type he sought, he was assuming that the law of linear superposition does hold, and that the magnetic flux density due to the current in a short element dl, is directly proportional to Idl. This assumption appears very reasonable, especially since it has been experimentally determined (Experiment 7) that the flux density at any point due to the current in any closed circuit is directly proportional to the current.[3]

Ampere arrived at the factor, sin (r, l), in Eq. (354) by considering a circuit made up of a straight conductor and a return conductor following the same general path but crooking back and forth across it as indicated in Fig. 205. The field due to such a circuit is practically zero at all points far enough away so that the distance is large compared to any distances between the crooked conductor and the straight one. It was necessary, therefore, to put a factor into Eq. (354) such that the calculated effect of the crooked conductor would exactly balance that of the straight conductor. It is evident that the factor sin (r,l) does this, for in Fig. 205, dl_1 sin $(r,l_1) =$ dl sin (r,l). Thus the effects of the two segments cut off by the radii r_1 and r_2 are balanced. In like manner other segments can be shown to balance, until the total effect of the two conductors has been shown to be negligibly small.

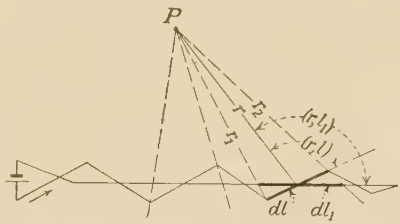


FIG. 205.—The field of twisted leads.

Ampere next assumed that the magnetic force between current elements varies inversely as some power (say the nth power) of the distance r between the elements. By applying his formula to a few cases in which measured results were available he found that the exponent n must be set equal to 2. We might arrive at this conclusion by the following argument which deals with the dimensions of Eq. (354). The equation for the magnetic flux density near a long, straight conductor is $B = \mu I/2\pi r$. We observe that the

[3] It should be noted that Ampere's work on electrodynamics (1820–1823) preceded the discovery of the laws of electrolysis by Faraday (1833) and of the law relating to metallic conduction by Ohm (1827) and of the law expressing the heating effect of the current (by Joule in 1841). Consequently, in the Amperian system (and in the Electromagnetic C.G.S. System of units which grew out of Ampere's work), electric current was not measured in terms of the movement of charge or of electrolytic or of heating effects; but the magnitude of the current in a circuit was defined to be directly proportional to the force it exerted on a magnet.

dimension, length, appears only in the denominator, and then as the first power. Now any other equation for magnetic intensity for any kind of circuit must be dimensionally equivalent to this equation. Equation (354) contains a length dl to the first power in the numerator, and, therefore, must contain a length to the second power in the denominator in order to be equivalent to the known equation, $B = \mu I/2\pi r$. The only length which would be expected to appear in the denominator is the distance r, and thus the denominator must contain r^2.

An equation derived by such a "guess and trial" process will contain a numerical coefficient which must be evaluated by comparisons of results calculated from the formula with the results of measurement. For example, we have the experimental result $B = \mu I/2\pi r$ for the value of B at a point near a long, straight conductor. Upon applying Ampere's formula to the calculation of this value, we find that, in order to give correct results, the numerical coefficient in the formula must have the value, $\mu/4\pi$. The numerical value of the constant μ is called the **permeability.**

In our present treatment this formula may be thought of as resting, not upon the suggestive but inconclusive arguments above, but upon the following fact:

Since Ampere's formula has been announced, it has been used in calculations of magnetic flux densities for all sorts of geometrical arrangements of conductors, and whenever calculated values of B have been compared with measured values they have been found to agree.

This fact is sufficient justification for accepting the formula. It is the only justification offered here.

243. Applications of Ampere's Formula.—We illustrate the application of Ampere's formula by two examples:

Example 1.—To find the magnetic flux density at a point P due to the current I in the long, straight conductor AB of Fig. 206.

The flux density at P caused by the current in the element dl which subtends at P the angle $d\lambda$ is

$$dB = \frac{\mu I d\lambda}{4\pi r}.$$

But

$$r = \frac{b}{\cos \lambda},$$

therefore

$$dB = \frac{\mu I \cos \lambda \, d\lambda}{4\pi b}.$$

Now the quantity dB, is a vector quantity and must be added vectorially to the similar vectors due to other segments of the conductor. But examination shows that in this case, all of the small vectors have the same direction and therefore the addition becomes algebraic and may be carried out by ordinary integration.

$$B = \int_{\lambda_1}^{\lambda_2} \frac{\mu I \cos \lambda d\lambda}{4\pi b} = \frac{\mu I}{4\pi b}\Big[\sin \lambda\Big]_{\lambda_1}^{\lambda_2},$$

$$B \text{ (webers per sq. cm.)} = \frac{\mu I}{4\pi b}(\sin \lambda_2 - \sin \lambda_1). \qquad (356)$$

The flux density at P is perpendicular to the plane PAB, in the direction given by Ampere's right-hand rule.

If the length of the wire AB is great in comparison with the perpendicular distance b from the point P to the wire, and if the point P is about equally distant from the two ends of the wire, $\lambda_2 = \pi/2$ and $\lambda_1 = -\pi/2$, and Eq. (356) reduces to

$$B = \frac{\mu I}{2\pi b}.$$

This is seen to be the Biot-Savart relation.

Example 2.—To find the magnetic flux density at a point P which lies in the plane of a small plane coil ABC of Fig. 207 at a distance r from the center of the coil, which is great in comparison with the largest dimension of the coil.

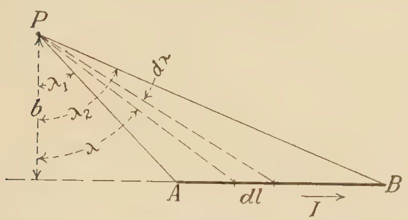


FIG. 206.—Magnetic flux density due to the current in a straight wire.

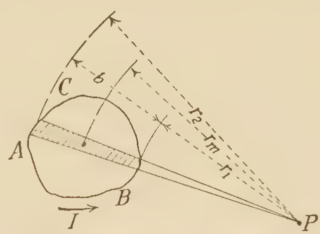


FIG. 207.—Magnetic flux density due to a small plane coil.

Consider the flux density at P due to the current in the two elementary lengths included between the two radii shown on Fig. 207. The flux densities due to the two elements are each perpendicular to the plane $PABC$, but are oppositely directed. Therefore

$$dB = \frac{\mu I d\lambda}{4\pi r_1} - \frac{\mu I d\lambda}{4\pi r_2} = \frac{\mu I d\lambda}{4\pi}\left[\frac{r_2 - r_1}{r_1 r_2}\right].$$

If r_m represents the mean radius, and b represents $r_2 - r_1$, the above expression can be written without appreciable error in the form

$$dB = \frac{\mu I}{4\pi}\frac{b d\lambda}{r_m^2} = \frac{\mu I}{4\pi}\frac{b(r_m d\lambda)}{r_m^3}.$$

But $b(r_m d\lambda)$ equals the area da of the cross-hatched portion of the plane which is enclosed by the two elementary lengths and the two radii. Therefore

$$dB = \frac{\mu I da}{4\pi r_m^3}$$

and the flux density B at P due to the current in all elements of the coil is seen to be

$$B \text{ (webers per sq. cm.)} = \frac{\mu I a}{4\pi r_m^3}, \tag{357}$$

in which a represents the area in square centimeters bounded by the coil. If the coil contains N closely wound turns enclosing a mean area a, Eq. (357) becomes

$$B \text{ (webers per sq. cm.)} = \frac{\mu N I a}{4\pi r_m{}^3} \tag{358}$$

The numerator NIa has in previous work (Sec. 232) been termed the magnetic moment of the coil. We see that the magnetic flux density set up by a small plane coil at a distant point P lying in the plane of the coil is directly proportional to the magnetic moment of the coil and inversely proportional to the cube of the distance from the coil to the point.

PART II—LINE-INTEGRALS OF MAGNETIC FLUX DENSITY

244. Line-integral of the Magnetic Flux Density in the Field of a Long Straight Conductor (EXP. DET. REL.).—We start the study of the line-integral[4] of the magnetic flux density in the magnetic field of a circuit of the simplest configuration—in the field in the immediate vicinity of a long, straight conductor. In experiment 8, it has been shown that the lines of flux density in the immediate vicinity of a long, straight wire of circular cross-section (with the return conductor in a remote region) are circles lying in planes at right angles to the length of the wire and having their centers in the axis of the wire. It has been shown that the value of the magnetic flux density at points in such a circle of radius r is

$$B = \frac{\mu I}{2\pi r}. \tag{353}$$

Let us compute the value of the **circuitation, or the line-integral, of the magnetic flux density around any of these circular lines of flux density.**

Since the flux density vectors are tangent to the circle and have the same value at all points of any one circle, the line-integral is very easily evaluated. It is

$$\int B \cos(B, l) dl = \frac{\mu I}{2\pi r}(2\pi r) = \mu I. \tag{359}$$

That is, the line-integral of the flux density taken around a line of magnetic flux density encircling a long, straight conductor is readily seen to have the same value for all circles, and this

[4] The definitions of line-integral and of circuitation contained in Sec. 47 should be reviewed at this point.

value is seen to be dependent upon one thing only—the value of the current which is linked with the circular path traversed in obtaining the line-integral.

244a. Line-integral Around Any Closed Line.—It can very readily be shown that the line-integral of the flux density taken around a closed line of any shape whatsoever, which is linked once with the long, straight conductor, has the same value as that deduced above for the circular path. Thus, in Fig. 208, let $ACDEA$ represent such a closed line linked with the long, straight conductor O.

To show that

$$\int^{\text{around } ACDEA} B \cos (B, l) dl = \mu I$$

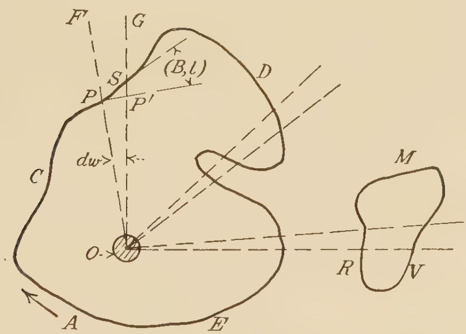


FIG. 208.—Line-integral of B around a long straight wire.

Let us consider the value of $B \cos (B, l)dl$, for any short, straight line which leads from the point P in the axial plane OF to the axial plane OG. These two planes intersect along the axis of the wire and make with each other the infinitesimal dihedral angle dw.

Let PP' represent a perpendicular from the plane OF to the plane OG. The length of the line PP' between the planes is $r(dw)$. The flux density vector at P is directed along PP', and has the value

$$B = \frac{\mu I}{2\pi r}.$$

Therefore the value of the line-integral over the line PP' is

$$B \cos (B,l)dl = \frac{\mu I}{2\pi r} r(dw) = \frac{\mu I}{2\pi}(dw) \qquad (359a)$$

Now suppose the short length PS of the path $ACDEA$ makes the angle (B,l) with the vector B at the point P. The length of the line PS is

$$PS = \frac{PP'}{\cos (B,l)} = \frac{rdw}{\cos (B,l)}.$$

The component of the flux density along PS is $\frac{\mu I}{2\pi r}\cos(B,l)$. Therefore the value of the line-integral over the PS is

$$B\cos(B,l)dl = \frac{\mu I}{2\pi r}\cos(B,l)\frac{rdw}{\cos(B,l)} = \frac{\mu I}{2\pi}\cdot dw \qquad (359a)$$

That is, the line-integral of the flux density over the lines PP' and PS leading from the radial plane OF to the radial plane OG have the same value, namely, $\frac{\mu I}{2\pi}(dw)$. But PP' represents any line drawn perpendicularly between the planes, and PS represents a line making any angle with the line PP'. The conclusion is, therefore, that the value of the line-integral over any straight line leading from OF to OG is $\frac{\mu}{2\pi}Idw$—a quantity whose value depends only upon the magnitudes of the current and of the dihedral angle dw between the planes.

But the line-integral over any closed line which links once with the conductor O may be found by dividing up the line into short lengths by a great number of radial planes, each plane making a small dihedral angle dw with the next plane. The line-integral over the closed path $ACDEA$ is the sum of the line-integrals in getting from a plane OF back to the plane OF by passing through all intermediate planes, and, since the sum of the dihedral angles in one complete circuit around the conductor is 2π radians, it follows that the line-integral of the flux density around the closed line $ACDEA$ is

$$\int B\cos(B,l)\,dl = \frac{\mu I}{2\pi}\int^{2\pi} dw = \mu I. \qquad (359)$$

A critical examination of the above demonstration will indicate that if the closed line does not link with the conductor, the line-integral is zero. $MVRM$ is such a line. On the other hand, if the line encircles the conductor N times before closing, the line-integral has the value $N\ (\mu I)$, or, for any closed line in the magnetic field of a long, straight conductor, we may write

$$\int^{\text{around any closed line}} B\cos(B,l)dl = \mu NI \qquad (360)$$

245. The Line-integral of the Flux Density in the Field of a Conductor of Any Configuration (EXP. DET. REL.).—The simple relation found to hold in the field of the long, straight conductor naturally leads to the question—Is the above relation between the flux densities and the current which sets up the field a general relation which applies to any magnetic field, or is it true only of the field of a long, straight conductor?

It will be shown in Sec. 263 that this relation may be deduced from Ampere's formula for the magnetic field due to a circuit of any configuration. Since, however, Ampere's formula is not

based upon direct measurements of the field set up by an elementary length of circuit, we may think of this relation as being confirmable by the following experimental procedure.

Let magnetic fields be set up by passing currents through conductors of any configuration, and let the values of the line-integrals of the flux density around a number of closed lines in these fields be worked out from experimental measurements of the value of the B vector at a great many points along these lines. Such an experimental study of the line-integral of magnetic flux densities leads to one of the most important of the experimental laws of the magnetic field. It is found that the value of the line-integral of the flux density around a closed line is directly proportional to the current which crosses any surface bounded by the line. The value of the proportionality constant, which is usually represented by μ, is found by these experiments to depend only upon the units in which B, I, and l are measured, and upon the medium in which the field is set up—whether air, evacuated space, oil, or similar media. This law may be expressed as follows:

245a. LAW OF CIRCUITATION OF MAGNETIC FLUX DENSITY (EXP. DET. REL.).—**The CIRCUITATION of the magnetic flux density, or the value of the LINE-INTEGRAL of the magnetic flux density around any closed line in a magnetic field (free of ferromagnetic materials), is equal to the product of the NET CURRENT ACROSS ANY SURFACE WHICH IS BOUNDED BY THE CLOSED PATH OF INTEGRATION multiplied by a proportionality constant μ, whose value depends upon the units used and upon the medium in which the field exists.**

$$\int^{\text{closed line}} B \cos (B,l) dl = \mu \Sigma NI. \tag{361}$$

The arrow direction **across** the surface is understood to be related to the arrow direction along the line by the right-hand screw convention of Sec. 225*b*. Since the current which crosses the surface usually flows in wires which may cross the surface more than once, the net current across the surface has been represented in Eq. (361) by the expression ΣNI. In this expression I represents the value of the current in a wire, N represents the net number of times the wire carries the current across the surface, and the summation sign indicates that the summation is to be carried over all the wires which cross the surface. For

example, if the field is set up by the current in a single coil, and if the conductors of this coil pass through the surface N times in the same direction, the net current across the surface is NI amperes. If the field is set up by the currents in three coils carrying currents of the values I_1, I_2, I_3, and if N_1, N_2, N_3 represent the net number of times each circuit carries the current across the surface in the positive direction, Eq. (361) may be written

$$\int^{\text{closed line}} B \cos (B, l) dl = \mu(N_1I_1 + N_2I_2 + N_3I_3) \qquad (361a)$$

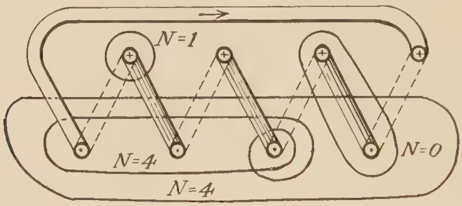


FIG. 209.—Net number of turns linked with the loops.

The values of N for a number of paths lying in the central plane of a solenoidal coil have been marked on Fig. 209.

246. Permeability, Magnetic Intensity, and Magnetomotive Force.—For the purpose of facilitating the expression and the application of the fundamental law contained in Eq. (361), it is convenient to so rewrite it that the constant μ appears on the other side of the equation, and then to assign names to several parts of the equation.

Upon dividing both members of Eq. (361) by the constant μ, which has been called the permeability of the medium surrounding the conductors, it takes the form

$$\frac{1}{\mu}\int^{\text{closed line}} B \cos (B,l)dl \text{ or } \int^{\text{closed line}} \frac{B}{\mu} \cos (B,l)\, dl = \Sigma NI. \qquad (362)$$

From this equation, we may frame the following definition of **permeability**.

246a. PERMEABILITY (DEFINITION).—By the (magnetic) PERMEABILITY μ of a medium is meant the ratio of the circuitation (line-integral) of the flux density taken around any specified closed line which lies wholly

within a homogeneous medium, to the net current crossing any open surface, or cap, of which the specified line is the boundary.

$$\mu = \frac{\int^{\text{closed line}} B \cos (B,l) dl}{\Sigma NI} \quad \text{(defining } \mu\text{)}. \tag{363}$$

The quantity B/μ appearing in Eq. (362) occurs in so many of the calculations dealing with magnetic fields, and likewise the line-integral of B/μ, that it is convenient to have names for them. Accordingly, we say that the current gives rise to a **magnetic intensity** at every point in the field, and that the current exerts a **magnetomotive force** along any path in the field, and we define these terms in the following manner:

246b. MAGNETIC INTENSITY (DEFINITION).[5]**—By the MAGNETIC INTENSITY at a point *P* in a magnetic field is meant the vector quantity obtained by dividing the magnetic flux density vector at the point by the permeability of the medium in which the point *P* is located. Magnetic intensity is represented by the symbol H.**

$$\mathsf{H} = \frac{\mathsf{B}}{\mu} \quad \text{(defining } \mathsf{H}\text{)}. \tag{364}$$

246c. MAGNETOMOTIVE FORCE (DEFINITION).**—The MAGNETOMOTIVE FORCE IN THE DIRECTION *AB* along a specified line from *A* to *B* in a magnetic field is the name applied to the line-integral of the magnetic intensity from *A* to *B* along the line. The MAGNETOMOTIVE FORCE around a specified CLOSED loop is another name for the CIRCUITATION OF THE MAGNETIC INTENSITY around the specified loop. Magnetomotive force (m.m.f.) is represented by the symbol $\mathfrak{F}$.**

$$\mathfrak{F} \text{ (ampere-turns)} = \int_A^B H \cos (H,l) dl \quad \text{(defining } \mathfrak{F}\text{)}. \tag{365}$$

The quantity which has just been called the magnetomotive force is not a force in any sense of the word. The name is not a good descriptive name, and has no justification except for the fact that the magnetomotive force along a line bears the same mathematical relation to the magnetic intensities at points along the line that the electromotive force along a conductor bears to the electromotive intensities at points along the conductor.

247. Units of Magnetomotive Force, Magnetic Intensity, and Permeability.—The names of these three units have been derived,

[5] B/μ is also called the **magnetizing force,** and the **magnetic force.**

one from the other, in an order which is exactly the reverse of the order in which the quantities have just been defined.

Since the magnetomotive force around a closed path is equal to the product of amperes times turns, the unit of magnetomotive force is called the **ampere-turn.**

247a. Unit of Magnetomotive Force (DEFINITION).[6]—*The unit of magnetomotive force, the ampere-turn, is the magnetomotive force around a path which links with one turn of wire carrying* 1 *ampere.*

Since magnetic intensity may be regarded as a magnetomotive force divided by a length, the derived name of the unit of intensity is the **ampere-turn per centimeter.**

247b. Unit of Magnetic Intensity (DEFINITION).[6]—*The unit of magnetic intensity, "the ampere-turn per centimeter," is the magnetic intensity at any point in a circular path* 1 *centimeter in circumference centered about a long straight wire carrying* 1 *ampere.*

Since permeability is used most frequently in the equation

$$B = \mu H \quad \text{or} \quad \mu = \frac{B}{H}, \tag{364}$$

and since names have been assigned to the units of B and H, we may derive the name of the unit of permeability from the names of these units, thus,

247c. Unit of Permeability (DEFINITION).—*A material is said to have a permeability of unity, or a permeability of one weber per sq. cm. per ampere-turn per cm., if a magnetic intensity of* 1 *ampere-turn per cm. is accompanied by a flux density of* 1 *weber per sq. cm.*

[6] At the present time, units of magnetic intensity and of magnetomotive force which are taken from the **Electromagnetic System of Units** are used by many writers. These units are named the **gilbert per centimeter** and the **gilbert,** respectively. The **gilbert per centimeter** (which is also known as the **gauss**) is defined as the magnetic intensity at unit distance from the unit pole of the E. M. system.

$$\begin{aligned} &1 \text{ amp.-turn per cm.} = 0.4\pi \text{ gilberts per cm.} = 0.4\pi \text{ gauss.} \\ &1 \text{ amp.-turn} = 0.4\pi \text{ gilberts.} \end{aligned} \tag{366}$$

The use of the gilbert per centimeter results in the appearance of 4π as a factor in an irrational manner in subsequent formulas. The use of the ampere-turn per centimeter banishes the irrational 4π from these formulae, and accomplishes the results sought in the rational systems of units.

The value in practical units of the permeability of free space (represented by μ_o), as determined by experiment is[7]

$$\mu_o = 0.4\pi \times 10^{-8} = 1.257 \times 10^{-8} \text{ (webers per sq. cm. per amp.-turn per cm.)}. \quad (367)$$

247d. Relative Permeability (DEFINITION).—*The relative permeability μ_r of a substance is defined to be the ratio of its permeability to the permeability of the standard medium, free space.*

To within 1 part in a million, the relative permeability of air is unity. The relative permeabilities of all other substances, save the ferromagnetic materials—iron, nickel, cobalt, and their magnetic alloys—differ from unity by less than a few parts in a thousand.

248. Law of Circuitation for Magnetic Intensity.—From the manner in which the terms "magnetic intensity" and "magnetomotive force" have been defined, the experimental law expressed by Eq. (361) may now be written in the forms

$$\mathfrak{F} \text{ (around a closed line)} = \int^{\text{closed line}} H \cos (H, l) dl = \Sigma NI \text{ (ampere-turns)}. \quad (368)$$

When expressed in terms of the newly defined quantities, the law of circuitation for the magnetic field becomes

[7] The permeability of free space has the value of $0.4\pi \times 10^{-8}$ because of the historical manner in which the practical unit of current was defined. The practical unit of current was defined to be one-tenth as great as the C.G.S. electromagnetic unit of current. This unit had been defined in terms of mechanical force. The definition of the C.G.S.E.M. unit was equivalent to defining it as that current which, when flowing in opposite directions in two long, straight, parallel wires 1 centimeter apart, will cause them to repel each other with a force of 2 dynes per centimeter of length. That is, the force per centimeter of length upon the wire of a force-finder placed 1 centimeter away from a long, straight wire carrying one E.M. unit of current would be 2 dynes. If the long wire and the force-finder each carry one practical unit of current or 1 ampere, the force will be 0.02 dyne or 2×10^{-9} dyne-sevens per centimeter of length. Therefore, the value of the flux density at a point 1 centimeter from the center of the long wire is 2×10^{-9} webers per square centimeter. From this it may be seen by substituting in Eq. (353), in which μ was introduced, that the value of μ_o is $0.4\pi \times 10^{-8}$.

248a. LAW OF CIRCUITATION IN TERMS OF MAGNETOMOTIVE FORCE.—The magnetomotive force $\mathfrak{F}$, around any closed line in a specified direction, or the line-integral of the magnetic intensity in the specified direction around the line, is equal to the net current (in amperes) in the arrow direction across any surface which is bounded by the closed line of integration. (The arrow direction across the surface is related to the specified direction around the line by the right-hand screw convention.)

249. The Law of Circuitation Expressed in Terms of Current Density (DEDUCTION).—Since the current crossing a surface is equal to the surface-integral of the current density, the law of circuitation may be stated in the following form.

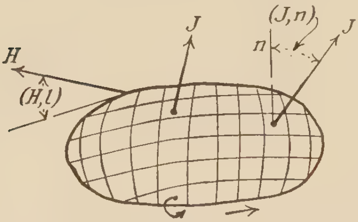


FIG. 210.—Line-integral of H = surface-integral of J.

The line-integral of the magnetic intensity H around any closed line is equal to the surface-integral of the current density J taken over any surface bounded by the line (see Fig. 210).

$$\int^{\text{closed line}} H \cos (H,l)(dl) = \int^{\text{bounded surface}} J \cos (J,n)(da). \tag{369}$$

250. The Law of Circuitation Expressed in Vector Notation.—In vector analysis, the **curl** of a vector is defined as follows:

250a. Curl of a Vector (DEFINITION).—*The* **curl of a vector H** *at a point P and* **in a specified plane** *passing through that point is defined as a vector* V *whose algebraic value is equal to the line-integral of the vector* H *taken around the boundary of an infinitesimal portion of the plane, divided by the area of the infinitesimal portion. The vector* V *is to be drawn normal to the plane; the arrow direction along the normal being related to the direction of integration by the right-hand screw convention. At the given point there will be some plane for which this quotient, or curl, has a maximum value. This maximum value is termed "the curl of the vector* H *at the point P.'*

$$\text{curl } \mathbf{H} = \frac{\int H \cos (H,l) dl}{a} \text{ (as } a \text{ approaches zero).} \tag{370}$$

The law of circuitation is thrown into vector notation in the following manner. Dividing both members of (369) by the area a of the surface bounded by the magnetic circuit.

$$\frac{\int H \cos (H,l) dl}{a} = \frac{\int J \cos (J,n) da}{a}.$$

If a becomes an infinitesimal area normal to the direction of current flow at a point P, the right member of this equation is the current density at the point P. The left member is seen to be **the curl of the vector** H **at the point** P.

Whence
$$\text{curl } \mathsf{H} = \mathsf{J}. \tag{371}$$

251. Use of the Law of Circuitation in the Calculation of Magnetic Intensities.—The law of circuitation gives the value of the line-integral of the magnetic intensity around any closed path, Eq. (368), but only under special conditions is it possible to calculate back from the known value of the integral of H along a line to the value of H at any particular point on that line. This is possible if two conditions are fulfilled.

1. If the direction of the H vectors are so well known that definite lines of magnetic intensity can be readily visualized and laid out, in order to determine their length.

2. If the value of the magnetic intensity is known to be constant along the lines so laid out.

Under these conditions, Eq. (368) reduces to the form

$$Hl = \Sigma NI. \tag{372}$$

From this the value of the magnetic intensity is readily obtained. The above conditions are fulfilled in the field surrounding a long, straight conductor, and in the field inside an annular core which is uniformly overwound with an exciting coil (see Fig. 211). They are approximately fulfilled in many of the magnetic fields found in dynamos and transformers. The above method of calculating magnetic intensities is therefore a very useful one. In fact, it is the only method used in calculations relating to the magnetic circuits of dynamos and transformers. If conditions 1 and 2 are not fulfilled, this method cannot be used, but the magnetic intensities may be calculated by means of Ampere's formula.

The application of the law of circuitation is illustrated in the solution of the following problem:

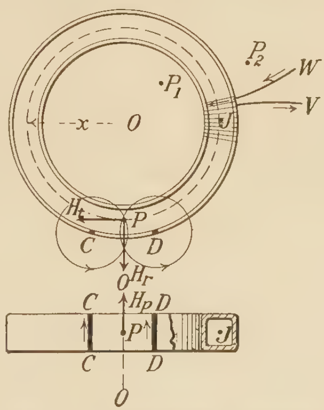


FIG. 211.—Magnetic intensities due to a ring-wound coil.

251a. Problem.—To compute the magnetic intensities at points in the field of the coil of N turns carrying I amperes which is uniformly distributed over the ring-shaped core, or form, shown in Fig. 211.

For simplicity assume that the wire is of rectangular section with so little space between turns that the current in the coil virtually constitutes a continuous **current sheet** flowing around the winding form.

Let us first compute the intensity at a point P which lies within the core at a distance x from the axis OO of the ring. Let a circle having its center on the axis and of radius x be passed through the point P. Let us compute the three mutually perpendicular components of the magnetic intensity at P, namely, H_t, the component tangential to the circle, H_r, the component along the radius OP and H_p, the component perpendicular to the plane of the circle.

The current crossing the surface bounded by the circle of radius x is NI amperes. Therefore, the magnetomotive force, or line-integral of H, around the circle is

$$\mathfrak{F} \text{ (ampere-turns)} = NI.$$

From the circular symmetry, the component of the magnetic intensity tangential to the circle has the same value at all points of the circle as at the point P. Therefore, the value of the tangential component of the magnetic intensity at P is

$$H_t(\text{amp.-turns per cm.}) = \frac{NI}{2\pi x}. \tag{373}$$

To determine the magnitude of the radial component, Ampere's formula may be used. The only current elements which can contribute a radial component to the magnetic intensity at P are the currents in the portions of the turns lying on the inside and outside surfaces of the ring. Consider the intensity at P due to the two elements CC and DD, symmetrically located with respect to P. The radial components are equal but oppositely directed and the resultant at P of the radial components of the two elements is zero. It is evident, therefore, that H_r, the radial component of H at the point P, is zero, since all the elements which could possibly contribute anything can be paired off in the above manner. Whence

$$H_r = 0.$$

By a similar argument, the effects of the current elements on the plain top and bottom surfaces of the coil may be paired off, to demonstrate that H_p, the component perpendicular to the plane of the circle, is zero, or $H_p = 0$.

If now the point P is located any place outside of the annular ring enclosed by the current sheet as at P_1 or at P_2, it may be readily seen that the same line of reasoning leads to the conclusion that[8]

$$H_t = H_r = H_p = 0.$$

Hence the following conclusion may be drawn:

A field coil so closely and uniformly wound over an annular ring that it is the equivalent of a continuous current sheet gives rise to no magnetic field whatsoever at points external to the ring.[8] *At any point lying within the ring at a*

[8] In an actual coil, the current stream lines do not lie in radial planes but the turns lying between the leads W and V each have a slight advance along the annulus. It may be seen that at distant points the magnetic field of such an N turn coil is the equivalent of N turns each lying in a radial plane and a single turn coinciding with the central filament JKL of the annulus.

distance x from the axis the magnetic intensity has the value $NI/2\pi x$ and is tangential to a circle centered about the axis and passing through the point.

PART III—SURFACE-INTEGRALS OF MAGNETIC FLUX DENSITY

252. Magnetic Flux.—The method of taking the surface-integral of a distributed vector has been described in Sec. 84. Illustrations of surface-integrals have been given, and it has been pointed out that the term **flux** in mathematical and physical science always designates the surface-integral of a distributed vector, an appropriate adjective being prefixed to flux to indicate the nature of the vector quantity involved. We now propose to study the (spacial mathematical) properties of the **flux of the B vector,** or of the **magnetic flux.**

In accord with general custom the term "magnetic flux" is defined in the following way:

252a. MAGNETIC FLUX (DEFINITION).—**The magnetic flux Φ in a specified direction across a given surface is defined to be the algebraic value of the surface-integral of the magnetic flux density over the given surface, with the specified direction as positive. That is to say, in the following integral, (B, n) represents the angle between the flux density vector B and the specified direction along the normal n to the elementary surface area da.**

$$\Phi \text{ (webers)} = \int B \cos (B, n) da. \tag{374}$$

It is evident that if the value of the B vector is constant over the surface under consideration, and is everywhere normal to the surface and is pointing in the positive direction, the above integral reduces to

$$\Phi \text{ (webers)} = Ba. \tag{375}$$

252b. Unit Magnetic Flux *is the flux (or surface-integral) which results from unit flux density normal to 1 square centimeter of surface. The unit of flux is called the "weber."*[9]

253. Historical Note.—The relations expressed in Eqs. (374) and (375) now make it possible to explain the origin of the term

[9] At the present time many writers carry on magnetic calculations in the electromagnetic system of units, in which the name of the unit of flux is the **maxwell, or the (magnetic) line.**

$$\left.\begin{array}{l} 1 \text{ weber} = 10^8 \text{ maxwells} \\ 1 \text{ eighth-weber} = 1 \text{ maxwell} = 1 \text{ line} \end{array}\right\}. \tag{375a}$$

magnetic flux density and of the name of the unit, namely, the **weber per square centimeter.** It is evident that the order in which the quantities B and Φ have been introduced and defined in this text is just the reverse of that in which the International Electrical Congresses have assigned names to the quantities and to the units. The vector quantity B which we now call the magnetic flux density was, by most of the early writers, termed the **magnetic induction.** The surface-integral of magnetic induction was termed the magnetic flux. The Second International Congress of Electricians in 1889 proposed that the unit of magnetic flux in the practical system be called the **weber,** in commemoration of the contributions of Wilhelm Weber to magnetic theory.

After the term **magnetic flux** became firmly estabished as the short name for **surface-integral of magnetic induction,** the relation between Φ and B expressed in Eqs. (374) and (375) led to the superseding of the original name of the B vector by the name **magnetic flux density.** Then, a name for the unit of magnetic flux density was derived from the **weber,** by making use of the mathematical relation between the two quantities, flux and flux density.

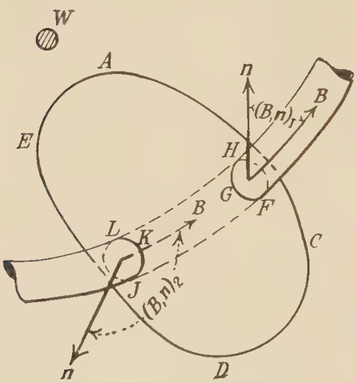


FIG. 212.—Surface-integral of B in the field of a long straight conductor.

254. Surface-integral of the Magnetic Flux Density in the Field of a Long Straight Conductor (DEDUCTION).—We start the study of the values assumed by the **surface-integral of the B vector over a closed surface** in the same simple field which furnished the clue to the properties of the **line-integral of the B vector around a closed line**; namely, in the magnetic field in the immediate vicinity of a long, straight wire carrying a current, with the return loop at a great distance. Such a closed surface is represented in plan view in Fig. 212. W represents the projection of the long, straight wire, which is perpendicular to the

plane of the page, and $ACDE$ is the projection of the closed surface on the paper.

Let FGH represent any closed curve drawn on the surface, and enclosing a surface area a_1 so small that it may be regarded as a plane surface. Imagine lines of magnetic flux density to be drawn through all points of the curve FGH. Since these lines are circles lying in planes at right angles to the length of the wire and having their centers on the axis of the wire, the lines through FGH form a tube which returns into itself, and the cross-section of this tube taken perpendicular to the lines has the same value a_0 at any point at which the tube may be cut.

The tube which enters the enclosed volume at FGH must emerge from the enclosed volume at some other point in order to return into itself. At the point of emergence the tube cuts on the surface $ACDE$ another small curve JKL which encloses a small plane portion of the surface having the area a_2.

The flux over the two small areas is

$$\Phi = B_1 a_1 \cos (B.n)_1 + B_2 a_2 \cos (B,n)_2$$

But $B_1 = B_2 = B,$

and since the flux density is normal to a_0.

$$-a_2 \cos (B,n)_2 = a_0 = a_1 \cos (B,n)_1$$

Therefore, $\Phi = Ba_0 - Ba_0 = 0.$

Now imagine any other small curve whatsoever which does not enclose any portion of the surface area a_1, to be drawn on the surface, enclosing the small area a_3. Lines of intensity passed through this curve form a second tube which cuts the surface in a second curve enclosing a small area a_4. The sum of the fluxes over a_3 and a_4 is zero. Since the tubes of magnetic intensity do not intersect, the area a_4 cannot include any portion of the areas a_1 or a_2. Hence the whole surface may be conceived to be divided up in this manner into corresponding pairs of patches, and since the net flux outward over any pair is zero, it follows that **the magnetic flux in an outward direction over the entire closed surface is zero.**

255. The Law of Continuity of Magnetic Flux Density.—The simple relation found above leads to the question: "Is the magnetic flux in a specified direction over a closed surface zero in any

field whatsoever, or is this relation true only in the field of the long, straight conductor?" The arguments, based on Ampere's formula, which are sketched below, and the indirect experimental studies, all lead to the conclusion that this relation holds in any magnetic field whatsoever. This relation means that unit tubes of magnetic flux (defined later in Sec. 260) do not terminate in any specific region (after the manner in which tubes of electrostatic flux terminate in regions containing charge) but each tube returns into itself. This relation is accordingly often called the **law of continuity for magnetic flux density.**

255a. LAW OF CONTINUITY FOR MAGNETIC FLUX DENSITY (EXP. DET. REL.).—**The surface-integral of the magnetic flux density vector in a specified direction (say, the outward), over any closed surface (that is, the magnetic flux over any closed surface) is zero.**

$$\Phi \text{ (over a closed surface) or } \int^{\text{closed surface}} B \cos (B, n) da = 0. \quad (376)$$

The following arguments indicate that the Law of Continuity may be deduced from Ampere's formula. According to Ampere's formula, the magnetic flux densities due to a circuit of known configuration may be computed by dividing the circuit into short, straight lengths, and then at a point vectorially adding the flux densities due to each elementary length. The formula states that the flux densities due to each short length are circuital vectors tangent to coaxial circles so situated that the short length of conductor lies on the common axis.

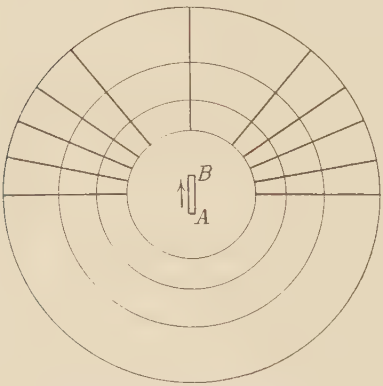


FIG. 213.—Tubes of equal magnetic flux due to the element of wire AB.

Figure 213 shows in section the geometry of the tubes of magnetic flux density surrounding a short length AB of wire. The tubes have been so laid out that the magnetic flux over the cross-section of any tube is equal to the flux over any other tube.

Now imagine any closed surface in this field. It lies in a field in which all lines of flux density are circles concentric with, and lying in planes perpendicular to, the same straight line. The arguments used for the case of the closed surface in the field of the long, straight wire may, therefore, be repeated to demonstrate that in this case the flux over the surface is zero. From this it follows that the flux over the surface due to the current in the entire coil (which is the algebraic sum of the fluxes due to all elementary lengths) is zero.

256. The Applications of the Line and Surface-integral Laws. We now have two relations of a fundamental nature dealing with the mode of variation of the flux density vector from point to point in the field, namely, the law of circuitation and the law of continuity.

The first law, the law of circuitation, relates the line-integral of the B vector around a closed line to the current causing the field. The application of this law comes in the calculation of the values of the magnetic flux density in certain symmetrical fields, as illustrated in Sec. 251.

In subsequent chapters we shall find that the work which is done when a conductor carrying a current moves through a magnetic field is proportional to the magnetic flux over the area swept over by the conductor, and that the electromotive force induced in the moving conductor is proportional to the magnetic flux over the area swept over per second. The law of continuity finds its application in calculations dealing with work and electromotive force. We next deduce several consequences of the law which will be useful in such calculations.

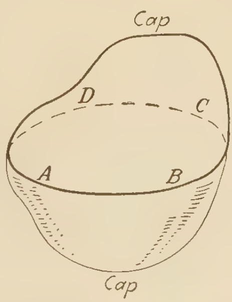


FIG. 214.—Caps having the same boundary.

257. Equality of Magnetic Fluxes over All Surfaces Bounded by the Same Closed Line (DEDUCTION).—Let *ABCD* of Fig. 214 represent any closed line. This line may be regarded as the common boundary or contour line of an infinite number of open surfaces or caps. Three of the possible caps have been represented in the figure. Since any two of these caps form a com-

pletely closed surface, and since the flux outward over any closed surface is zero, it follows that:

The magnetic flux over any surface bounded by a given closed line must be equal to the flux over any other surface bounded by the same closed line.

The flux over one surface is directed outward and that over the other is directed inward, or both bear the same relation to the positive direction around the circuit. In problems requiring the computation of the flux over a given surface, we are at liberty to substitute any other surface for which the computation happens to be easier, **provided** the two surfaces have a common boundary.

258. Expression of the Law of Continuity by a Differential Equation (DEDUCTION).—The law of continuity for magnetic flux density may be

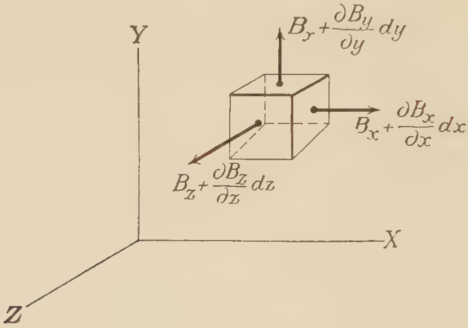


FIG. 215.—Surface-integral of B over a cube.

put in the form of a differential equation involving the X, Y, and Z components of the B vector. This is done by writing the expression for the surface-integral of the magnetic flux density in the outward direction over the surface of the infinitesimal cubical volume shown in Fig. 215, and putting this equal to zero. Thus

$$\Phi \text{ (outward over the cube)} = -B_x dy\,dz + \left(B_x + \frac{\partial B_x}{\partial x} dx\right) dy\,dz$$

$$-B_y dz\,dx + \left(B_y + \frac{\partial B_y}{\partial y} dy\right) dz\,dx - B_z dx\,dy + \left(B_z + \frac{\partial B_z}{\partial z} dz\right) dx\,dy = 0.$$

$$\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right] dx\,dy\,dz = 0. \qquad (377)$$

By dividing both members by the volume of the cube (namely, $dx\,dy\,dz$), the following differential equation is obtained

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \qquad (378)$$

259. Law of Continuity in Vector Notation.—The left member of Eq. (378) is the quotient obtained by dividing the surface-integral of the B vector in the outward direction over an infinitesimal closed surface by the volume enclosed by the surface. From the definition of the **divergence of a vector** contained in Sec. 87*a*, we see that this quotient is the divergence of the vector B at the point P at the center of the cube.

Whence the law of continuity when written in the notation of vector analysis takes the form

$$\text{div } \mathsf{B} = 0. \tag{379}$$

260. Tubes of Constant Magnetic Flux (DEFINITION).—In problems (such as the demonstration that the flux over any closed surface is zero) which require us to visualize the mathematical relations between the spacial quantities, current, flux density, and magnetic flux, it is extremely helpful to conceive of **tubes** whose walls are formed by lines of magnetic flux density drawn through all points of any small, closed curve. These tubes were called **tubes of force** by Faraday. They are frequently called **tubes of induction.** We will call them **tubes of constant (magnetic) flux.** Their usefulness arises from the following features.

a. Since the lines of magnetic flux density have been shown by experiment to form endless loops, it follows that these tubes are endless tubes returning into themselves.

b. Since lines of flux density do not intersect, no tube intersects any other tube.

c. Since the flux density vector at every point in the wall is an element of the tube wall, the magnetic flux over any portion of the wall of a tube is zero.

d. Any two diaphragms in a tube together with the connecting tube walls (taken either the short way or the long way between diaphragms) form a closed surface. Since the flux over any closed surface is zero and since the flux over the walls is zero, the flux outward over one diaphragm must be exactly equal to the flux inward over the other. That is to say, the flux over any diaphragm—plain or bulged—in a given tube is equal to the flux over any other diaphragm in the tube.

e. Imagine that any surface in a magnetic field has been so divided into patches that the magnetic fluxes over all patches are equal. Now visualize tubes of constant flux to be passed through the lines of division between the patches. Any other

surface which the tubes cut across is divided by the tubes into patches of equal magnetic flux. These tubes are a convenient **geometrical construction** (fiction) for dividing the field into compartments and dividing surfaces into patches of equal flux.

f. If one is picturing or visualizing a magnetic field in this manner and if he has taken the tubes so that the magnetic flux over the patches cut out by the tubes is 1 weber each (or any submultiple, say 1 eighth-weber each), he may make the following statement as to the flux over any specified surface area:

The magnetic flux over any specified surface area is equal to the number of (weber) unit tubes of flux which the surface cuts through.

g. In diagrams representing a magnetic field, the walls of the tubes are not shown but each tube is represented by a line of magnetic flux density drawn along its center line.

PART IV—FURTHER CALCULATIONS OF MAGNETIC FLUX DENSITY

261. The Computation of Magnetic Intensities by Means of Solid Angles.—We proceed to use Ampere's formula to deduce a

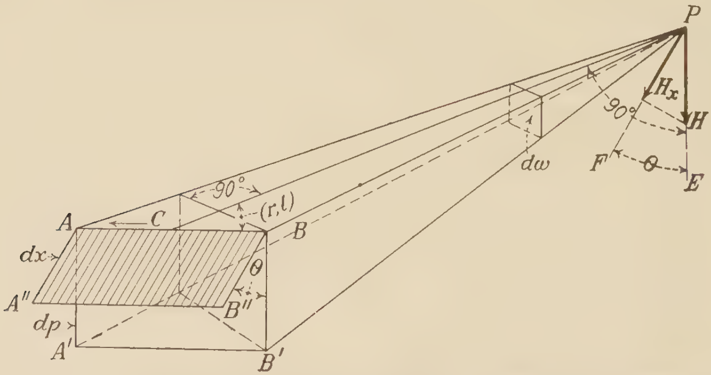


FIG. 216.—Calculation of magnetic intensities by means of solid angles.

method of computing the magnetic intensity at a specified point P in the field of a circuit in terms of the solid angle subtended at P by the area bounded by the circuit.

The Intensities Due to an Elementary Length of the Circuit.—Let AB (Fig. 216) represent any short elementary length dl of a

conductor carrying a current I in the direction shown by the arrow. Let (r,l) represent the angle which the line CP drawn from the conductor to the point P makes with the elementary length AB.

The current in AB gives rise to an elementary magnetic intensity vector dH at P which is perpendicular to the plane ABP and has the direction indicated by the arrow PE.

Its value is

$$dH = \frac{Idl \sin (r,l)}{4\pi r^2}. \tag{354}$$

Imagine the wire AB to be **translated**[10] a distance dp measured in a direction parallel to the intensity vector dH. In its motion,

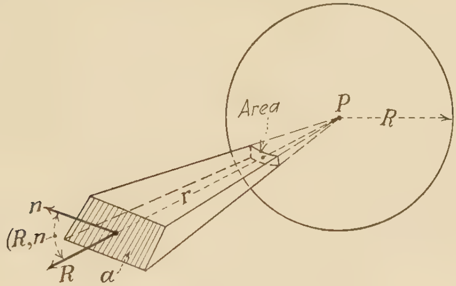


FIG. 217.—The measure of a solid angle.

the translated wire sweeps out an area $ABB'A'$ which may be seen to subtend at P a solid angle $d\omega$ having the value[11]

$$d\omega(\text{steradians}) = \frac{dp\ dl \sin (r,l)}{r^2}. \tag{380}$$

[10] A line or an object is said to be translated (or to undergo a motion of translation) when all points of the line or the object are moved by equal amounts in the same direction.

[11] By the solid angle subtended at a point P (Fig. 217) by a closed line or by a surface is meant the solid angle of the cone which is formed by drawing straight lines from the point P to all points of the closed line or to all points of the closed line which is the boundary or edge of the surface.

A numerical value is assigned to this angle in the following manner. Imagine a spherical surface of radius R to be described about P as a center. The cone cuts through the surface of this sphere, dividing the surface into two regions. By definition, the numerical value of the solid angle of the

A comparison of Eqs. (354) and (380) enables us to write

$$dH = \frac{Id\omega}{4\pi dp}. \tag{381}$$

Suppose that the position of the wire AB and of the point P are specified in some system of space coordinates, as the rectangular, and that we wish to find the X, Y, and Z components of the vector dH.

To find the X component, imagine that PF is a line drawn through P parallel to the X axis, and that PF makes the angle θ with PE. The component of dH along PF, or $(dH)_x$ is

$$dH_x = dH \cos\theta$$

$$dH_x = \frac{I}{4\pi}\frac{d\omega}{dp}\cos\theta. \tag{382}$$

Now imagine the wire AB to be translated from its original position by the distance dx measured in a direction which is parallel to the X axis, or to PF. Then the translation dx makes the angle θ with the translation dp. A study of the figure will show that if dx is equal to dp, the wire in moving the distance

---

cone is declared to be equal to the ratio of the area of the smaller region to the square of the radius of the sphere.

$$\omega \text{ (steradians)} = \frac{\text{area}}{R^2}. \tag{380a}$$

The name assigned to the unit angle is the *steradian* (solid radian). Thus when the area of the surface of the sphere enclosed by the elements of the cone is R^2 square centimeters, the value of the solid angle is said to be 1 steradian.

Since the area of a sphere is $4\pi R^2$, the sum of the solid angles which may be drawn about a center is 4π steradians.

If the loop subtending the solid angle is a plane loop and if the point P lies in the plane of the loop and inside the loop, the loop subtends an angle of 2π steradians. Under the convention of always using the smaller region it is evident that no one solid angle can have a value greater than 2π steradians.

If a small, plane surface of area da lies at a distance R from P which is very great in comparison with the greatest linear dimension of the surface, and if a normal to the surface (in the arrow direction across it) makes the angle (R,n) with the radius R from P to the center of the surface, then the algebraic value of the solid angle subtended by the surface is

$$d\omega = \frac{da}{R^2}\cos(R,n). \tag{380b}$$

dx sweeps out an area $ABB''A''$ which subtends at P a solid angle $d\omega'$ which is less than the solid angle it swept out when translated parallel to the dH vector in the ratio of $\cos\theta$ to 1. That is,

$$\frac{d\omega'}{dx} = \frac{d\omega}{dp}\cos\theta \tag{383}$$

Substituting the value of $d\omega/dp$ from Eq. (383) in Eq. (382), we obtain

$$dH_x = \frac{I}{4\pi}\frac{d\omega'}{dx}. \tag{384}$$

From this we may formulate the following statement:

261a. Rule for the Use of Solid Angles in Computing the Magnetic Intensities Due to the Current in a Short Element of a Wire (DEDUCTION).—*To find the X component of the magnetic intensity at a point P due to the current I in a short element AB of a wire. Imagine the wire to be translated the distance dx in the positive direction parallel to the X axis. The X component of the intensity is to be obtained by taking the product of the current I in the element times the solid angle which is subtended at P by the area swept over by the element AB, and dividing this product by 4π times the distance of translation dx.*

The above rule gives the absolute magnitude of the X component, but it does not enable us to tell whether it is a positive or a negative quantity. The positive direction at the H vector may be found by Ampere's rule (Sec. 226a). By the study of a number of cases it will be found that the following rule will correctly fix the sign which should be given to the solid angle appearing in Eq. (384).

261b. Rule for the Algebraic Sign of the Solid Angle.—*The wire in its "ultimate" position, after the translation, may be regarded as the magnetic equivalent of the wire in its "initial" position plus an "added" circuit carrying a current I around the boundary of the swept-over area. The current in the added circuit is imagined to flow in such a direction as to wipe out the current in the initial position and to leave the current in the ultimate position. (This is ignoring the effect of the oppositely directed currents along those two*

edges of the boundary which are traced out by the two end points of the wire.)

If the current in the "added circuit" when looked at from P is seen to flow around the boundary of the area in the counterclockwise or positive direction, then the positive sign should be assigned to the solid angle in Eq. (384), *otherwise the negative sign should be used.*

The Magnetic Intensities Due to a Complete Loop of Wire.—Since the X component of the magnetic intensity at a point P due to the current in a **complete loop of wire** is the algebraic sum of the X components due to all the elementary lengths of wire which go to make up the loop, the effect of the current in the complete loop may be computed by the following rules.

261c. Rule for the Use of Solid Angles in Computing the Magnetic Intensities Due to the Current in a Complete Loop of Wire (DEDUCTION).—*To find the X component of the magnetic intensity at a point P due to the current I in a loop of wire. Imagine the loop to be translated the distance dx in the positive direction parallel to the X axis. The X component of the intensity at P is to be obtained by taking the product of the current I times the solid angle which is subtended at P by the area swept over by the wire, and dividing this product by* 4π *times the distance of translation dx.*

261d. Rule for the Algebraic Sign of the Solid Angle.—*The loop in its "ultimate" position (after the translation) may be regarded as the magnetic equivalent of the loop in its "initial" position after it has been cut in one or more places and has had connected in series with it at each cut an "added" loop carrying a current around the boundary of the "swept-over" area. The current in the added circuits is imagined to flow in such a direction as to wipe out the current in the initial position and to leave the current in the ultimate position.*

If the current in any "added circuit" when looked at from P is seen to flow around the boundary of the area in the counterclockwise or positive direction, then the positive sign should be assigned to the solid angle subtended by that added circuit, otherwise the negative sign should be used in Eq. (384).

The loop in its initial position subtends at P a certain solid angle ω_0, which may be a positive or a negative quantity. In

its ultimate postion after the translation, it subtends a solid angle ω_u, which in general differs in value from ω_0. For example, when looked at from P, the loop in its initial position may appear as indicated by the heavy lines in Figs. 217*a* and 217*b*. In its translated position it may appear as shown by the light lines. The areas swept over during the translation are indicated by the cross-hatching, and the added circuits are shown around these areas. A study of the solid geometry involved in this and other cases will show that the algebraic sum of the solid angles subtended by

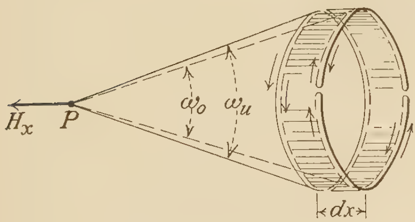


Fig. 217*a*.—The angle subtended by the swept-over area.

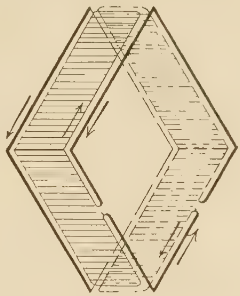


Fig 217*b*.—The swept-over area.

the added circuits bounding the cross-hatched area is equal to $\omega_u - \omega_0$, that is, the sum is equal to the increment $\Delta\omega$ in the solid angle subtended at P by the loop.

$$\Delta\omega = \omega_u - \omega_0. \tag{385}$$

Therefore, we may formulate the following alternative form of rule 261*c*.

261e. RULE FOR THE USE OF SOLID ANGLES IN COMPUTING THE MAGNETIC INTENSITIES DUE TO THE CURRENT IN A LOOP OF WIRE.—The X component of the magnetic intensity at a point P due to the current I in a loop of wire is to be obtained by taking the product of the current I in the coil times the algebraic value of the INCREMENT in the solid angle at P which is subtended by the loop when it is translated by the distance dx in the positive direction parallel to the X axis, and dividing this product by 4π times distance of translation dx.

The algebraic value of the solid angle subtended by the loop is to be taken as positive if the current in the loop, when looked at from P, is seen to flow around the loop in the counterclockwise or positive direction, otherwise it is to be taken as negative.

262. The Magnetic Potential Function.—Let $MNOM$ in Fig. 218 represent a closed loop of wire carrying a current of value I. Let APB represent any definitely marked or specified line in the magnetic field of the loop, and let P represent a point on this line.

Suppose that it is desired to find the value of that component of the H vector at P which is tangential to the line APB in the direction PE.

From the previous section it follows that this tangential component H_t is to be found by imagining the loop of wire to be translated an infinitesimal distance dl in a direction parallel to the tangent PE, finding the solid angle ω subtended at P by the loop before and after the translation, and then computing the tangential component from the equation

$$H_t \text{ (in direction of translation)} = \frac{I}{4\pi}\frac{d\omega}{dl} \text{ (circuit being translated).} \quad (386)$$

But the **increase** in the angle subtended at P by the loop when it is translated the distance dl is exactly equal but opposite in sign to the **increase** which occurs in the subtended angle if the loop is kept stationary and the point of view P is translated by the amount dl in the same direction to a new position P_1 on the curve. Hence we may write

$$H_t \text{ (in direction of translation)} = -\frac{I}{4\pi}\frac{d\omega}{dl} \text{ (the point of view being translated).} \quad (387)$$

In electrostatics, we have found the notion of (electric) potential function to be of great service in the calculation of electric intensities. The potential increase ΔE along a path from a point A to a point B is the negative of the line-integral of the electric intensity F from A to B

$$\Delta E = -\int_{l_A}^{l_B} F \cos(F, l)dl. \quad (27)$$

Let us now conceive of another **scalar point function** which is to bear to the **magnetic intensities** a mathematical relation analogous to the relation which the electric potential function bears to the electric intensities. This function, which will be

called the **magnetic potential function,** may be defined in the following manner:

262a. MAGNETIC POTENTIAL INCREASE (DEFINITION).—**The (magnetic) potential increase along a specified path from a point** A **to a point** B **is defined to be the negative of the line-integral of the magnetic intensity along the path from** A **to** B**.**

$$\Delta U \text{ (ampere-turns)} = -\int_{l_A}^{l_B} H \cos (H, l) dl. \tag{388}$$

A comparison of the defining equation for magnetic potential increase (Eq. (388)) with the defining equation for magnetomotive force will show that magnetic potential increase and magnetomotive force are two names for the line-integral of magnetic intensity. It is for this reason that we use for both quantities the same name for the unit, namely, the **ampere-turn.**

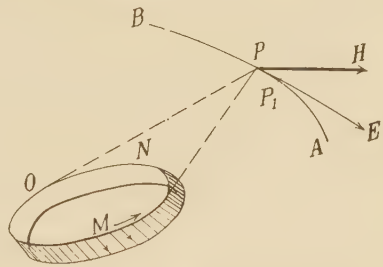


FIG. 218.—Magnetic potential increase.

Since magnetic potential increase has been defined to be the negative of the line-integral of the magnetic intensity, then, inversely, the component of the magnetic intensity in a specified direction is the negative of the rate of increase of the magnetic potential in the specified direction, or it is the negative of the (magnetic) potential gradient in the specified direction. In other words, the removal of the integration sign from Eq. (388) gives

$$dU = -H \cos (H, l) dl,$$

$$H \cos (H, l) = -\frac{dU}{dl},$$

or

$$H_x = -\frac{dU}{dx} = -\text{grad}_x\, U. \tag{389}$$

By substituting the value of H_t from Eq. (387) for $H \cos (H,l)$ in Eq. (388), an expression in terms of solid angles is obtained

for the increase in the magnetic potential from A to B along the path of Fig. 218. Thus

$$\Delta U = -\int H \cos(H, l)\, dl = -\int H_t\, dl = \frac{I}{4\pi}\int_{\omega_A}^{\omega_B} d\omega,$$

$$\Delta U \text{ (ampere-turns)} = \frac{I}{4\pi}(\omega_B - \omega_A), \tag{390a}$$

$$\Delta U \text{ (ampere-turns)} = \frac{I}{4\pi}\Delta\omega. \tag{390b}$$

That is to say,

262b. MAGNETIC POTENTIAL INCREASE EXPRESSED IN TERMS OF SOLID ANGLES (DEDUCTION).—**The increase in the magnetic potential from A to B along a line in the magnetic field of a loop of wire carrying a current I is equal to $I/4\pi$ times the increase in the solid angle subtended by the loop when the point of view is shifted from A to B.**

We now have a method of determining, from the configuration of the loop and of the points in the field, the increase in the potential from any point to any other point. It remains to select and to specify a datum point, or rather a datum surface, to be arbitrarily called the surface at zero magnetic potential. This we do by adopting the following convention.

262c. Convention Specifying the Surface at Zero Potential.—*The algebraic value of the magnetic potential at a point P due to the current I in a loop of wire will be taken to be the product of $I/4\pi$ times the solid angle subtended at P by the loop.*

$$U \text{ (ampere-turns)} = \frac{I}{4\pi}\omega. \tag{391}$$

This convention is in reality an indirect way of stating that all points of view from which the solid angle subtended by the loop is zero, are to be called points at zero potential. This specification of points of zero potential coupled with the definition of potential increase as given in Sec. 262*b* completes the process of defining magnetic potential.

263. Deduction of the Law of Circuitation by Means of the Magnetic Potential Function.—The values of the magnetic potentials at points along a line $ABCDEFA$ in the field of a

plane loop of wire have been indicated in Fig. 219. It will be noted that if we start at the point A at which the potential is zero and move along the line, the potential increases continuously until a point is reached at which the loop subtends a solid angle of 2π steradians. At this point the potential has the value $+I/2$. If we continue an infinitesimal distance further, the loop subtends a solid angle of -2π steradians, and consequently the potential jumps abruptly to the value of $-I/2$. This discontinuity comes from the convention by which we assign the algebraic sign to the solid angle subtended by the loop. There is no discontinuity in the values of the magnetic intensity at these points. The H vector has substantially the same value and direction at the three points on the line CDF from which the solid angle subtended by the coil is $(2\pi - \theta)$, 2π, and $(-2\pi + \theta)$, in which θ represents an infinitesimal quantity.

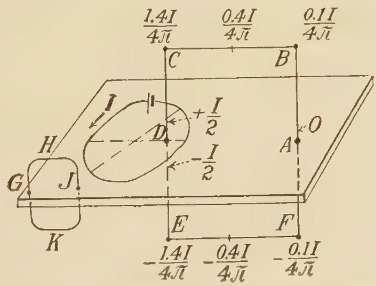


FIG. 219.—Magnetic potentials in the field of a plane coil.

From the above it is evident that the line-integral of the magnetic intensity around the closed line $ABCDEFA$ is equal to the current flowing in the loop. But $ABCDEFA$ represents any line which is linked with the closed loop of current, and consequently it represents a closed line which bounds a surface across which the current I flows. On the other hand, it will be seen that the line-integral of the magnetic intensity around any closed line which does not link with the current loop (such as the line $GHJKG$ of Fig. 219) is zero. If the closed line should thread through N closed loops carrying the currents I_1, I_2, I_3, etc., in the arrow direction across the surface bounded by the line, it will be seen that the line-integral of the magnetic intensity around the line will be equal to $I_1 + I_2 + \ . \ . \ . + I_n$. Or if the closed line should thread through a coil of N turns carrying the current I, as in Fig. 220a, it will be seen that the line-integral of H around the line will be equal to NI, since the N turn coil of Fig. 220a is the magnetic equivalent of the $N + 1$ closed loops of Fig. 220b.

From this discussion the conclusion may be drawn that

The CIRCUITATION of the magnetic intensity, or the value of the line-integral of the magnetic intensity around any closed line in a magnetic field is equal to the net current across any surface which is bounded by the closed line.

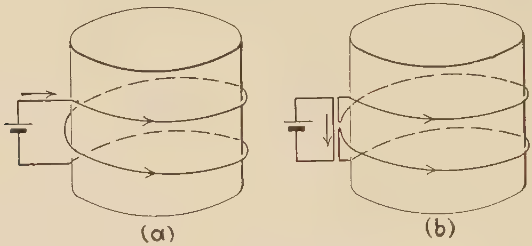


FIG. 220.—Resolution of an N turn coil into $(N + 1)$ closed loops.

The law of circuitation has thus been deduced from Ampere's formula for the magnetic flux density due to an elementary length of circuit.

264. The Magnetic Intensities at Great Distances from a Small Plane Coil or a Small Solenoid.—A **solenoid** is a coil of wire formed by winding an insulated wire upon a cylindrical winding form, with the turns closely spaced at equal intervals along the length of the cylinder. The solenoid may consist of a single layer of wire or of two or more layers wound one upon the other. The right cross-section of the cylinder may be of any shape, but it is generally either circular or rectangular.

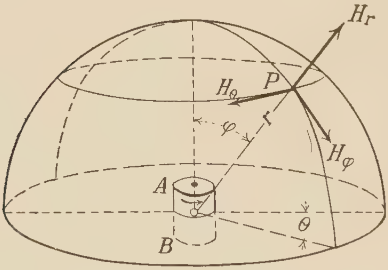


FIG. 221.—Magnetic intensities at a great distance from a solenoid.

From Fig. 220 it will be seen that if the solenoid contains N closely wound turns of wire and carries a current of I amperes, it is the magnetic equivalent of N loops each carrying a current I, each approximately plane and having its plane perpendicular to the axis of the solenoid, or it is approximately the magnetic equivalent of a **current sheet** of NI amperes circulating around the cylinder.

Let P in Fig. 221 represent any point in the field of a small solenoid AB, at a distance from the solenoid which is very great in comparison with the greatest linear dimensions of the solenoid. Let the axis of the solenoid be taken as the polar axis of a system of spherical coordinates and let the coordinates of the point P be represented by r, φ, and θ.

Let us find the three mutually perpendicular components of the magnetic intensity at P, namely H_r, H_φ, and H_θ.

Let the coil consist of N closely wound turns bounding a cross-section of any shape. Let I represent the current flowing in the coil and let a represent the area of the cross-section bounded by the mean turn.

From Eq. (391), the value of the magnetic potential at P is

$$U \text{ (ampere-turns)} = \frac{NI\,a}{4\pi r^2} \cos \varphi. \tag{392}$$

From Eq. (389), the three components of the magnetic intensity at P are

$$H_r = -\frac{dU}{dr} = +\frac{2NI\,a}{4\pi r^3} \cos \varphi, \tag{393}$$

$$H_\varphi = -\frac{dU}{d(r\varphi)} = -\frac{1}{r}\frac{dU}{d\varphi} = \frac{NI\,a}{4\pi r^3} \sin \varphi, \tag{394}$$
$$H_\theta = 0.$$

It will be noted that the length of the solenoid does not enter into this expression for the magnetic intensity at distant points. The only dimensions of the solenoid in the formula are the cross-sectional area and the number of turns. The solenoid may be made long or short or it may be made a single turn carrying a current of NI amperes without affecting the field at **distant** points.

265. The Conception of Magnetic Poles. Strength of a Pole.—A solenoid and a permanent bar magnet of the same dimensions have been shown by experiment to set up similar magnetic fields at distant points. Now in the **magnetic fluid theory,** or the **magnetic pole theory,** by which the properties of permanent magnets were at first described, the magnetic intensities in the field of a bar magnet were computed by postulating two **magnetic fluids or two magnetic poles** on, or close to, the end faces of the bar. The so-called north (seeking) pole at one end was postulated to give rise to a magnetic intensity at any distant point which was directed **radially away from** the north pole and was inversely proportional to the square of the distance from the pole to the point. The south pole gave rise to a similar set of intensities which were directed **radially toward** the south pole.

That is, the postulated poles were postulated to give rise to centrally directed magnetic intensities whose magnitudes were expressed by the formula

$$H = \frac{m}{4\pi\mu r^2}, \tag{395}$$

in which m is a constant for any particular magnet and is called the **strength of the pole.**

r represents the distance from the pole to the point.

Let us determine whether these postulates would lead to the distribution of magnetic intensities which the solenoid has been shown to cause. If so,

let us determine the relation of the **strength** of the postulated poles to the dimensions of the solenoid.

In Fig. 222, the poles are represented as N and S.

Let l represent the distance between the poles.

r represent the distance from the center point of the solenoid to the distant point P.

Then the components of the magnetic intensity at P, as computed from Eq. (395), are

$$H_r = \frac{m}{4\pi\mu\left(r - \frac{l \cos \varphi}{2}\right)^2} - \frac{m}{4\pi\mu\left(r + \frac{l \cos \varphi}{2}\right)^2}$$

$$H_r = \frac{m}{4\pi\mu}\,\frac{2lr \cos \varphi}{r^4} = \frac{2ml}{4\pi\mu r^3} \cos \varphi. \tag{396}$$

$$H_\varphi = 2\left(\frac{m}{4\pi\mu r^2}\,\frac{l \sin \varphi}{2r} = \frac{ml}{4\pi\mu r^3} \sin \varphi.\right) \tag{397}$$

$$H_\theta = 0.$$

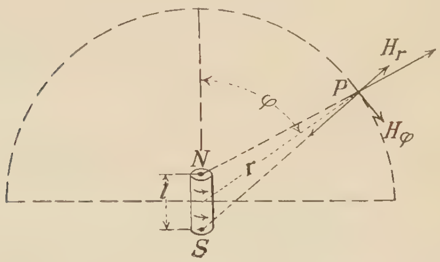


FIG. 222.—Magnetic intensities calculated from the magnetic pole concept.

A comparison of those equations with Eqs. (393) and (394) shows that the two distributions will be identical in value if ml/μ is put equal to NIa. Hence

$$m = \frac{\mu NIa}{l}. \tag{398}$$

266. Magnetic Potentials and Intensities in the Field of a Long Solenoid.—The use of the potential function and of the method of solid angles is illustrated by the following application to the calculation of magnetic intensities in the field of a long solenoid.

A longitudinal cross-section of a single layer solenoid of circular cross-section is illustrated in Fig. 223.

Let h represent the half length of the solenoid.

r represent the radius to the center of the turns.

n represent the number of turns per centimeter of length.

I represent the current in the winding.

To find the component of the magnetic intensity parallel to the axis of the solenoid at any point P in the field, let us imagine that the solenoid is displaced a distance $dx = 1/n$ in a direction parallel to its axis. During this displacement each filament of current sweeps over the cylindrical surface between it and the corresponding filament in the adjacent turn. Consequently, during this displacement, the cylindrical surface of the solenoid is swept over by the current I.

Therefore, if the point P is located anywhere within the solenoid, as at P_1, the solid angle subtended at P_1 by the area swept over will be equal to 4π steradians less the sum of the solid angles subtended at P_1 by the two end faces of the solenoid. That is, the solid angle subtended at P_1 by the area swept over by the wire during a displacement of $1/n$ centimeters is,

$$d\omega = 4\pi - (\omega_1 + \omega_2).$$

Substituting this value in Eq. (384)

$$H \text{ (axial)} = \frac{I}{4\pi}\frac{d\omega}{dx} = \frac{I}{4\pi}\left[\frac{4\pi - (\omega_1 + \omega_2)}{\frac{1}{n}}\right],$$

$$H \text{ (axial)} = nI\left[1 - \frac{\omega_1 + \omega_2}{4\pi}\right]. \tag{399}$$

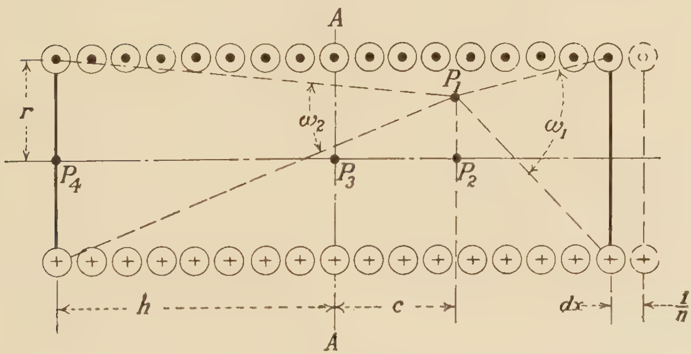


FIG. 223.—Points in the field of a long solenoid.

If the point P is at P_2 on the axis of the solenoid at a distance c from the center, the solid angles subtended by the circular end faces are

$$\omega_1 = \frac{2\pi\sqrt{(h-c)^2 + r^2}\left[\sqrt{(h-c)^2 + r^2} - (h-c)\right]}{(h-c)^2 + r^2},$$

$$\left.\begin{aligned} \omega_1 &= 2\pi\left[1 - \frac{h-c}{\sqrt{(h-c)^2 + r^2}}\right], \\ \omega_2 &= 2\pi\left[1 - \frac{h+c}{\sqrt{(h+c)^2 + r^2}}\right]. \end{aligned}\right\} \tag{400}$$

Substituting these values in Eq. (399), the axial component of the magnetic intensity at a point on the axis of the solenoid at a distance c from the center is

$$H\text{ (axial)} = \frac{nI}{2}\left[\frac{h-c}{\sqrt{(h-c)^2+r^2}} + \frac{h+c}{\sqrt{(h+c)^2+r^2}}\right]. \tag{401}$$

For a point P_3 at the center of the solenoid, the axial component of the intensity is

$$H\text{ (axial)} = nI\frac{h}{\sqrt{h^2+r^2}}. \tag{402}$$

$$= nI \quad \text{(to within 0.5 per cent if } h > 10r). \tag{402a}$$

For a point P_4 at the center of either end face, the axial component of the intensity is

$$H\text{ (axial)} = \frac{nIh}{\sqrt{4h^2+r^2}}. \tag{403}$$

$$H\text{ (axial)} = \frac{nI}{2} \quad \text{(to within 0.12 per cent if } h > 10r). \tag{403a}$$

That is to say, in a very long solenoid the magnetic intensity at either end face is just one-half as great as at the center of the solenoid. If $h - c$ is great in comparison with r, that is, if the point P_2 is at a distance from either end face which is greater than $10r$, the equation may be written without appreciable error in the following approximate form[12]

$$H\text{ (axial)} = \frac{nI}{2}\left[\frac{h-c}{(h-c)\left[1+\dfrac{r^2}{2(h-c)^2}\right]} + \frac{h+c}{(h+c)\left[1+\dfrac{r^2}{2(h+c)^2}\right]}\right]$$

(approximately)

$$H\text{ (axial)} = nI - \left[\frac{nI\pi r^2}{4\pi(h-c)^2} + \frac{nI\pi r^2}{4\pi(h+c)^2}\right] \text{ (approximately).} \tag{404}$$

If the point P lies **outside** the solenoid, as at P_5 in Fig. 224, the current in the added circuit around the boundary of the surface swept out by that

[12] We have shown in the discussion of the magnetic pole concept in Sec. 265, that under the pole concept, the strength of the poles of a solenoid would be taken as

$$m = \frac{\mu NIa}{l} = \mu nIa. \tag{398}$$

By substituting m/μ for $nI\pi r^2$ in Eq. (404) the following expression is obtained:

$$H\text{ (axial)} = nI - \left[\frac{m}{4\pi\mu(h-c)^2} + \frac{m}{4\pi\mu(h+c)^2}\right] \text{ (approximately).} \tag{404a}$$

That is to say, at any point on the axis of the solenoid between the two end faces of a long solenoid, but not too close to the end faces, the magnetic intensity is less than the constant value nI by an amount which may be calculated under the magnetic pole concepts by taking the strength of the poles distributed over each end face to be as given in Eq. (398) and applying the inverse square law expressed in Eq. (395).

portion of the turns which is visible from P_5 traverses the added circuit in a clockwise or negative direction, while the current in the added circuit around the surface swept out by the portion of the turns which is invisible from P_5 traverses the circuit in a counterclockwise direction. Therefore the solid angles subtended at P by the two swept-over surfaces are of opposite sign. Consequently, the net solid angle has only the small negative value which results from the fact that the negative solid angle is slightly larger than the positive angle. In the following paragraphs the intensity at P_5 will be calculated by means of the potential function.

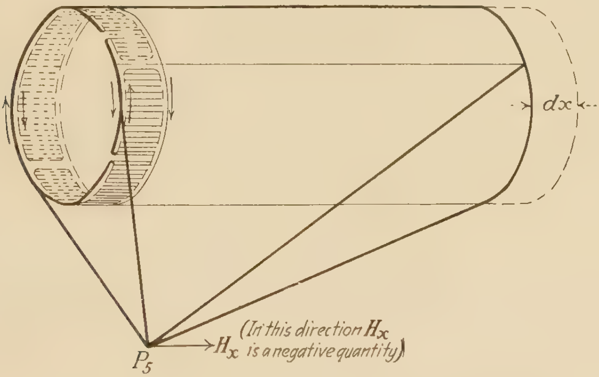


FIG. 224.—Only the area swept over by the end turn has been indicated.

266a. Calculation by Means of the Potential Function.—Let us calculate the value of the magnetic potential function at the point P_2 in Fig. 225 which lies on the axis of the circular solenoid at a distance c from the center. We assume that the turns are so closely spaced that the current in the winding is the equivalent of a current sheet of nI amperes per centimeter of length of the solenoid.

The current circulating around the elementary strip shown in the figure (namely $nIdx$ amperes) subtends at P a solid angle ω whose value is

$$\omega = -\frac{2\pi\sqrt{x^2 + r^2}[\sqrt{x^2 + r^2} - x]}{x^2 + r^2} = -2\pi\left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right).$$

By substituting in Eq. (391), the magnetic potential at P due to this elementary current is found to be

$$dU = -\frac{nI}{2}\left[1 - \frac{x}{\sqrt{x^2 + r^2}}\right]dx.$$

The potential at P due to the entire solenoid is

$$U = \int_{(h-c)}^{(h+c)} -\frac{nI}{2}\left[1 - \frac{x}{\sqrt{x^2 + r^2}}\right]dx = -\frac{nI}{2}\left[x - \sqrt{x^2 + r^2}\right]_{(h-c)}^{(h+c)},$$

$$U = -\frac{nI}{2}\left[2c - \sqrt{(h + c)^2 + r^2} + \sqrt{(h - c)^2 + r^2}\right]. \qquad (405)$$

For a point P which lies at the center of the end face, the general expression for the potential becomes

$$U_e = \frac{nI}{2}\left[2h - \sqrt{(2h)^2 + r^2} + r\right].$$

If h is greater than $10r$, this expression simplifies to

$$U_e = \frac{nIr}{2} \text{ (approximately).} \tag{406}$$

The rate of increase of the potential as the point P is moved along the axis away from the center is

$$\frac{dU}{dc} = -\frac{nI}{2}\left[2 - \frac{h+c}{\sqrt{(h+c)^2 + r^2}} - \frac{h-c}{\sqrt{(h-c)^2 + r^2}}\right]. \tag{407}$$

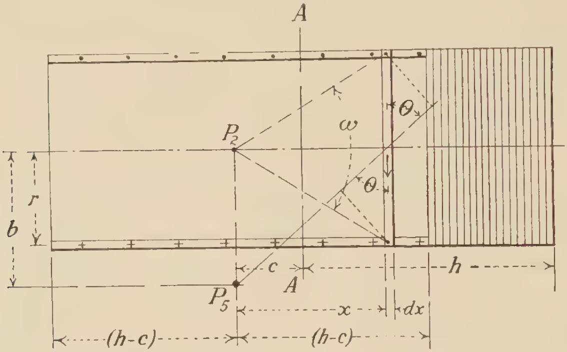


FIG. 225.—Points in the field of a solenoid.

Now the component of the magnetic intensity tangent to a given line at a given point is the negative of the rate of increase of potential along the line at the point, unless the potential function is discontinuous at the particular point in question. If the potential function is discontinuous at the point, we must subtract from the gross rate of increase of the potential, that portion of the rate of increase which is to be attributed to the discontinuities in the function. Now there is a discontinuity equal to I in the potential function at any point in which the line passes through a plane area bounded by a current filament of strength I. The point P in shifting along the axis by a distance of 1 centimeter passes through the planes of filaments in which the total current is nI. As a consequence, the rate of increase of the potential along the axis due to the discontinuities in the potential function is nI ampere-turns per centimeter. On substracting this from the members of Eq. (407), the following expression is obtained for the magnetic intensity at any point on that portion of the axis of the solenoid which lies between the two end faces.

$$H = -\frac{dU}{dc} - nI = -\frac{nI}{2}\left[\frac{h+c}{\sqrt{(h+c)^2 + r^2}} + \frac{h-c}{\sqrt{(h-c)^2 + r^2}}\right]. \tag{401}$$

This equation is identical with Eq. (401) obtained before.

Let us now calculate the potential at a point P_5 in Fig. 225 which lies outside the solenoid, but not at a great distance from the solenoid or from the central plane AA.

Let b represent the distance of the point P_5 from the axis of the solenoid.
c " the distance of the point from the central plane AA.

All the current in the portion of the solenoid from the plane $P_5 P_2$ to the left face gives rise to a certain positive potential at P_5. The length of this portion is $h - c$. This positive potential is exactly neutralized by the negative potential at P_5 caused by the current in a portion of the solenoid of length $h - c$ immediately to the right of the plane $P_5 P_2$. Consequently, to calculate the value of the potential at P_5 all current may be ignored save that in the right end portion of the solenoid between planes at distances $h - c$ and $h + c$ to the right of the plane $P_5 P_2$.

The angle subtended at P_5 by the circular element of the solenoid at distance x to the right of the plane $P_5 P_2$ is

$$\omega = -\frac{\pi r^2 \sin \theta}{b^2 + x^2} \text{ (approximately).}$$

The potential at P_5 caused by the current in this element is

$$dU = -\frac{nI}{4\pi}\frac{(\pi r^2) \sin \theta}{b^2 + x^2}dx = -\frac{nI}{4\pi}\frac{\pi r^2}{(b^2 + x^2)}\frac{x}{(b^2 + x^2)^{1/2}}dx.$$

The potential due to all current between the planes at distance $h - c$ and $h + c$ is

$$U = \int_{(h-c)}^{(h+c)} -\frac{nI(\pi r^2)}{4\pi}\frac{xdx}{(b^2 + x^2)^{3/2}} = \frac{nI\pi r^2}{4\pi}\left[\frac{1}{\sqrt{b^2 + x^2}}\right]_{(h-c)}^{(h+c)}$$

$$U = -\frac{nI\pi r^2}{4\pi}\left[\frac{1}{\sqrt{(h - c)^2 + b^2}} - \frac{1}{\sqrt{(h + c)^2 + b^2}}\right] \text{(approximately).}$$

The component of the magnetic intensity parallel to the axis and toward the left is,

$$H = -\frac{dU}{dc} = \frac{nI\pi r^2}{4\pi}\left[\frac{h - c}{[(h - c)^2 + b^2]^{3/2}} + \frac{h + c}{[(h + c)^2 + b^2]^{3/2}}\right] \text{(approximately).} \quad (408)$$

If b and c are small in comparison with h, that is, for points outside but close to the solenoid near its midsection, this equation reduces to

$$H \text{ (axial)} = \left[\frac{nI\pi r^2}{4\pi(h - c)^2} + \frac{nI\pi r^2}{4\pi(h + c)^2}\right] \text{(approximately).} \quad (409)$$

This result should be compared with the equations which give the axial intensities at points inside the solenoid, namely, Eqs. (404) and (404a). As the length of the solenoid increases indefinitely the axial component of the intensity approaches nI for points inside the solenoid and approaches zero for points outside the solenoid, provided these points are at distances from the end greater than $10r$.

267. The Vector Potential Function.—We have found the scalar point function—the magnetic potential—to be a useful function from which to compute the values of magnetic intensities. We now proceed to define a **vector** point function and to demon-

strate that it may be used in the computation of the magnetic intensities and flux densities. The function is called the **vector potential function.** It is defined as follows:

267a. VECTOR POTENTIAL AT A POINT (DEFINITION).—**The vector potential A at a point P due to the current I in an elementary length l of a circuit, is defined to be a vector drawn from the point P parallel to the length in the direction of flow of the current in l, and having a magnitude equal to**

$$\mathsf{A} = \frac{\mu I l}{4\pi r} \tag{410a}$$

in which r is the distance from P to the length l.

The vector potential A at a point P due to the current flowing in the entire circuit, or due to the currents in any number of circuits, is the vector through the point which results from the following construction:

1. *Dividing the circuit or circuits into lengths each so short that it may be regarded as a straight conductor.*
2. *Determining the vector potential* A *at* P *due to each of these differential lengths.*
3. *Taking the resultant of all the infinitesimal vectors thus determined.*

$$\mathsf{A} = \Sigma\frac{\mu I dl}{4\pi r}. \tag{410}$$

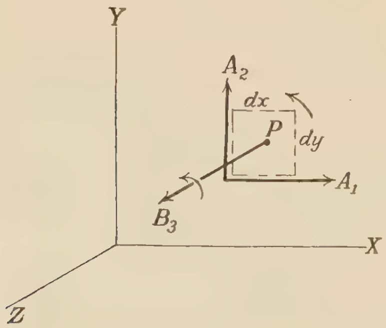


FIG. 226.—The curl of a vector.

We proceed to demonstrate the following proposition.

267b. (DEDUCTION).—**The magnetic flux density, B, at any point P is equal to the curl of the vector potential A at the point.**

$$\mathsf{B} = \text{curl } \mathsf{A}. \tag{411}$$

Let us first write the expressions for the X, Y, and Z components of the curl of any vector in terms of the X, Y, and Z components of the vector itself.

By definition (Sec. 250) the value at a point P of the Z component of the curl of the distributed vector A is equal to the line-integral of the vector A taken around the boundary of an infinitesimal patch lying in the XY plane at P, divided by the area of the patch.

$$\text{curl}_3\, \mathsf{A} = \frac{\int A \cos(A,l)dl}{a}. \tag{370a}$$

An examination of Fig. 226 will show that the line-integral of a distributed vector **A** around the small rectangular area shown around P is given by the following expression

$$\int A \cos (A,l)\, dl = A_1 dx + \left[A_2 + \frac{\partial A_2}{\partial x}\, dx \right] dy - \left[A_1 + \frac{\partial A_1}{\partial y}\, dy \right] dx - A_2 dy$$

$$\int A \cos (A,l) dl \doteq \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right] dx\, dy.$$

Whence

$$\left.\begin{aligned} \mathrm{curl}_3 A &= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \\ \mathrm{curl}_1 A &= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \mathrm{curl}_2 A &= \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \end{aligned}\right\} \cdot \tag{412}$$

Let CD in Fig. 227 represent a short segment (of length l) of the conductor contributing to the magnetic intensity at point P. Let us imagine the

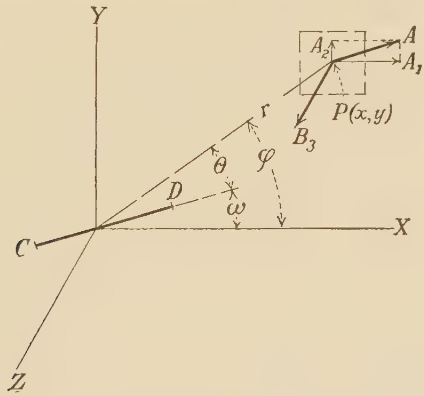


FIG. 227.—The vector potential.

midpoint of CD to be the origin of a system of rectangular axis so oriented that CD lies in the XY plane, making the angle ω with the x axis.

Then the value of the vector potential at any point P (x, y) in the XY plane is

$$A = \frac{\mu I l}{4\pi\sqrt{x^2 + y^2}}.$$

Its X, Y, and Z components are

$$A_1 = \frac{\mu I l}{4\pi\sqrt{x^2 + y^2}} \cos \omega,$$

$$A_2 = \frac{\mu I l}{4\pi\sqrt{x^2 + y^2}} \sin \omega,$$

$$A_3 = 0.$$

Consequently

$$\frac{\partial A_3}{\partial y} = \frac{\partial A_2}{\partial z} = \frac{\partial A_1}{\partial z} = \frac{\partial A_3}{\partial x} = 0.$$

Therefore, $\text{curl}_1 A = \text{curl}_2 A = 0$

$$\text{curl}_3 A = \left[\frac{\partial A_2}{dx} - \frac{\partial A_1}{dy}\right] = \frac{\mu Il}{4\pi}\left[\frac{-x}{(x^2+y^2)^{3/2}}\sin\omega + \frac{y}{(x^2+y^2)^{3/2}}\cos\omega\right]$$

$$= \frac{\mu Il}{4\pi(x^2+y^2)}\left[\frac{y}{\sqrt{x^2+y^2}}\cos\omega - \frac{x}{\sqrt{x^2+y^2}}\sin\omega\right].$$

Since $$\frac{y}{\sqrt{x^2+y^2}} = \sin\varphi \text{ and } \frac{x}{\sqrt{x^2+y^2}} = \cos\varphi,$$

this may be written

$$\text{curl}_3 A = \frac{\mu Il}{4\pi(x^2+y^2)}\sin(\varphi - \omega)$$

$$= \frac{\mu Il}{4\pi r^2}\sin\theta.$$

Whence

$$B_3 = \text{curl}_3 A = \frac{\mu Il}{4\pi r^2}\sin\theta.$$

This value is in agreement both in magnitude and direction with the value which may be written by applying Ampere's formula directly to Fig. 227.

267c. Exercises.

1. A wire carrying a current of 240 amperes is bent into a rectangle 40 by 70 centimeters. Calculate the value of the flux density at the center of this rectangle.

2. Calculate by Ampere's formula the flux density at the center of a circle of wire around which a current of I amperes is flowing. Let the radius of the circle be r.

3. Two parallel wires, A and B, are 8 centimeters apart. They carry currents of 20 and 50 amperes respectively in the same direction. Calculate the force exerted on A per centimeter of length; on B.

4. Two parallel bus bars are 6 inches apart. An accidental short circuit causes a current of 100,000 amperes to flow momentarily in each conductor—opposite directions. Calculate force exerted on one of the bus bars in pounds per foot.

5. A long cylindrical shell carries current parallel to the axis. Find the flux density at a point inside the hollow shell.

6. A long solid cylinder r centimeters in radius carries a current of I amperes parallel to the axis. Calculate the flux density at a point in the conductor at a distance x from the axis.

7. Assume that two copper wires each 2 centimeters in diameter are strung parallel to each other, as in a telephone line, with the wires 30 centimeters apart center to center, and that each wire is carrying a current of I amperes (in opposite directions). Plot a curve showing the value of the

magnetic flux density at all points along a straight line connecting the centers of the two wires.

8. Assume that a torque-finding coil consisting of 30 turns, each carrying 5 amperes, and each turn bounding an area of 12 square centimeters, is placed at the center of the rectangle of exercise 1. What torque must be exerted on this small coil to hold its plane perpendicular to the plane of the large coil?

9. A ring-shaped wooden coil is uniformly covered with a winding of 400 turns each carrying 2 amperes. The core has a square cross-section 1 centimeter on a side. The inside and outside radii of the core are 9 and 10 centimeters, respectively. (Draw sketches similar to Fig. 211, and show a circular path C corresponding to the circle of radius x, Fig. 211.)

What is the magnetomotive force around the circle C?

What is the magnetic intensity at a point P on the circle C?

What is the flux density at the point P?

What is the flux across any cross-section of the core?

10. A ring-shaped core with a rectangular cross-section is uniformly wound with 1500 turns of wire carrying a current of 5 amperes. The inner and outer radii of the core are 14 and 20 centimeters. The axial thickness is 10 centimeters.

a. Determine the maximum and the minimum values of the magnetic intensity in the cross-section and on the mean circumference.

b. Determine by integration the average value of the magnetic intensity over the cross-section, and compare it with that for the mean circumference. Under what conditions do these two values become practically equal?

11. Let the surface S lie in the plane containing the axes of conductors A and B, exercise 7, and let it be the strip 1 centimeter long bounded by the inside surfaces of the conductors. Calculate the flux across S, due to the current in the two conductors.

12. A straight air-core solenoid of circular cross-sectional area is wound with 400 turns in a single layer of wire. The solenoid has an axial length of 50 centimeters and a diameter of 2 centimeters center to center of the winding. A current of 3 amperes flows in the winding.

Plot a curve showing the values of the magnetic intensities at all points along the axis of the solenoid. From this curve, determine the value of the magnetomotive force along the axis between the end surfaces of the solenoid. Determine the percentage by which this magnetomotive force differs from NI.

13. Determine the magnetic pole strength of the air-core solenoid of exercise 12 when it is carrying a current of 3 amperes.

14. In the known magnetic field around a long straight conductor-carrying current, choose a convenient rectangular set of axes, choose a point x, y, z, and for this point obtain expression for the x, y, and z components of B. Substitute these expressions into Eq. (378), Sec. 258 as a means of checking the equation.

CHAPTER XII

MOTIONAL ELECTROMOTIVE FORCES AND ENERGY TRANSFORMATIONS IN MOVING CONDUCTORS

268. Purpose of This Chapter.—In the first chapter dealing with magnetic theory (Chap. X), the four apparently distinct experimental effects observed in the magnetic field were named and briefly described. A choice was then made of one of these effects—namely, of the force exerted upon a short, straight element of a conductor carrying a current—as the effect to be used in defining magnetic quantities. Accordingly, this effect was made the first subject for further experimental study. This study led to the introduction of the vector quantity, **magnetic flux density,** a quantity which was defined in terms of the force exerted on a centimeter length of wire carrying unit current.

The experimental studies of Chap. XI have resulted in the formulation of the laws by means of which it is possible to compute the value of the magnetic flux density at any point in the field of circuits of known configuration carrying known currents. We are, therefore, in a position to calculate the forces which one circuit will exert upon another.

The present chapter will deal with a second of the four effects observed in magnetic fields—the electromotive force induced in conductors which are in motion relative to a magnetic field. The chapter following this will deal with the third effect—the electromotive force induced in stationary coils. A still later chapter will deal with the fourth effect—the forces on magnets and magnetic materials.

The general laws relating to the electromotive force generated in conductors moving in magnetic fields may be obtained in two ways:

a. They may be derived from experimental measurements of the electromotive force generated in moving conductors.

b. They may be deduced, provided one postulates that the formula deduced in Sec. 233 for the force on a charge moving

in a magnetic field, namely, $f = QVB \sin (V,B)$, is of universal application.

The first method is the historical method. It is followed in Part I of this chapter. Part II deals with the energy transformations which occur where conductors carrying current move in a magnetic field.

PART I—THE EXPERIMENTAL BASIS FOR THE LAWS OF MOTIONAL ELECTROMOTIVE FORCE

269. The Discovery of Electromagnetically Induced Electromotive Forces.—The electromagnetic generators described in Chap. VII and the experiments about to be described grew out of the discovery by Faraday in 1831[1] of what we now call **electromagnetically induced electromotive forces.** A charged body was known to **induce** charges on other bodies. A magnet was known to **induce** the magnetic state in soft iron. Reasoning by analogy, Faraday conceived the notion that a current in one circuit might set up an **induced** current in an adjacent circuit, which would continue as long as the inducing current continued. At intervals extending over a period of 6 years, he carried on experiments in the attempt to verify this notion. The early experiments took the form of attempts to detect induced currents in **stationary circuits** mounted adjacent to another circuit carrying a large **steady** current. Under these conditions—that is, with stationary circuits and with a steady current—Faraday was unable to obtain any evidence of an induced current. Finally, he made the discovery that currents may indeed be induced in other circuits but only under the following conditions:

a. At the moments of starting and of interrupting the current in one circuit, momentary currents are induced in adjacent circuits. Or, in general, while the current in one circuit **is varying** *in value, an electromotive force is induced in any adjacent circuit.*

b. If the current in the inducing circuit A is constant in value, an electromotive force is, in general, induced in any adjacent circuit B which is in motion **relative** *to the inducing circuit. The relative motion of B with respect to A may be the result of the motion of either or of both circuits relative to the earth.*

[1] Faraday: *Experimental Researches*, Vol. I, Pars. 1–139.

The electromotive forces induced in conductors whether by reason of the relative motion of the conductor and the magnetic field, or by reason of a time variation in the flux densities of the field are termed **electromagnetically induced electromotive forces.** The term "electromagnetic" signifies simply a magnetic field which is attributed to the differential motion of electricity. The statements ot the conditions under which an electromotive force will be induced in a circuit by the above variations, combined with the further statement of the precise or quantitative relation which exists between the magnitude of the induced electromotive force and the magnitude of the change in the magnetic field or in the configuration of the circuit, are called **the laws of electromagnetic induction.** In Chap. VII the conditions under which electromotive forces are induced in coils have been considered in a qualitative way. We now propose to consider the quantitative experiments from which the laws relating to motional electromotive forces may be deduced.

270. Methods of Measuring Motional Electromotive Forces.—By a **motional electromotive force** is meant an electromotive force which results from the motion of a **body** relative to the magnetic field in which the body lies.

There are three ways of moving a conductor or a coil in a magnetic field and of measuring the induced electromotive force for the purpose of discovering the underlying relations.

a. The coil or the conductor may be jerked or moved suddenly from one position of rest to another position of rest, and the integrated value of the momentary electromotive force may be measured by a ballistic instrument.

b. The conductor or coil may be moved in a known manner and the electromotive force may be measured by an instrument (an oscillograph) designed to give a continuous record of the instantaneous values of the variable electromotive force.

c. The coil may be given a cyclic motion which it repeats indefinitely—for example, the coil may rotate or oscillate in the field—and some average effect of the induced electromotive force may be measured by a suitable designed instrument (voltmeter).

Faraday deduced the laws of electromagnetic induction mainly from measurements made with a ballistic galvanometer by the

first method, although he also made important deductions from the currents set up in metal disks and rods during their rotation in a magnetic field.

271. The Ballistic Galvanometer.—The ballistic galvanometer is designed to measure the time-integral of a variable transient current or electromotive force which lasts for a small fractional part of a second. The features of the ballistic instrument, its properties, and the method of using it, as given in Sec. 132, should be reviewed. Figure 228 shows a circuit for the experimental determination of the properties of the instrument in measuring transient electromotive forces of short duration. S represents a switch or commutator by means of which the battery circuit may

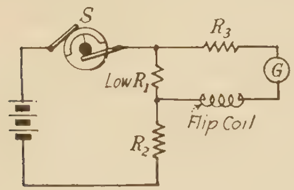


FIG. 228.—Circuit for calibration of ballistic galvanometer.

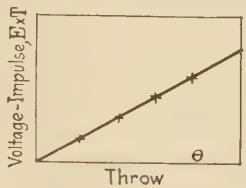


FIG. 229.—Calibration curve of ballistic galvanometer.

be closed for definite short intervals of time, such as 0.01, 0.05 second, etc. A voltage E, which may be readily calculated or measured, may thus be impressed in the galvanometer circuit for any chosen interval T. The values of E and T may be varied at will, except that T must always be small compared to the period of oscillation of the galvanometer. If the throw of the instrument for each combination of E and T is read, and if the throw θ is plotted against the product $E \times T$ a straight line is obtained, as in Fig. 229. That is to say, the throw of a ballistic galvanometer is directly proportional to the product of the impressed voltage and the length of time it is impressed; or vice versa, the voltage-time product is directly proportional to the throw. This property is expressed by the equation

$$ET \text{ (volt-seconds)} = K\theta, \qquad (413a)$$

in which, the proportionality factor K may be called the **voltage-impulse** constant of the particular instrument and test circuit. K may be readily calculated from the experimental data.

When the voltage is not constant throughout the interval T, but is a variable e, the more general Eq. (413) is found to hold

$$\int_0^T e\,dt = K\theta. \tag{413}$$

This reduces to Eq. (413a) when e is constant. These equations express the property of the ballistic galvanometer which makes this instrument so useful in the study of magnetic fields.

272. Voltage-impulse (Definition).—We may express the useful property of the ballistic galvanometer as follows. "A ballistic galvanometer measures the **voltage-impulses** in its circuit." By the term **voltage-impulse** is meant the product of voltage by the time during which it acts, et, or the time-integral of the voltage, $\int e\,dt$.

The name **voltage-impulse** is suggested by the practice in physics and mechanics of calling the product of force by the time during which it acts, the **force-impulse.** For example, if a suspended weight is struck by a sledge, the weight is acted upon by a variable force for a short interval of time. Neither the force nor the time can be determined, but the impulse $\int f\,dt$ may be quite easily determined by measurements of the momentum imparted to the suspended body. Likewise, when a circuit is suddenly moved in a magnetic field, a voltage is induced in the circuit. If the time period is short, neither the period nor the variable voltage can be measured accurately but the voltage impulse $\int e\,dt$ may be measured by means of a ballistic galvanometer.

Since voltage-impulses represent products of voltage and time they are measured in **volt-seconds.** If $\int e\,dt = 1$, the impulse is 1 **volt-second.** An e.m.f. of 50 volts applied for 0.02 second results in an impulse of 1 volt-second. A ballistic galvanometer calibrated and used in this manner may be called a **volt-second meter,** just as the ordinary house meter which measures watt-hours of energy, is called a watt-hour meter.

273. Faraday's Motionally Induced Eddy Current Experiments.—The following experiments, selected mainly from the numerous experiments performed by Faraday, serve to bring

out the conditions under which currents are set up in conductors while in motion in a magnetic field.

273a. Experiment 1.[2]—In Fig. 230 is represented a rectangular plate of copper (say, 0.5 centimeter thick by 5 centimeters wide by 30 centimeters long), which is mounted so that it can be drawn through the magnetic field in the narrow air gap between two coaxial solenoids or bar magnets. These solenoids are mounted one on either side of the plate with their common axis perpendicular to the plane of the plate. The cross-sectional outline of the solenoids and the direction of the flow of the exciting current around them is indicated by the heavy circle and its arrow. This means that the B vectors are perpendicular to the plane of the paper (or of the copper plate) and point in the downward direction.

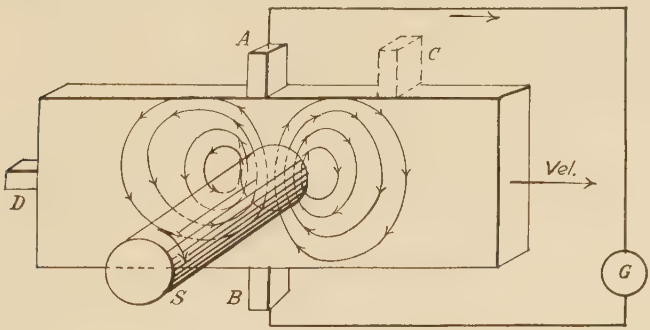


FIG. 230.—Faraday's moving plate experiment.

The galvanometer terminals are connected to two copper bars called **brushes** which are so mounted on the frame holding the solenoids that they make sliding contact with the plate during its motion.

If the two brushes are mounted in the positions A and B (Fig. 230) and if the plate is drawn to the right, the galvanometer immediately deflects in a direction which indicates that a current is set up **in the galvanometer circuit** in the direction of the arrow, that is, from the top brush to the bottom brush. If the plate is drawn toward the left, the galvanometer deflects in the opposite direction.

If the plate and brushes are now held stationary and the solenoids are moved toward the left, the galvanometer deflects in the same direction as that caused by moving the plate toward the right.

By noting the direction and the magnitude of the galvanometer deflection with the brushes mounted in many different positions (for example, at A and C, at A and D, etc.), Faraday was able to conclude that the motion of the plate toward the right

[2] FARADAY: *Experimental Researches*, Vol. I, Pars. 101–111.

through the magnetic field between the solenoids causes a circulating current within the plate along the stream lines sketched in Fig. 230.

273b. Experiment 2. Faraday's Direct-current Disk Generator.[3]—In this experiment, Faraday mounted a circular copper disk so that when rotated on its axis some portion of the disk would be at all times cutting or sweeping across the tubes of magnetic flux in the air gap between the two solenoids or bar magnets of the previous experiment. By applying the two brushes, one to the axle of the rotating disk and the other to its periphery in the position shown in Fig. 231, a continuous current was obtained in the galvanometer circuit. Upon reversing the direction of rotation of the disk, or upon reversing the direction of the exciting current in the solenoids, the direction of the current in the galvanometer circuit reversed.

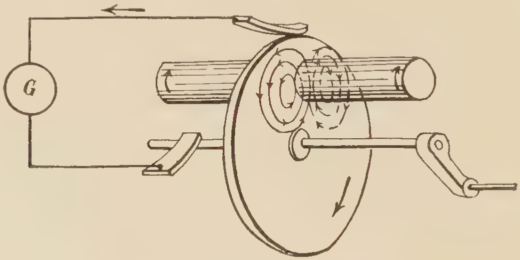


Fig. 231.—Faraday's disk generator.

By applying the brushes to many different positions on the disk and noting the direction and magnitude of the galvanometer deflection for each pair of positions, Faraday was able to conclude that the rotation of the disk in the direction shown by the arrow caused a circulating current within the copper disk along the stream lines sketched in Fig. 231.

Twenty years after this work, Foucault found that if the copper disk of Fig. 231 is continued in rapid rotation between the poles of a strong magnet, it quickly becomes very hot. This heating may be regarded as additional evidence of the existence of the eddy currents within the disk.

The wasteful and objectionable eddy currents which may exist within conductors or conducting masses are sometimes referred to as Foucault currents. In view of the history of the discovery of these currents, and of the lack of physical significance attaching to the qualifier *Foucault* as contrasted with *eddy*, the designation Foucault current would seem to be inappropriate and unwarranted.

273c. Experiment 3. Arago's Magnetic Phenomena.—About 1824, or some 7 years before Faraday's discovery of motionally induced currents, it was observed that a compass needle, if disturbed and left to oscillate on its

[3] Faraday: *Experimental Researches*, Vol. I, Pars. 83–100.

pivot, is damped more rapidly, or comes to rest after a shorter number of oscillations, if the bottom of the compass box is of copper than if it is of wood or other non-conducting material.

Arago, who among others studied the phenomenon, remarked that it gives evidence of a force which acts only while there is relative motion between the needle and the nearby conducting body. He reasoned that a moving conducting mass ought to exert a force on any nearby stationary magnetic needles. Accordingly, he suspended a compass needle in a glass jar near its bottom, and held the jar and its needle directly over a circular disk of copper which could be rotated in a horizontal plane. He found (1824) that at low speeds of rotation, the needle deflects from the magnetic meridian by a slight angle in the direction of rotation of the disk. As the speed of rotation of the disk is increased, the needle deflects more and more until the angle of deflection becomes 90 degrees. At disk speeds higher than the speed which causes the 90-degree deflection, the needle itself rotates continuously in the same direction as the disk, but at a lower speed than the disk. In the following year, Babbage and Hershell, by driving a magnet on its pivot beneath a pivoted copper disk, caused the latter to rotate continuously.

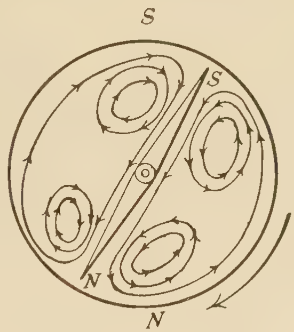


Fig. 232.—Arago's experiment.

Although many attempts were made to account for these phenomena, their explanation remained unknown until Faraday, after his discovery of motionally induced currents, pointed out that the motion of the metal disk in the field of the magnetic needle would give rise to eddy currents in the disk along the lines sketched in Fig. 232, and that the force exerted on the needle is the Oersted effect of these circulating currents.

These eddy-current experiments and the many other equally striking experiments described in Faraday's **Experimental Researches** combine to justify the following conclusions.[4]

[4] We have compiled from Pars. 119 and 3087 of *Experimental Researches* the following conclusion stated in Faraday's own language save that we have substituted "tubes of magnetic flux" where Faraday wrote "lines of magnetic force."

A piece of metal or of conducting matter which moves across tubes of magnetic flux has, or tends to have, a current of electricity produced in it traverse to the direction of motion. A more restricted and precise expression of the full effect is the following: If a continuous circuit of conducting matter be traced out, or conceived of, either in a solid or fluid mass of metal or conducting matter, or in wires or bars of metal arranged in non-conducting matter or space, which, being moved, crosses tubes of magnetic flux, or, being still, is by the translation

273d. MOTIONAL FORCES ON ELECTRICITY (Exp. Det. Rel.).—**The electricity (electrons and protons) in any small portion of a conducting body which is in motion relative to magnetic field, is subject to forces whose directions are perpendicular both to the direction of motion of that portion of the conductor and to the direction of the B vectors.**

The magnitude of the force depends upon the value of the flux density and upon the velocity of the body relative to the field, and upon the direction of the motion relative to the B vectors.

The precise nature of this dependence is the subject of the next section.

274. Faraday's Voltage-impulse Experiments (1831).—The simplest measurements in the study of the electromotive forces induced in a circuit during its motion in a magnetic field are measurements of the voltage-impulses caused by sudden movements. For this purpose the ballistic galvanometer may be connected in any circuit. The circuit or any part of the circuit may then be moved rapidly (jerked) from various initial to various ultimate positions, and the voltage-impulse caused by each displacement of the circuit may be determined from the throw of the galvanometer.

As pointed out in other cases, the experimental method consists in taking measurements of the desired effect under different sets of known conditions, and then searching for a law or formula which agrees with all the experimental data. When possible, the conditions of the experiment are varied one at a time. Following this plan, the Faraday experiments show clearly that the magnitude of the voltage-impulse is independent of many of the conditions of the experiment. For example the magnitude of the voltage-impulse is:

a. Independent of the material constituting the conductors of the circuit. The voltage-impulse is the same for a circuit in which the wires are of high resistance alloy as for a circuit of copper wire.

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of a magnet (or electromagnet) crossed by such tubes of magnetic flux, and further, if, by inequality of angular motion, or by contrary motion of different parts of the circuit, or by inequality of the motion in the same direction, one part (of the circuit) crosses either more or fewer unit tubes of magnetic flux than the other, then a current will exist around the circuit, due to the differential relation of the two or more intersecting parts during the time of motion.

b. Independent of the cross-sectional shape and area of the wires of the circuit. The voltage-impulse is the same for a circuit in which the wire has a cross-sectional area of 0.1 square millimeter as for one in which the area is 1 square millimeter. It is the same for round, flat, and stranded wires.

c. Independent of the time taken to displace the circuit from a given initial to a given ultimate position. On account of the limitations of the ballistic galvanometer, however, this time cannot be allowed to exceed a small fractional part (one-one-hundredth) of the period of oscillation of the galvanometer.

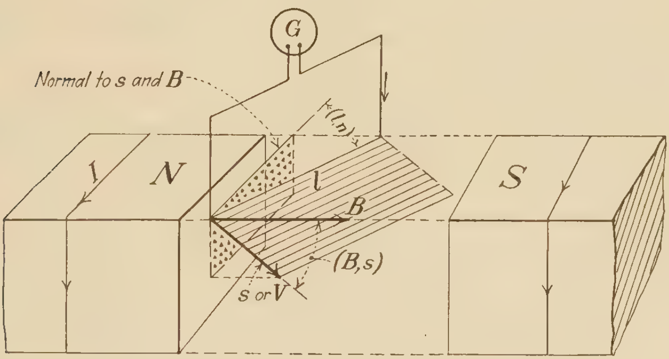

FIG. 233.—Arrangement for voltage-impulse experiments.

By determining the directions of the induced electromotive forces and the values of the voltage-impulses when one side of a rectangular coil is **displaced parallel to its initial position** in various directions and by various amounts across the flux density vectors in the uniform field in the gap between the rectangular cross-sectioned solenoids or bar magnets shown in Fig. 233, and by repeating the determinations for different values of flux density and with coils of different dimensions and number of turns, the following conclusions are reached:

d. From the fact that the movements in parts of the coil not lying in the magnetic field of the gap give rise to no deflection of the galvanometer, the conclusion is drawn that the driving force which tends to cause a circulation of electricity in the moving circuit is exerted upon the electricity in those portions of the

conductor which lie within, and move through, the magnetic field, or that the electromotive force is induced or localized in those portions of the conductor.

e. The direction of the induced electromotive force may be determined by the following empirically derived rule.

274e. Fleming's Right-hand Rule for the Direction of a Motional Electromotive Force (Exp. Det. Rel.).—*The direction of the motional electromotive force induced in a wire moving relative to a magnetic field may be determined as follows:*

Point the first finger of the right hand in the direction of the **B** *vectors (see Fig.* 234).

Point the thumb in the direction of motion of the wire **relative** *to the field.*

The center finger if pointed along the wire will then point in the direction of the motional electromotive force, that is, in the direction of the motional force on positive electricity.

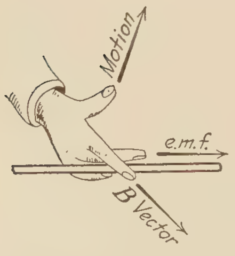

Fig. 234.—Right hand rule for induced e.m.f.

This rule is frequently called "Fleming's rule," since Fleming formulated it. A comparison of this rule with the rule for the force on a moving charge contained in Sec. 233*b* will show that the two are identical in substance.

The magnitude of the voltage-impulse is:

f. Directly proportional to the value of the flux density in the portion of the field swept over by the conductors.

g. Directly proportional to the number of turns (in series) in the coil, provided the N turns are so close together that they sweep over substantially the same surface.

h. Directly proportional to that component of the displacement s, of the wire which is perpendicular to the **B** vectors, namely, $s \sin (B,s)$, in which

> (B,s) is the angle between the direction of **B** vectors and the direction of the displacement s.

i. Directly proportional to the projection of the length of the conductor upon a normal erected to a plane determined by the

direction of the B vectors and the direction of the wire displacement, namely, $l \cos (l,n)$, in which

l is the length of the portion of the straight conductor which lies within the field.

(l,n) is the angle which the conductor makes with a normal to the plane determined by the direction of the B vectors and the direction of the displacement s.

Conclusions f to i combine into the single statement that the value of the voltage-impulse is directly proportional to the product $NsBl \sin(s,B) \cos(l,n)$.

Precise measurements of all the quantities involved in the experimental determinations serve to bring out a relation which, if looked at simply as an experimentally developed relation, must be regarded as most extraordinary in its uniqueness; namely, the fact that the value of the voltage-impulse is **precisely equal to** the product written above. That is,

$$\int e\,dt \text{ (volt-seconds)} = NsBl \sin(s,B) \cos(l,n) \quad \text{(webers per sq. cm., cm.)} \qquad (414)$$

The unique feature of this relation is that the voltage-impulse of a motionally induced electromotive force (which at this stage in this study is a new phenomenon, as yet unaccounted for in terms of other electrical phenomena) should turn out to be **exactly equal to** the product of the factors appearing in the right member of the equation. Things rarely happen this way in nature. Proportionality constants always turn out to have some odd decimal value, such as 3.1416 or 2.54 or 32.16, and never turn out to be unity except under one of two conditions:

1. *When the new quantity has been defined in terms of the more familiar antecedent quantities in such a manner as deliberately to make the proportionality constant in the defining equation unity.*

2. *When the new phenomenon turns out to be but another aspect of a phenomenon whose measure appears on the right side of the equation.*

Now we have not consciously defined the "voltage-impulse of a motionally induced electromotive force" in terms of any of the factors appearing in the right member of Eq. (414). The experimentally determined equality should, therefore, have an impor-

tant physical significance. In Sec. 282 and 286 we will show that a motional electromotive force, or (a line-integral of) a force on the electrons in a moving conductor, is but another aspect of the phenomena in terms of which we have defined the B vector namely, the force on a conductor which itself is stationary but through which electrons are moving.

The measurements described above are the experimental basis for the alternative statements of Faraday's Law for Motional Electromotive Force formulated below.

275. Faraday's Law for the Motional Voltage-impulse in a Wire (EXP. DET. REL.).—The law which is contained in algebraic form in Eq. 414 may be expressed in the following language.

275a. *The voltage-impulse induced in a straight wire of length l during its displacement parallel to its initial position through the distance s in a uniform magnetic field is equal to the magnetic flux density B times the component of the displacement which is perpendicular to the direction of the B vectors, times the projection of the length l on a line normal to both the s and the B vectors. The direction of the induced electromotive force is given by Fleming's right-hand rule of Sec. 274a.*

$$\int e\,dt \text{ (volt-seconds)} = sBl \sin(s,B) \cos(l,n).$$
$$\text{(webers per sq. cm., cm.)} \quad (415)$$

in which

(B,s) represents the angle between the B and the s vectors.

(l,n) " the angle between the normal to these two vectors and the length of the straight wire.

By a study of Fig. 233, it may be seen that the product $sBl \sin(s,B) \cos(l,n)$ appearing in the right member of Eq. 415 is the expression for the magnetic flux over the area which is swept over by the conductor of length l when it is displaced the distance s. One of Faraday's most remarkable achievements was to arrive at, and to confirm, the notion that in every case in which a wire of any shape moves in any manner in a magnetic field the "tendency for current to flow depends upon, and is proportional to, the number of lines of force cut by the wire in its motion." The phrase "number of lines of force" was Faraday's equivalent of the quantity herein called the "magnetic flux." Subsequent

experiments have abundantly verified Faraday's notion, and justify the acceptance of the following alternative form as the more general statement of the law.

275b. MOTIONAL VOLTAGE-IMPULSE INDUCED IN A WIRE (EXP. DET. REL.).—**The voltage-impulse induced in a wire of any shape during its motion in any manner in any magnetic field is equal to the magnetic flux swept over by the wire during its motion; or it is equal to the number of weber tubes of magnetic flux cut by the wire in its motion.**

$$\int e\,dt \text{ (volt-seconds)} = \Phi \text{ over surface swept over,} \tag{416}$$

$$\int e\,dt \text{ (volt-seconds)} = \int B \cos (B,n)\, da. \tag{416}$$

276. Faraday's Law for the Motional Electromotive Force Induced in a Wire (EXP. DET. REL.).—Equations (415) and (416), for the value of the voltage-impulse, apply not only for a certain initial and a certain ultimate position of the wire, but they apply between all intermediate positions which the wire assumes during its motion from the initial to the ultimate position. Therefore, we may equate the derivatives with respect to time of both members of the equations. Taking derivatives and equating, we have

$$e \text{ (volts)} = Bl\frac{ds}{dt} \sin (s,B) \cos (l,n)$$

or,

$$e \text{ (volts)} = VBl \sin (V,B) \cos (l,n) \text{ (webers per sq. cm.),} \tag{417}$$

in which (V,B) represents the angle between the V and B vectors.

(l,n) represents the angle between the normal to these two vectors and the length of the straight wire.

This result may be expressed in the following language:

276a. *The electromotive force e induced in a straight wire of length l moving parallel to itself with the velocity V in a uniform magnetic field is equal to the magnetic flux density B times the component of the velocity perpendicular to the direction of the magnetic flux density times the projection of the length l on a normal to both the V and the B vectors.*

In like manner, by equating the derivatives of Eq. (416), the following general statement of the law of motional electromotive force is obtained:

276b. MOTIONAL ELECTROMOTIVE FORCE INDUCED IN A WIRE.—The electromotive force e **induced in a wire by reason of the**

motion in a magnetic field is equal to the TIME RATE at which the wire cuts across weber tubes of magnetic flux.

$$e \text{ (volts)} = \frac{\Phi}{t} = \frac{d\Phi}{dt} \frac{\text{(webers)}.}{\text{(sec)}.} \tag{418}$$

277. Faraday's Law for the Motional Voltage-impulse and Electromotive Force in a Circuit (EXP. DET. REL., 1831).—Fleming's right-hand rule may be used to find the direction of the electromotive force induced in any short portion of a circuit. In the general case of the motion of a complete circuit, the e.m.f. induced in some portions of the circuit may be opposite in direction to the e.m.f. induced in other portions. Thus it is that Fleming's rule cannot be used in a brief, easily applied manner to predict the direction of the net or resultant electromotive force in terms of the net rate at which the moving circuit is cutting across the tubes of magnetic flux. It therefore becomes necessary to adopt such comprehensive conventions as to the algebraic signs of the factors e, Φ, B, and (B,n) entering into the electromotive force equations, that these equations will automatically give the net value and direction of the induced electromotive force. To this end, we adopt the following conventions relating algebraic sign to direction.

a. For convenience in specifying directions around the circuit and across (through) any surface bounded by the circuit, arrows will be drawn on the diagram around the circuit and through the surface. The directions indicated by the arrows will be referred to as the **arrow direction** or as the **specified direction** around the circuit and through the surface, respectively.

b. The direction of the arrow around the circuit will be arbitrarily chosen, but the direction of the arrow through the surface (or through the circuit) will be related to the direction of the arrow around the circuit by the right-hand screw convention; namely, the arrow direction through the circuit will bear to the arrow direction around the circuit the same relation that the direction of advancement of a right-hand screw bears to its direction of rotation.

c. In the equation defining the magnetic flux over a surface bounded by the circuit, namely,

$$\Phi = \int B \cos (B,n) da \tag{374}$$

(B,n) is the angle between the direction of the B vector and the arrow direction along the normal to the area da. That is to say, the algebraic value of the flux over the area da is positive if the B vector points across the surface in the arrow direction.

Using these conventions, the laws for the direction and magnitude of the voltage-impulse and electromotive force induced in a circuit by reason of its motion relative to a magnetic field may be thus expressed.[5]

277d. *The algebraic value of the voltage-impulse in the arrow direction induced in a circuit during its motion from one position to another in a magnetic field is equal to the decrement in the magnetic flux (in the arrow direction) over any surface bounded by the circuit (see Sec. 279 also).*

$$\int e\,dt \text{ (volt-seconds)} = -\Delta\Phi \text{ webers.} \qquad (419)$$

277e. MOTIONAL ELECTROMOTIVE FORCE INDUCED IN A CIRCUIT.—The algebraic value of the electromotive force in the arrow direction induced in a circuit by reason of its motion in a magnetic field is equal to the time-rate of decrease of the magnetic flux over any surface bounded by the circuit (see Sec. 279 also).

$$e \text{ (volts)} = -\frac{d\Phi \text{ (webers)}}{dt \text{ (seconds)}}. \qquad (420)$$

278. Magnetic Flux-linkage (DEFINITION).—When a circuit is not in the form of a single turn forming the contour of a simple surface, but is in the form of a coil consisting of many turns, it is customary to term the surface-integral of the magnetic flux density over the many-folded surface (each fold being bounded by a turn of the coil) the **flux-linkage between the coil and the field,** or, briefly, the **flux-linkage** (Λ).

$$\Lambda = \Phi_1 + \Phi_2 + \Phi_3 + \ldots \ldots \qquad (421)$$

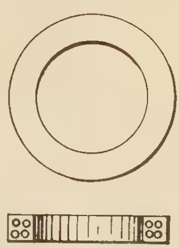

FIG. 235.—Ring coil.

In a coil of N turns of fine wire in which all of the turns bound substantially the same surface (Fig. 235), or in

[5] The reader should satisfy himself that the statement of directions contained in these laws is consistent with that contained in Fleming's rule. This may be done by picturing definite circuits to move in specified ways in known fields, and by comparing the directions of the motional electromotive forces as predicted by the two methods.

which the magnetic flux over the surface bounded by one turn is equal to the flux over the surface bounded by any other turn (as in a long solenoid or in the ring coil of Fig. 235), the total flux Φ_t or the flux-linkage Λ over the multifolded surface is approximately equal to N times the flux Φ_1, over the surface bounded by a single turn.

$$\Lambda \text{ (or } \Phi_t\text{, webers)} = N\Phi_1. \tag{421a}$$

The quantity flux-linkage differs in no respect from magnetic flux. It is simply another name for flux, a name which it is convenient to use in the case of those circuits which bound multifolded surfaces.[6]

279. The Motional Laws for the Multiturn Coil.—If the circuit which moves in the magnetic field is not in the form of a single turn forming the boundary of a simple, unfolded surface, but is in the form of many turns of wire, the equation for the motional voltage-impulse may be written

$$\int e\,dt \text{ (volt-seconds)} = -(\Delta\Phi_1 + \Delta\Phi_2 + \Delta\Phi_3 + \ . \ . \ . \ ) \tag{419a}$$

[6] In order to see how a simple circuit bounding a plane area is folded over upon itself when it is converted into an N-turn circuit, the student should take an endless cord or rubber band and after selecting an arrow direction around the band he should place it once over a pencil to form a single turn

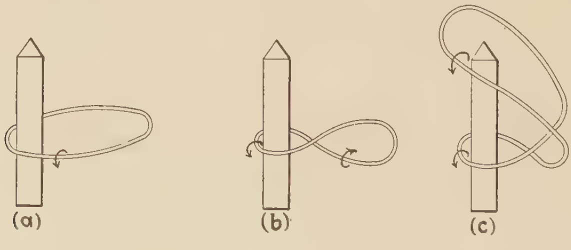

Fig. 236.—Formation of multi-folded surface.

around the pencil, with the point of the pencil pointing in the arrow direction through the loop (Fig. 236a). Then the loop should be twisted over to form two loops, a small loop and a large loop as in Fig. 236b. Then the large loop should be folded over on the small loop as in Fig. 236c, and passed over the pencil with the point passing through in the arrow direction. This places two turns around the pencil. This process of twisting to form loops and then folding the surface over on itself if repeated N times places N turns around the pencil and the pencil passes through the surface bounded by each turn in the arrow direction.

in which Φ_1, Φ_2, Φ_3, etc. represent the fluxes over the surfaces bounded by turns 1, 2, 3, etc.

For this case, the term **flux-linkage of the circuit** may be substituted for the phrase **magnetic flux over any surface bounded by the circuit** in Secs. 277*d* and 277*e*, and the equations for the voltage-impulse and for the electromotive force may be written

$$\int e\,dt \text{ (volt-seconds)} = -\Delta\Lambda \text{ (webers)}, \tag{419b}$$

$$e \text{ (volts)} = -\frac{d\Lambda \text{ (webers)}}{dt \text{ (seconds)}}. \tag{420a}$$

280. Lenz's Law for the Direction of the Induced Electromotive Force (EXP. DET. REL., 1834).—The foregoing algebraic statements of the general law of motional electromotive force are complete in themselves and carry the information as to directions in their algebraic signs, that is, in the use of the negative sign in the equations, and in the use of the word **decrease** rather than increase. It is customary, however, to state separate rules which relate to directions alone, and to make use of these rules when interested only in relative directions.

In 1834, Lenz formulated a very comprehensive law for determining the direction of the electromotive force electromagnetically induced in a circuit.[7] It may be stated thus:

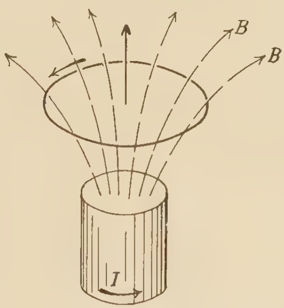

FIG. 237.

280a. LENZ'S LAW (EXP. DET. REL.).—**The electromotive force which is induced in a circuit as a result of any variation of the magnetic field with reference to the circuit is in SUCH A DIRECTION that the current which results tends to PREVENT THE CHANGE which occasions the induced electromotive force.**

To show that the direction of the induced electromotive force, as predicted from Eqs. (419) and (420) or from the wording of Faraday's law, is in agreement with the predictions from Lenz's law, consider the circuit shown in Fig. 237.

Let arrows be so drawn **around** and **through** the circuit that the two arrow directions are related by the right-hand screw convention. Let the circuit

[7] A philosophical discussion of the work of Lenz and his contemporaries will be found in *The Contributions of H. F. E. Lenz to Electromagnetism*, by W. M. STINE.

move so that the flux linking with the circuit in the arrow direction is increased. From Faraday's law we predict that the induced electromotive force will be a negative quantity, that is, its direction will be against the arrow around the circuit.

The argument based on the Lenz's law is as follows: The law is, that the electromotive force is induced in such a direction that the current which results tends to prevent the **change** which occasions the e.m.f. Again let us say that the change which occasions the electromotive force is an increase in the flux linking with the circuit in the arrow direction. According to the law, the induced current should flow in such a direction as to set up a flux in the opposite direction. Upon applying Ampere's rule (Sec. 226*a*) to find the direction in which the current must flow to do this, we find that the current must flow against the arrow around the circuit; therefore the induced e.m.f. must be in this direction. The prediction from Faraday's law agrees with this.

281. Flip-coil Method of Measuring and Defining Magnetic Flux Density.—The magnetic units discussed thus far have all been defined in terms of magnetic flux density. The flux density at a point has been defined as the force per unit length upon a test wire carrying 1 ampere. Accordingly the primary method of determining the flux density at a given point in the field is to set up either a wire force-finder or a coil torque-finder and to read the force or the torque on these instruments. Now the fact that any displacement of a coil in a magnetic field induces a voltage-impulse which is equal to the decrease in the flux-linkage caused by the displacement, makes possible the following alternative method of measuring flux density—a method which is more convenient and accurate than the primary method of measuring a small force.

A circular coil containing N turns of fine wire[8] bounding a small plane surface of known area a is pivoted in a suitable framework on an axis which coincides with one of the coil diameters (see Fig. 238). The coil is provided with a trigger, a spring, and a stop, which are so arranged that when the trigger is released, the spring **flips** the coil over on its diameter, and the stop brings it to rest after the plane of the coil has turned through 180 degrees. The terminals of the flip coil are connected by suitable lead wires to the terminals of a ballistic galvanometer which has been previously calibrated as a volt-second meter.

[8] For example, 300 turns of copper wire 0.1 millimeter in diameter in a ring coil 3 centimeters in mean diameter.

Suppose the flip coil is set up with its center at a point P in a magnetic field, and with its normal (not mounting) axis pointing in the positive direction along the B vector. If the throw of the galvanometer θ caused by a 180-degree flip is now read, the value of the voltage-impulse induced, and the change in flux-linkage may be computed from the voltage-impulse equations of the galvanometer and the circuit, namely,

$$\int e\,dt \text{ (volt-seconds)} = K\theta \tag{413}$$

and

$$\int e\,dt \text{ (volt-seconds)} = -\Delta\Lambda \tag{419b}$$

in which K is the known volt-second constant of the instrument.

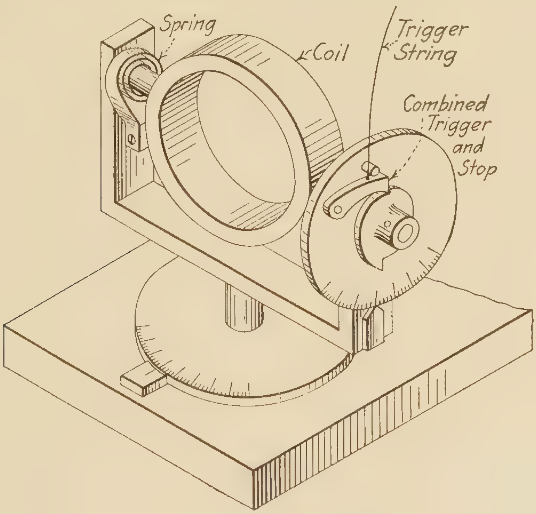

FIG. 238.—Flip coil.

But the magnetic flux over the area bounded by the coil in the original position is

$$\Phi \text{ (webers)} = Ba \tag{375}$$

and the value of the flux-linkage is

$$\Lambda_1 = NBa \text{ webers.}$$

After the coil has flipped through 180 degrees the value of the flux linkage is

$$\Lambda_2 = -NBa$$

Therefore $\Delta\Lambda = (\Lambda_2 - \Lambda_1) = -2NBa.$

Hence $$B = -\frac{\Delta\Lambda}{2Na} = \frac{K\theta}{2Na} \text{ webers per sq. cm.} \tag{422}$$

The measurement as outlined above necessitates preliminary testing at P to find the direction of the B vector. Its direction may be found by using a direction-finding coil, or by shifting the direction of the axis of the flip coil and by noting the throw corresponding to a 180-degree flip from each initial position, until the position which gives the maximum throw is determined.

A better way to map out a field is to select a set of X, Y, and Z axes (for example, the edges of the laboratory) and at each point of interest to measure the X, Y, and Z components of the B vector. The X component B_x at a given point is determined by pointing the normal axis of the coil along the X axis, and noting the throw θ_x caused by a 180-degree flip.

Whence, $$B_x = \frac{K\theta_x}{2Na}.$$

The value of B at at any point is then computed from the relation

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2}.$$

We have chosen to define magnetic flux density in terms of mechanical force, but it is now evident that it is possible to define the same quantity in terms of the electromotive force induced in a coil. For example, the following definition leads to the same unit.

281a. *The magnetic flux density at a point may be defined as a vector quantity whose magnitude is equal to one-half of the maximum voltage-impulse induced by a* 180-*degree flip of a* 1-*turn flip coil bounding a plane area of* 1 *square centimeter centered at the point. The direction of the vector is defined as the arrow direction along the normal axis of the flip coil, when the flip coil occupies that initial position which will give the maximum voltage-impulse. The arrow direction along the normal is defined to be related to the direction of the induced electromotive force around the coil by the right-hand screw convention.*

282. Law of Motional Electromotive Intensities. (Deduced from the Law for the Force on a Moving Charge).—In the preceding sections, the laws dealing with motional electromotive forces have been deduced from voltage-impulse experiments. The arguments have contained no hypotheses as to the nature of the

driving forces acting on the electricity in the moving circuit, and have not correlated the induced e.m.fs. with other electrical phenomena. We now propose to account for these electromotive forces in terms of the driving forces which act on the electrons in the moving conductors.

The experiments outlined in Chap. X dealing with the effect of a magnetic field upon an arc stream and upon the cathode stream in an evacuated tube have shown that an ion in motion in a magnetic field is acted upon by a force which, at every point of the path of the ion, is at right angles to both the V and B vectors. This force upon moving electrons has been used to account for the mechanical force upon a conductor carrying a current.

We now advance the proposition that this force upon moving charges, which has enabled us to render an account of the mechanical force upon conductors carrying current, should also enable us to account for the electromotive force induced in moving conductors. That is, we consider that when any body—conducting or non-conducting—is moving with a given velocity in a given magnetic field, the electrons and positive nuclei of which the body is composed are subject to the same driving force as if they were free ions moving in an evacuated space with the given velocity. The electrons are urged in one direction and the nuclei in the opposite. If the moving body is part of a complete (conducting) circuit, the forces may cause the atmosphere of free electrons to circulate around the circuit. If the circuit is open so that the electrons cannot circulate, the forces slightly displace the atmosphere of free electrons toward one end of the circuit, charging that end negatively and leaving the other end positively charged. If the moving body is a non-conductor of electricity, the forces cause an elastic displacement (within the molecular systems) of the electrons relative to the nuclei, and the body becomes polarized—one end showing a negative charge and the other a positive.

We propose to justify this proposition by showing that the equations for the motional electromotive force which have been experimentally obtained in this chapter directly from voltage-impulse measurements may be deduced or predicted from the expression (obtained by mechanical measurements in Chap. X) for the force on a moving charge, namely

$$f \text{ (dyne-sevens)} = QVB \sin (V,B). \qquad (349)$$

In Sec. 286 this proposition will receive further justification from the manner in which it may be used to describe the process by which energy is supplied to a moving motor coil.

In electrostatic fields, when a small charge Q is acted upon by a force f, we have called the ratio f/Q the electric intensity F at the point. Likewise, if a charge Q in a body moving in a magnetic field is, by reason of its motion, acted upon by a force f we may say that there is **induced in** the moving body a motional electromotive intensity F_m, defined by a similar equation, namely

$$F_m \text{ (volts per cm.)} = \frac{f \text{ (dyne-sevens)}}{Q \text{ (coulombs)}}. \qquad (423)$$

Since the charge Q in a body moving in a magnetic field with a velocity V is acted upon by the force

$$f = QVB \sin (V,B),$$

the motional electromotive intensity F_m induced in the moving body is

$$F_m\text{(volts per cm.)} = VB \sin (V,B), \qquad (424)$$

in which, (V,B) represents the angle between the V and B vectors, and in which the direction of the vector F_m is that of the force on a moving positive charge.[9]

The complete statement of this relation is as follows:

282a. LAW OF MOTIONAL ELECTRIC INTENSITY.—When a body moves relatively to a magnetic field which is itself unvarying when referred to axes fixed with reference to the circuit or magnetized body setting up the field, a motional electric intensity is induced in the moving body. The magnitude of the motional intensity is equal to the product of the magnetic flux density B, times the component of the velocity normal to the magnetic flux density.

The motional electric intensity at any point in the moving body is normal to the plane determined by the two vectors representing, respectively, the velocity of the body at the point (relative to the field) and the magnetic flux density. The motional intensity is in that direction along the normal in which a right-hand screw would advance if rotated in the direction in which the velocity vector must be turned to bring it into parallelism with the B vector.

[9] In vector notation, this equation may be written

$$\mathbf{F}_m \text{ (volts per cm.)} = \mathbf{V} \times \mathbf{B} \text{ (vector product)}. \qquad (425)$$

An alternative rule for determining the direction of the induced intensity is the general right-hand rule, namely:

Point the first finger of the right hand in the direction of the B vector, and the thumb in the direction of motion of the body relative to the field. The second finger indicates the direction of the induced intensity along the normal.

From the expression for the motional electric intensity let us derive an expression for the electromotive force induced in the moving conductor shown in Fig. 233.

From the definition of electromotive force in Sec. 147*b*, the e.m.f. of the motional forces in the wire is equal to the work done by the forces per unit (or equivalent unit) of positive electricity which flows through the wire.

Let us suppose the wire of Fig. 233 contains q coulombs of free electrons per centimeter of length. Then the quantity of free electricity in the wire is ql, and from Eq. (424) the force on this atmosphere of free electricity is

$$f = (ql)F_m = qlVB \sin (V,B).$$

To find the work done when the electrons move, let us suppose the electron atmosphere is moving through the wire with a velocity of v centimeters per second. The quantity Q which moves through the wire per second is

$$Q = qv.$$

Now the force on this electricity by reason of the motion of the wire is normal to both the B and the V vectors, but the electrons in moving through the wire move, not in the direction of this force, but along a wire which makes an angle (l, n) with the force. Therefore the expression for the work done per second will be

$$\begin{aligned} W \text{ (joules)} &= fv \cos (l,n) \\ &= qVBl \sin (V,B)v \cos (l,n), \end{aligned}$$

and the expression for the electromotive force will be

$$e \text{ (volts)} = \frac{W}{Q} = \frac{qVBl \sin (V,B)\ v \cos (l,n)}{qv},$$

$$e \text{ (volts)} = VBl \sin (V,B) \cos (l,n).$$

This equation for the motional e.m.f. is identical with Eq. (417) which was derived from the voltage-impulse measurements,

and thus justifies the proposition that a motional electromotive force is but another aspect of that force upon moving charges which was investigated in Chap. X.

PART II—ENERGY TRANSFORMATIONS OCCURRING DURING THE MOTION IN A MAGNETIC FIELD OF A CONDUCTOR CARRYING A CURRENT

283. Work Done When a Conductor Carrying a Current Moves in a Magnetic Field (Deduction).

283a. The Case of a Straight Slider.—For the purpose of deriving expressions for the work which is done when a conductor carrying a current in a magnetic field moves under the action of the forces of the field, suppose current is supplied from a generator G (Fig. 239) through two parallel conducting rails AK and CD to a straight conductor AC which bridges across the rails, thus completing the circuit. Suppose AC is free to slide along the rails save that it is constrained so to move that all parts of AC move in straight lines parallel to each other; that is, the conductor always remains parallel to the initial position. (This can readily be accomplished by suitably attaching the conductor to a truck—which it pulls along the rails.)

Suppose the conductor slides in a uniform magnetic field in which the value of the magnetic flux density is represented by B. The most general directions which straight rails and a slider may assume relative to the B vector have been illustrated in Fig. 239. The YZ plane of the figure has been taken parallel to the plane determined by the slider and the B vectors. Since the force on a straight conductor is always perpendicular to the plane determined by the conductor and the B vectors, the force will be parallel to the X axis. The slider is shown as making an angle (B,l) with the B vector. If the slider carries the constant current I in the direction shown by the arrows, the force of the field on the slider is in the direction indicated, and from Eq. (341) has the value

$$f \text{ (dyne-sevens)} = IBl \sin (B,l), \tag{341}$$

in which l is the length of the slider, and

(B, l) is the angle between the direction of the slider and the B vectors.

The slider has been illustrated as moving along the rails a distance s from an initial position AC to a final position A_1C_1. The rails may make any angle (s,n) with the direction of the force (the X axis) and therefore the slider in moving along the rails moves the distance s, not in the direction of the force, but in a direction making the angle (s,n) with the force. Therefore the expression for the work done by the force on the slider is

$$W \text{ (joules)} = fs \cos (s,n) = IBls \sin (B,l) \cos (s,n). \quad (426)$$

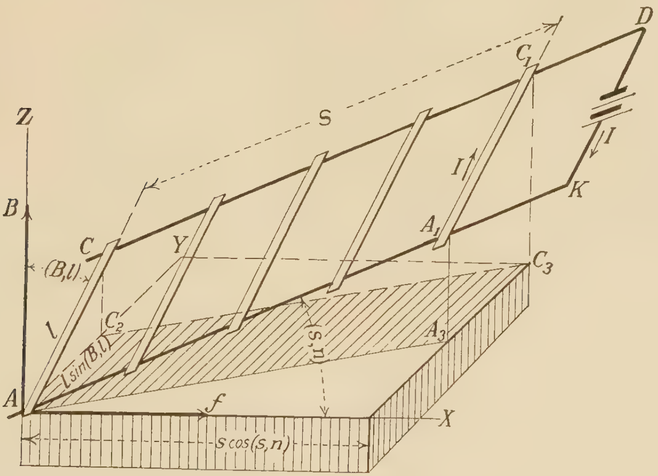

FIG. 239.—Work done in sliding a conductor.

In Fig. 239, the projection $AC_2C_3A_3$ of the surface swept over by the slider has been shown on a plane perpendicular to the B vectors, the XY plane. An examination of the figure will disclose the fact that $l \sin (B, l)$ is the projection AC_2 of the length l on the XY plane, and $s \cos (s, n)$ is the projection of s on the X axis, or it is the perpendicular distance between the initial and final projections of AC on the XY plane.

Therefore it follows that:

a. The expression $sl \sin (B, l) \cos (s, n)$ represents the projected area $AC_2C_3A_3$ upon a plane perpendicular to the B vector of the surface area a swept over by the slider.

b. The expression $Bsl \sin (B, l) \cos (s, n)$ represents the magnetic flux Φ_s over the surface swept over by the slider during its motion.

c. The whole right member of the equation for the work is simply the product of the current I in the moving wire, times this flux Φ_s.

That is to say:

283b. WORK DONE IN MOVING A WIRE.—The work done by the forces of the magnetic field upon a wire which carries a constant current I and moves from one position to another is equal to the product of the current times the magnetic flux over the surface swept over by the wire in its movement.

$$W \text{ (joules)} = I\Phi_s \text{ (amperes, webers)}, \tag{427}$$

$$W \text{ (joules)} = I \int^{\text{swept surface}} B \cos (B,n) da. \tag{428}$$

283c. The Case of a Wire of Any Configuration.—For the simplest derivation of the equations for the work, we have assumed a straight conductor with sliding contacts to move from one position to another, keeping always parallel to its initial position. We have further assumed that the surface swept over by the conductor lies in a region of the field in which the magnetic flux density is uniform. Let us now consider the work done by the forces of the field when a conductor carrying a constant current moves in any manner whatsoever in a field in which the flux density is not uniform but varies from point to point. The movement in question may be the movement of a conductor which makes sliding contact with other conductors of the circuit, as in Fig. 240, or it may be the bending or the stretching of the conductors, as in Fig. 241, or it may be the rotation or translation of the circuit as a rigid body, as in Figs. 217 and 242.

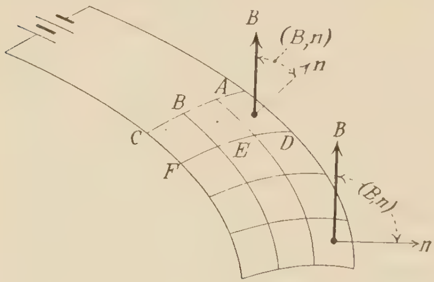

FIG. 240.—Work done on a sliding conductor.

To calculate the work done in this case by the forces of the field, we may proceed as follows. Imagine marks to be placed on the conductor dividing it into many short portions of length dl. Now imagine the conductor to move from any position ABC

(Fig. 240) to a nearby position $ADEFC$. As it moves, each differential length sweeps over a small area da.

From a reconsideration of the arguments by which Eq. (427) was derived, it may be seen that the work which will be done by the force on a short length dl of a conductor as the length dl moves a short distance ds, sweeping over the small area da, will be equal to the product of the current times the magnetic flux, $d\Phi_s$, over the differential area da.

Since the work done by the forces of the field on the entire conductor is the algebraic sum of the amounts of work done by the force on each differential length as the conductor moves by differential steps from the initial to the final position, it follows that the total work will be found by the integrating operation

$$W(\text{joules}) = \Sigma I d\Phi_s = I\int d\Phi_s. \tag{429}$$

By adopting the conventions of the next section, it may be shown that the value of this expression for the work is equal to the current times the **net** magnetic flux over the area swept over.

283d. Conventions Relating to the Algebraic Signs of W, (B, n), I and Φ.—It is to be noted that the work done by the forces of the magnetic field upon a short conductor during motion, may be a positive or a negative quantity, positive if the conductor has moved with the force, and negative if the conductor has been pushed by some other agency in a direction opposite to the force. In the case of a long curved conductor, the work may be positive for some portions of the conductor and negative for other portions. It remains to adopt such comprehensive conventions for the algebraic signs of the factors (B, n), I, and Φ entering into the expressions for work that Eqs. (427) to (429) will automatically give the **net value** and the correct sign for the work W. Let us adopt the following notation and conventions.

Let da represent the area of an elementary portion of the surface swept over by the moving wire.

(B,n) represent the angle between the B vector and the arrow direction along the normal to the elementary portion of the surface swept over.

Φ_s represent the magnetic flux in the arrow direction across the surface swept over.

Φ represent the flux in the arrow direction across any surface bounded by the circuit of which the moving wire is a part.

Φ_u and Φ_0 represent the fluxes in the arrow directions across (any) surfaces bounded by this circuit in its **ultimate** and **initial** positions, respectively.

The convention relating the arrow direction through a circuit to the arrow direction around the circuit is the general right-hand screw convention of Sec. 225. The conventions relating the algebraic signs of (B,n), I, and Φ to the arrow directions have been stated in Secs. 277 and 198. The only new convention needed is a convention defining the arrow direction across the **surface swept over.** In the more general cases of circuit move-

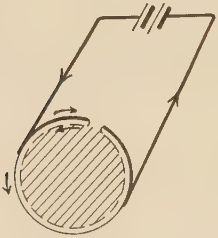

Fig. 241.—Work done on a shifting conductor.

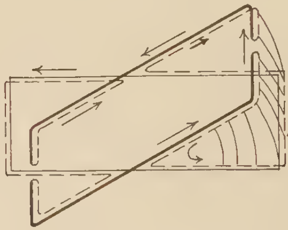

Fig. 242.—Work done on a rotating coil.

ment illustrated in Figs. 240, 241, 242, and 217, the surface swept over by the moving conductor (the cross-hatched surface) is not bounded by a circuit. Therefore, to have a definite relation between the arrow direction across this surface and the arrow direction (for current) around the circuit, this **swept-over surface** must be bounded by an imaginary circuit, by the adoption of the following convention patterned after the convention found so useful in Sec. 261*d*.

283e. Convention Defining the Arrow Direction across a Swept-over Surface.—*The circuit in its ultimate position (after the motion) may be regarded as the magnetic equivalent of the circuit in its "initial" position after it has been cut in one or more places and has had connected in series with it at each cut an added loop which forms the contour or boundary of a swept-over surface. The "added" loops are to be so connected to the initial circuit that*

the current in them flows in such a direction as to wipe out the current in the initial position and to leave the current in the ultimate position. (The added loops are indicated by the dashed lines of the figures.)

Each swept-over surface is now bounded by an added loop with the arrow direction around each loop known. The arrow direction across the swept-over surface bounded by any loop is now defined to be related to the arrow direction (for current) around the loop by the right-hand screw convention (Sec. 225).

The construction called for by these conventions is illustrated in Fig. 243 for the case of the straight slider and also in Figs. 217, 241, and 242 for the other types of motion. In these figures, the initial position of the circuit is indicated by the heavy lines, the ultimate position, by the light lines; the swept-over surface by the cross-hatching; and the added loops by the dotted lines bounding the cross-hatched areas.

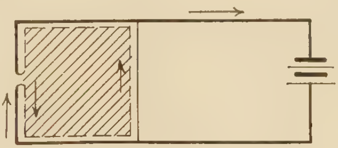

FIG. 243.—Coil with added loop.

By applying Fleming's right-hand rule to these circuits to determine the direction of the forces on the different portions of the wire and the algebraic sign of the work done when the circuits are imagined to move in any imaginable way in known magnetic fields, the reader may verify the statement that, under the above conventions, Eqs. (427) to (429) will always give the net value and the correct sign for the work done on the wire by the forces of the field.

283f. Work in Terms of Increment in Flux-linkage.—In the above equations for the work, Φ_s represents the flux over the surface swept over by the wire. Now an examination of Figs. 217, and 241 to 243 will show that the following three surfaces always form a completely closed surface:

a. The surface swept over by the circuit in its motion.

b. Any convenient surface or cap of which the circuit in its ultimate position is the boundary.

c. Any convenient surface of which the circuit in its initial position is the boundary.

Let Φ_s, Φ_u, and Φ_o represent the respective values of the magnetic flux over these three surfaces. A further examination of the figures will show that if the positive direction for Φ_u is outward from the enclosed space the positive direction for both Φ_o and Φ_s is inward, or vice versa. Therefore, since the flux outward over any closed surface is zero

$$(\Phi_o + \Phi_s) + (-\Phi_u) = 0,$$

or

$$\Phi_s = \Phi_u - \Phi_o = \Delta\Phi. \qquad (430)$$

That is to say, Φ_s, the flux over the surface swept over, may be computed by computing the flux Φ_u over any surface whatsoever which is bounded by the circuit in its ultimate position and subtracting from this the flux Φ_0 over any surface bounded by the circuit in its initial position. Accordingly, Eq. (427) may be written in the following alternative form:

$$W \text{ (joules)} = I\,(\Phi_u - \Phi_0) = I\Delta\Phi \text{ (ampere, webers)}. \qquad (431)$$

If the circuit which moves in the magnetic field is not in the form of a single turn forming the boundary of a simple surface but is in the form of a coil of many turns of wire occupying considerable volume, the work Eq. (431) may be written

$$W = I(\Delta\Phi_1 + \Delta\Phi_2 + \Delta\Phi_3 +), \qquad (432)$$

in which $\Delta\Phi_1$, $\Delta\Phi_2$, etc. represent the increments in the fluxes over the surfaces bounded by turn No. 1, turn No. 2, etc.

If there are N turns, and if they are all so close together that they sweep over substantially the same surface, the work done is

$$W = IN\Delta\Phi_1. \qquad (433)$$

But from the definition of flux-linkage (Sec. 278) the expressions $\Delta\Phi_1 + \Delta\Phi_2 + \Delta\Phi_3 +$ and $N\Delta\Phi_1$ are expressions for the **increment in the flux-linkage** of the multiturn coil. Consequently, the most general relation for the work done on a coil of any kind during its motion may be thus stated:

283g. WORK DONE UPON A MOVING COIL.—The work done by the forces of the magnetic field upon a coil which carries a constant current I and moves from one position to another is equal to the product of the current times the INCREMENT in the flux-linkage of the coil.

$$W \text{ (joules)} = I(\Lambda_u - \Lambda_0) = I\Delta\Lambda \text{ (amperes, webers)}. \qquad (434)$$

284. Power Delivered by the Forces of the Magnetic Field (DEDUCTION).—If both members of the general equation for the work done by the forces of the magnetic field are divided by the time Δt taken to move the circuit from one position to the next, the equality assumes the form

$$\frac{W \text{ (joules)}}{\Delta t \text{ (seconds)}} = I \frac{\Delta\Lambda}{\Delta t}.$$

The left member is the average rate over the interval Δt at which work is done by the forces. If the time interval is very small (infinitesimal), the average rate over the interval is the instantaneous value of the power P delivered by the forces. Accordingly we draw the conclusion:

The power delivered by the forces of the field to a moving circuit carrying a current I is equal to the product of the current times the time rate of increase of the flux-linkage of the circuit.

$$P \text{ (watts)} = I \frac{d\Lambda}{dt} \frac{\text{(amperes, webers)}}{\text{(seconds)}}. \tag{435}$$

285. Force and Torque Equations in Terms of Magnetic Flux (DEDUCTIONS).—Suppose a circuit carrying a current I moves in a magnetic field as a rigid body with a pure motion of translation through the short distance Δx. The work done is

$$W \text{ (joules)} = I\Delta\Lambda. \tag{434}$$

Dividing both members by Δx, this equality becomes

$$\frac{W}{\Delta x} = I \frac{\Delta\Lambda}{\Delta x}.$$

But the left member is the average value of the component in the direction Δx of the force acting on the circuit; therefore, we have the following rule:

285a. Force on a Circuit in the Direction of Translation.—*In the case of the translation of a circuit parallel to itself, the component f of the force tending to move the translated circuit in the direction of the translation dx from the initial to the final position is equal to the current times the rate of increase of the flux-linkage with motion in the direction dx.*

$$f_x \text{ (dyne-sevens)} = I \frac{d\Lambda}{dx} \frac{\text{(amperes, webers)}}{\text{(centimeters)}}. \tag{436}$$

If one portion of a circuit is provided with sliding contacts and slides with a pure motion of translation over a fixed portion, the above expression gives the component of the force on the moving member in the direction dx.

Suppose a circuit carrying a current I in a magnetic field rotates as a rigid body about a fixed axis through the small angle $\Delta\theta$. The work done is

$$W(\text{joules}) = I\Delta\Lambda. \tag{434}$$

Dividing both members by $\Delta\theta$, this equality becomes

$$\frac{W}{\Delta\theta} = I\frac{\Delta\Lambda}{\Delta\theta}.$$

But the left member is the torque or turning moment τ of the forces of the field about the fixed axis; therefore, we have the following rule:

285b. Torque on a Coil.—*The torque or turning moment τ of the forces on a coil about a given axis and in the direction of increase of θ is equal to the current times the rate of increase of the flux-linkage with increase in θ. (θ is to be expressed in radians.)*

$$\tau\ (\text{dyne-seven, cm.}) = I\frac{d\Lambda}{d\theta}\ \frac{(\text{amperes, webers})}{(\text{radians})}. \tag{437}$$

286. Correlation of Motor and Generator Action in Moving Coils. A Summary of the Energy Transformations.—In the derivation of the expressions for the work done by the forces of the magnetic field upon a moving circuit, two assumptions have been made, and the question has not been raised as to the source from which the energy is obtained. It has been assumed:

First, that the component in the direction of motion of the force on the moving conductor in any given position is the same as on a stationary conductor in the same position.

Second, that the current I and the flux densities of the field remain constant during the motion of the conductor.

We now propose to inquire as to measures which must be taken to keep the current constant, as to the source of the energy, and as to the process by which the energy is obtained. During this inquiry, the two assumptions will be examined.

A brief and quite general explanation of the process by which energy is supplied to a motor coil while it is doing work by moving in a magnetic field is contained in the expressions which have been obtained for the power of a moving coil and for the motional electromotive force generated in it.

The power, or the rate at which mechanical work is delivered by the moving wire, is equal to the current times the rate of increase of the flux-linkage of the coil.

$$P \text{ (watts)} = I\frac{d\Lambda}{dt}. \tag{435}$$

That is, if the current is in the arrow direction and the flux-linkage is increasing (in the arrow direction), the coil acts as a motor coil—it does mechanical work.

On the other hand, the motional e.m.f. induced in the arrow direction is the **negative** of the rate of increase of the flux-linkage of the coil.

$$e \text{ (volts)} = -\frac{d\Lambda}{dt}. \tag{420a}$$

That is to say, the motional electromotive force generated in a motor coil is always in a direction to reduce the current in the coil. Therefore, if the current is to be maintained unchanged in value during the motion of the coil, the electromotive force of the generator which is supplying the coil with current must be **greater than** the electromotive force required when the coil is stationary by $d\Lambda/dt$ volts. Under these conditions, the generator supplies the moving coil with the same current as the stationary coil, but at this greater electromotive force. It, therefore, supplies more energy than is expended as I^2R loss in the coil at the excess rate of

$$I\left(IR + \frac{d\Lambda}{dt}\right) - (IR)I = I\frac{d\Lambda}{dt} \text{ watts.}$$

This additional power delivery of the generator is exactly equal to the mechanical work done per second by the moving motor coil and accounts for it.

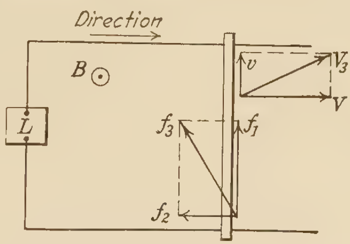

FIG. 244.—The force and velocity parallelograms for electrons in a slider.

The above account is quite conclusive but it lacks detail, because the details are all contained in the study leading up to the final formulas for work and motional electromotive force. It is worth while to review in greater detail the part played by the forces of the magnetic field in these energy transformations.

Consider the simple case shown in Fig. 244, in which a slide AC slides to the right with a velocity of V centimeters per second along the two conducting rails DE and GK. Assume that the rails lie in the plane of the page in a uniform magnetic field, the B vectors pointing vertically upward from the plane of the page. The rail circuit is bridged at one end by the slider and at the left by a device L. In some cases L may be a battery or generator to send current through the slider, thus driving the slider as a motor. In other cases L may represent a resistor to utilize the energy developed by the slider when some external force pushes the slider along, thus converting

it into a generator. Let the direction indicated by the arrow be selected as the arrow direction around the circuit. Let q represent the quantity of electricity (free electrons) per unit length of the slider.

From Eq. (349), the mechanical force on the electrons in the slider of length l due to the velocity V is

$$f_1 = qlVB.$$

This force is in the direction of the length of the slider, and it tends to cause the electrons to circulate in the counter arrow direction around the circuit, as shown on the diagram.

286a. The Case of the Generator.—Let us first suppose that the circuit is completed at L through some inert power-utilizing device such as lamps or heating coils; and that, under the above driving force, the electron atmosphere does circulate, moving through the slider with a velocity v. Then in 1 second the points of application of the driving force f_1 move in the direction of the force v centimeters. Therefore the work done per second by the force f_1, or the power expended in the resistance of the circuit and of the device L, is

$$P = qlVBv.$$

(It may be noted that since qv is the current in the circuit, this expression is the equivalent of the customary expression for power, namely $P = fV = BIlV$.)

The question now arises—What is the source of the energy which is expended as I^2R loss in the circuit in forcing the electrons to drift through it? When the electrons drift through the slider with the velocity v, the electrons in each differential length are subject to a force, $vBqdl$, tending to push the electrons out through the surface of the slider in the direction f_2. The electrons are held in the slider by the surface forces previously discussed, and this force is transmitted to the rigid slider. The total force f_2 distributed along the entire length of the slider is

$$f_2 = qlvB.$$

(It may be noted that since qv is the current I, this expression is the equivalent of the customary expression for the force, namely IBl.)

Therefore if the slider moves to the right, some external agency must be exerting a force equal and opposite to f_2. Since the slider moves to the right with the velocity V, the work done per second, or the power expended upon the slider by the external agency is

$$P = qlvBV.$$

This is seen to equal the power expended in the circuit. That is, the power expended in forcing the electrons to circulate is exactly equal to the power delivered by the external agency which pushes the slider along.[10]

[10] It may be well to call attention to the fact that our account of the energy transformations is not complete. We have not considered the effect of the motion upon the field circuit which gives rise to the magnetic field. Reflec-

In the velocity and force parallelograms of Fig. 244, the following relation always exists between the forces and the velocities.

$$\frac{f_2}{f_1} = \frac{v}{V}.$$

From the diagrams, it follows that the resultant force of the field f_3 is always at right angles to the resultant velocity of the electrons V_3; and, therefore, the resultant force of the magnetic field does no work upon the moving electrons. The field plays a rôle in electrical mechanisms, analogous to the rôle of the framework supporting a bell crank. If we can exert a horizontally directed force f_1 (Fig. 245), this may be changed in direction to balance a vertically directed force f_2 by the bell crank illustrated, provided a framework is available to exert a balancing reaction against the bearing of the bell crank. The reacting force f_3 exerted by the framework is at right angles to the direction of motion and does no work. In somewhat analogous fashion, the magnetic field enables us to change the direction of application of a force. In the electric generator, the steam engine pushes the slider broadside on, and as a result the electrons in the slider are subjected to a force along the length of the slider—a force at right angles to the force applied to the slider. The initial and the final forces, corresponding to f_1 and f_2 of the bell crank, are the mechanical force applied to the conductor and the frictional force of ohmic resistance. The intermediate forces, corresponding to the reaction of the framework and the intermolecular forces of tension and compression in the arms of the bell crank, are the reacting force of the magnetic field upon the moving electrons, and the electrostatic forces between electrons and nuclei by which the force is transmitted through the electron atmosphere.

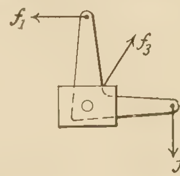

FIG. 245.—Bell crank relations.

As stated above, the resultant force of the magnetic field is at right angles to the direction of motion of the electrons and does no work. This does not invalidate the formulas derived at the beginning of this chapter for the work done by the forces of the magnetic field upon a moving conductor carrying a current. It is well to recognize, however, that the force used in deriving these formulas is not the resultant force of the field upon the electrons, but simply that component which makes possible a mechanical force at right angles to the length of the wire; the other component makes possible the electromotive force along the wire. In other words, the motional electromotive force induced in a moving conductor and the mechanical force upon a conductor carrying a current are but different aspects of the force acting on electricity in motion in a magnetic field.

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tion will indicate that an e.m.f. will be generated in the field circuit, and therefore other energy transformations will accompany the movement of the slider. The more complete account will follow in the next chapter. It supplements but does not invalidate this discussion.

286b. The Case of the Motor.—Let us now briefly consider the case in which the circuit of Fig. 244 is completed at L through a battery which is so connected that it drives the electron atmosphere against the driving forces in the slider with the velocity v. That is, the current I is now a negative quantity. The only respect in which the force and velocity parallelograms will now differ from those of Fig. 244, is that the vectors representing the velocity v and the force f_2 on the electrons due to this velocity will both be reversed. (The direction of the resultants f_3 and V_3 will be altered to correspond.) The field now exerts a force in the direction of motion of the slider, and the slider may be used to do mechanical work upon some external agency, for example, to pull a car up a grade.

The motional e.m.f. generated in the slider has the same value and direction as in the former case. Hence we see that to cause a given current to flow against the motional e.m.f., the e.m.f. of the battery must be greater than the IR voltage necessary when the circuit is stationary, by the amount of the motional e.m.f. At this higher e.m.f. the power output of the battery is greater than the power expended as I^2R loss in the ohmic resistance of the circuit by the power expenditure of the forces of the field upon the moving slider.

287. Exercises.

1. In the circuit for the calibration of a ballistic galvanometer, Fig. 228, Sec. 271, R_2 was 50 ohms, R_1 was 1 ohm, and R_3, including the resistance of the galvanometer, was 200 ohms. With a battery terminal e.m.f. of 6 volts, and the switch closed for 0.02 second, the deflection of the galvanometer was found to be 21.3 divisions on the scale. Determine the "voltage-impulse constant of this galvanometer for a 200-ohm circuit."

2. At a certain place the magnetic flux density of the earth's field has the value 0.6 eighth-weber per square centimeter and is directed downward toward the north (approximately) at an angle of 30 degrees with the vertical. A railroad train is moving at the rate of 60 miles an hour. What motional e.m.f. is generated in each car axle? The distance between rails is 54.5 inches. Show that the direction of travel of the train is immaterial.

3. In the earth's field described in exercise 2, a flip coil was placed in a horizontal plane and suddenly rotated about its axis through an angle of 180 degrees. The coil contained 2500 turns each bounding an area of 8 square centimeters. What voltage impulse was produced in the coil? What deflection would be produced in the ballistic galvanometer of exercise 1 if it were in the circuit with the flip coil during this period (total circuit resistance of 200 ohms)?

4. The flip coil of exercise 3 connected to the galvanometer of exercise 1 was placed at a chosen point in the magnetic field of a given coil, with the normal axis of the flip coil pointing in turn to the north, to the east, and straight upward. In each position it was flipped through 180 degrees

and the readings of the galvanometer were 22.8 divisions positive swing, 14.7 divisions negative swing, and 8.6 divisions positive swing, respectively. Determine the magnitude and direction of the flux-density vector at this point in the field. (The specified direction along the normal axis is related to the specified direction around the coil by the right-hand screw convention. The coil is connected to the galvanometer with proper polarity so that a positive voltage impulse in the coil gives a positive deflection on the galvanometer.)

5. A coil of 75 turns was connected to the galvanometer of exercise 1 with a circuit resistance of 200 ohms. The coil was placed over the end of a bar magnet having a cross-section of 1.8 square centimeters and then suddenly removed to a considerable distance. The deflection produced on the ballistic galvanometer was 32.6 divisions. Compute the values of the magnetic flux over the section of the magnet, and of the flux density in the iron?

6. Let the train of exercise 3 travel toward the east, and let a current of 2 amperes pass through the car axle from one rail to the other toward the south. Is the mechanical work done by the forces of the magnetic field on the moving axle a positive or a negative quantity? How much work is done while the train travels 1 mile?

7. A pole-line circuit 60 meters long consists of copper conductors 1 centimeter in diameter spaced 30 centimeters between axes. The circuit carries a current of 220 amperes. How much work is done by the forces of the magnetic field on the conductors as they are separated to a distance of the 18 inches?

8. A straight conductor extending north and south is moving bodily to the east at the rate of 60 centimeters per second. At the same time let us assume a definite group of electrons (total charge q) to be moving along the conductor toward the south at the rate of 20 centimeters per second. Calculate the component of the force along the conductor on this charge, the component perpendicular to the conductor, and the magnitude and direction of the resultant force. Also find the resultant velocity and compare its direction with that of the resultant force.

9. In a certain four-pole generator, the magnetic flux passing from a pole face across the air gap into the armature is 0.04 weber. With the armature running at 1800 r.p.m., what is the average voltage induced in a single armature conductor during the time it sweeps past the face of one pole? If there are 46 conductors connected in series between positive and negative brushes, what is the terminal voltage of the machine?

10. A concentrated circular coil of 10 turns and radius 10 centimeters is revolved on a horizontal axis pointing east and west at the rate of 10 r. p. s. The flux-density vector of the earth's field is directed north and makes an angle of 60 degrees with the horizontal. The value of the flux density is 0.42 eighth-weber per square centimeter.

a. What is the equation of the e.m.f. generated in the coil in terms of the maximum value and the angle which the position makes with the position of zero e.m.f.?

b. What is the maximum value of the e.m.f., and what is the position of the coil when the e.m.f. has its maximum value?

c. What is the average e.m.f. generated in the coil?

11. If the dynamo described in exercise 9 operates as a motor at the same speed, and if each of the 184 armature conductors is carrying a current of 30 amperes, compute the mechanical power developed. What torque is developed? Express the result in dyne-seven-centimeters and in pound-feet.

CHAPTER XIII

ELECTROMOTIVE FORCES INDUCED IN STATIONARY CIRCUITS

288. Electromotive Forces of Mutual and of Self-inductance.—The discussion of the properties of conductors has dealt with the electromotive force necessary to keep electricity circulating in an **unvarying** manner against the frictional forces of resistance. The electromotive force of resistance in any portion of the circuit multiplied by the current flowing is equal to the rate at which energy is dissipated in the form of heat in that portion of the circuit.

The chapter pertaining to motional electromotive forces has dealt with the electromotive force **generated** in a circuit which moves relative to an unvarying magnetic field. The motional electromotive force generated in a conductor multiplied by the current flowing is equal to the rate at which mechanical energy is converted into electrical energy in the form of a stream of electrons propelled along the conductor by the **side push** of the magnetic field, or vice versa.

This chapter will deal with the forces involved in the acceleration of electricity. It will deal with the electromotive forces **induced** in circuits, fixed with reference to each other, when the current in one of the circuits varies. These electromotive forces are known as the electromotive forces of **self-inductance** and of **mutual inductance.** We shall find that these electromotive forces multiplied by the currents flowing are equal to the rate at which energy is stored in the magnetic field.

289. Phenomena of Mutual Inductance.—Faraday's discovery,[1] in 1831, that an electromotive force is generated in any circuit

[1] FARADAY: *Experimental Researches*, Vol. I, Pars. 1–40. Faraday's discovery of the e.m.f. of mutual inductance anticipated the independent discovery of the same phenomena by Joseph Henry by only a few months. For some time prior to 1832, Henry, then professor of physics at Albany,

while it is in motion in a magnetic field, was accompanied by the discovery that at the moment the current in a circuit is started or stopped, momentary electromotive forces are induced in other circuits which lie in the magnetic field of the first circuit.

The discovery was made with the circuits illustrated in Fig. 248. Current from a battery B is supplied to a coil P through a switch K. A second coil S has its terminals connected to the ballistic galvanometer G. The two coils may be mounted one within the other, or some distance apart. When the switch K is closed, sending a current through P, a throw of the galvanometer takes place, thus showing that an electromotive force has

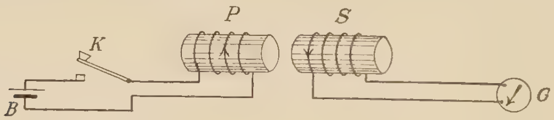

FIG. 248.—Circuits for showing mutual inductive effects.

been induced in the coil S. This electromotive force is transient, for the galvanometer does not give a permanent deflection, but, after oscillating, settles and remains at zero as long as the current through P remains constant. When the battery current is stopped by opening the switch K, an equal throw of the galvanometer takes place, but in the opposite direction. The voltage-impulses induced in S on starting and stopping a given current in P are, therefore, equal, but the electromotive forces are induced in opposite directions. If the connections of the coil P to the battery are reversed, thereby reversing the direction of flow of the current through the coil P, the direction of throw of the galvanometer upon closing or opening the switch K is the reverse of the throw

N. Y., had dwelt on the possibility of obtaining an "electric current from magnetism." In 1832 he observed, by a galvanometer, the current which is induced in a coil wound around the armature of an electromagnet when the current in the primary is started, or interrupted, or when the armature is pulled away from the pole pieces. At about the same time, he studied the effect both of the length of wire and of winding the wire in coils on the brightness of the spark and the severity of the shock which may be obtained upon interrupting the current, thus anticipating Faraday's independent study of self-induction. See HENRY, JOSEPH, *On the Production of Currents and Sparks of Electricity from Magnetism*, Silliman's Am. J. Sci., July, 1832, Vol. XXII, p. 403; also Vol. XXVIII, 1835, p. 327.

caused by the corresponding switching operation with the original battery connection.

An electromotive force which is **induced** in a secondary circuit by reason of a variation in the value of the current in a primary circuit, is called an **electromotive force of mutual inductance.**

Faraday accounted, in a bookkeeping sense, for the electromotive forces of mutual inductance and the motional electromotive forces in the same manner—namely, in terms of the change in the flux-linkage of the circuit. There are two ways in which the flux-linkage of a secondary circuit with the field of a primary circuit may be changed. The secondary circuit may be moved in the field, or the secondary circuit may be held stationary and the flux densities throughout the field may be altered by changing the value of the current flowing in the primary. The quantitative experiments recited in Sec. 292 demonstrate that the voltage-impulses of mutual inductance are likewise equal to the decrease in the flux-linkage of the secondary circuit.

290. Phenomena of Self-inductance.—Three years after Faraday's discovery[1] of the electromotive forces of mutual inductance, his attention was called to the fact that if current from a low-voltage battery (20 volts or less) flows through a wire of short length, it is impossible to obtain an electric shock from the circuit no matter how it may be manipulated. On the other hand, if the battery supplies a current to a longer wire wound in the form of a coil or a solenoid of many turns, a shock is felt each time the circuit is opened, provided the wires on each side of the break are grasped, one in each hand. Another effect (which had long been known) is observed at the same time, namely, a bright spark occurs at the break in the circuit.

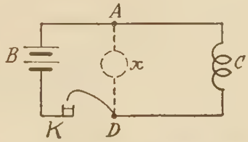

FIG. 249.—Circuit for showing self-inductive effects.

To study these phenomena, Faraday used the circuit shown in Fig. 249. Current from a battery B was supplied through a switch K to the conductor ACD whose properties were to be studied. The circuit was made or was broken at K by a copper wire which was inserted in, or withdrawn from, a small mercury

cup. Observations were made of the brilliancy and volume of the spark or arc at K upon breaking the circuit. Observations were also made of the intensity of the shock at "break" to a person who grasps the wires on each side of the break. If ACD is a short wire no shock is obtained and the spark is almost imperceptible. If ACD is a long wire which has been doubled back upon itself at its midpoint to form a loop having two parallel sides (along which the current flows in opposite directions) separated only by the thickness of the cotton insulation of the wires, the spark, and the shock are still quite weak. If the narrow loop is now opened out so that it bounds a surface of considerable area, the spark and the shock become stronger. If the long wire is now wound in the form of a closely wound coil or a solenoid of many turns, the brilliancy of the spark and the intensity of the shock are greatly increased. Finally, a further increase is observed if the solenoid is provided with an iron core. It is evident that the effects are enhanced by arranging the wire ACD. so that the flux-linkage of the circuit will be as great as possible.

When the galvanometer is connected at X, in parallel with a coil of many turns at ACD, with the switch K closed, the galvanometer assumes a small, steady deflection. Upon opening the switch K, a very large **throw** of the galvanometer occurs. The throw is always in a direction opposite to the steady deflection caused by the battery current. This shows that after the main circuit containing the battery has been opened, and while the current is dropping to zero or the magnetic field is vanishing, a transient electromotive force is induced in the coil in such a direction as momentarily to maintain the current in the coil. This current flows through the galvanometer in a direction which is opposite to the direction of the current previously caused by the battery. Other experiments indicate that when the circuit is closed and while the current in the coil ACD is increasing in value or the magnetic field is building up, a transient electromotive force is induced in the coil which is in opposition to the battery electromotive force, thus causing the current to increase gradually, and not instantaneously, from zero to the final value—the value given by Ohm's law.

These effects are seen to be analogous to the water-hammer effects which are obtained from water circulating in a system of

pipes. We may say that electricity circulates with something akin to **momentum**, and it exhibits effects akin to the effects classed as **inertial** effects in ordinary matter; that is, forces arise during the acceleration and deceleration of the moving stream of electrons. The property which is akin to **mass** is not related in a simple manner to the dimensions of the wire through which the electricity circulates, but is greatly influenced by the manner in which the wire is coiled; it is seen to be closely associated with the flux-linkage of the circuit per unit of current.

An electromotive force which is induced in a circuit by reason of a variation in the value of the current in the circuit is called an **electromotive force of self-inductance.**

Further experiments lead to the conclusion that the electromotive forces of mutual and of self-inductance are to be accounted for in the same manner, namely, in terms of the change in the flux-linkage of the circuit in which the electromotive force is induced. If the turns of a primary and secondary are wound side by side, any change in the primary current is found to induce voltage-impulses of substantially equal values in both the primary and secondary.

291. The Direction of the Induced Electromotive Forces. Lenz's Law (EXP. DET. REL.).—The direction of the electromotive forces of mutual and of self-inductance may be predicted by the application of the comprehensive form of Lenz's law presented in Sec. 280, namely:

The electromotive force which is induced in a body as a result of any variation of the magnetic field with reference to the body, is in such a direction that the current which results tends to prevent the change which occasions the induced e.m.f.

The following is an illustration of the application of the law. If a current is started in the primary of Fig. 248 in the direction shown by the arrow, it builds up a magnetic field. We regard the change in the flux-linkage of the secondary and primary coils as the change which occasions the e.m.fs. of mutual and self-inductance. To retard this change in the flux-linkage, the transient current induced in the secondary must flow around the

secondary in the direction of the arrow on the secondary. Therefore, this arrow shows the direction of the e.m.f. of mutual inductance. Likewise the transient electromotive force induced in the primary must be in direction to subtract from the e.m.f. of the battery supplying the current, or the e.m.f. of self-inductance in the primary must also be in the direction shown by the arrow on the secondary. The application of the law will show that at the interruption of the primary current, the induced e.m.fs. will both be in the opposite direction to that predicted above. That is, the e.m.f. induced in the primary, is in a direction to keep the current from dying out.

292. Quantitative Experiments on Voltage-impulses Induced in a Secondary.

Experiment 1.—Consider the circuit shown in Fig. 250. By means of the reversing switch K, the circuit may be opened and closed and the direction of the current through the primary coil

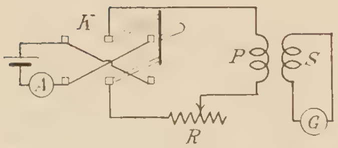

Fig. 250.—Circuit for measuring mutual inductive effects.

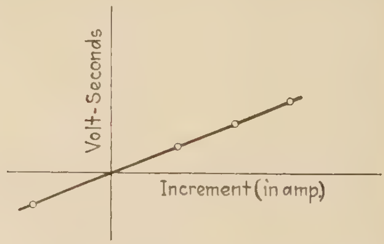

Fig. 251.—Relation between voltage-impulse in secondary and increment in primary current.

P may be quickly reversed. By manipulating the rheostat R, the value of the current in the primary coil may be quickly changed from one value to another. The **change** in the primary current resulting from any manipulation of the switch or rheostat may be determined from the readings of the ammeter A before and after; and the magnitude of the voltage-impulse induced in the secondary coil S may be determined from the throw of the ballistic galvanometer G.

If the field around the coils is free of ferromagnetic material, a plot of the readings resulting from many manipulations of the

switch and rheostat discloses the straight-line relation shown in Fig. 251 and expressed by the following statement.

292a. RELATION OF INDUCED VOLTAGE-IMPULSE TO CHANGE IN CURRENT (EXP. DET. REL.).—**When the current in a primary circuit changes in value, the voltage-impulse induced in any secondary circuit is directly proportional to the decrease in the value of the current in the primary.**

$$\int e_2 dt \text{ (volt-seconds)} = -M\Delta i_1 \text{ (amperes)}. \tag{438}$$

The proportionality constant M appearing in this equation is used so frequently that it has received a name. It is called the (coefficient of) **mutual inductance of the circuits,** and it is invariably represented by the same symbol M. The value of M may be positive, negative, or zero, depending upon the dimensions, number of turns, arrow directions, and relative position of the primary and secondary circuits. We write Eq. (438) with a minus sign and use the word **decrease** in stating the law because we will find that the use of the minus sign enables us to put the definition of M in a more elegant form (see Sec. 298).

Experiment 2.—If, during the course of the tests outlined above, the secondary is replaced by another secondary which differs from the first only in the material, diameter, or cross-sectional shape of the wire, the voltage-impulse is found to be the same as with the original secondary. If, however, any change is made in the form, dimensions, or location of the secondary, the voltage-impulse is different; unless by chance it happens that the changes are mutually compensating in their effects. From this we conclude.

292b. (LAW).—*The electromotive force induced in a secondary circuit by the change in the current in the primary depends upon the form and dimensions of the circuit, and not upon the material and cross-section of the wire.*

Experiment 3.—The experiments of Chap. XI have shown that in a magnetic field free of ferromagnetic materials, the flux density at any point is directly proportional to the exciting current. It follows from this that the changes in the flux-link-

age in the secondary in experiment 1 are directly proportional to the changes in the exciting current.

Consequently, the voltage-impulse induced in a stationary coil is directly proportional to the change in the flux-linkage of the coil.

$$\int e_2 dt = K\Delta\Lambda_2. \tag{439a}$$

The value of the proportionality constant is readily determined by the following experiment.

Let the conditions be as illustrated in Fig. 250 except that the secondary is now to consist of a flip coil (described in Sec. 281) connected by twisted leads to the galvanometer. Set up the flip coil at any point in the field of a primary circuit carrying a given current. Flip the coil through 180 degrees and observe the throw. Now reset the flip coil in its first position. Then suddenly reverse the primary current and observe the throw. The two throws will be found to occur in the same direction and to be equal in value. Now either operation—reversing the primary current, or flipping the secondary—reverses the direction of the flux density vectors with reference to the arrow direction through the secondary, and therefore they both produce the same change in the flux-linkage of the secondary. (In either case, the change in the flux-linkage of the flip coil is from Λ_0, the initial value, to $-\Lambda_0$, or an algebraic increment of $-2\Lambda_0$ weber-turns.)

This experiment demonstrates that a given change in the flux-linkage of a coil induces the same voltage-impulse, whether the change is brought about by motion of the circuit in a magnetic field or by a variation in the strength of the magnetic field. We therefore conclude that the value of the voltage-impulse induced in a stationary circuit by reason of the change in the field strength will be expressed by a formula identical with that previously derived for motional voltage-impulses, namely;

$$\int e_2 dt \text{ (volt-seconds)} = -\Delta\,\Lambda_2 \text{ (webers)}. \tag{439}$$

That is to say,

292c. RELATION OF INDUCED VOLTAGE-IMPULSE TO CHANGE IN FLUX-LINKAGE (Exp. Det. Rel.).—**In all cases in which the flux-linkage of a secondary circuit changes by reason of a change in the current in a neighboring circuit, the algebraic value of the voltage-impulse induced**

in the secondary is equal to the algebraic value of the DECREMENT in the flux-linkage.[2]

293. Quantitative Experiments on Voltage-impulses Induced in the Primary Itself.—Quantitative measurements of the voltage-impulse induced in a coil when its own current is interrupted may be made as outlined in Sec. 290 by connecting a ballistic galvanometer at X in Fig. 249 in parallel with a many-turn coil, ACD. These measurements and others all lead to the conclusion that any variation in the current gives rise to an induced voltage in the primary, and that the laws relating to this voltage are indentically the same as those relating to the voltage induced in other circuits which lie in the field of the primary current. These laws, when reworded to apply to the primary, read as follows:

293a. (EXP. DET. REL.).—**When the current in a circuit changes in value, a voltage-impulse is induced in the circuit which is directly proportional to the DECREASE in the value of the current.**

$$\int e_1 dt \text{ (volt-seconds)} = -L\Delta i_1 \text{ (amperes)}. \qquad (440)$$

293b. (EXP. DET. REL.).—**In all cases in which the flux-linkage of a circuit changes by reason of a change in the current in the circuit, the voltage-impulse induced in the circuit is equal to the DECREASE in the flux-linkage.**

$$\int e_1 dt \text{ (volt-seconds)} = -\Delta\Lambda_1 \text{ (weber-turns)}. \qquad (441)$$

The proportionality constant L in Eq. (440) is called the **self-inductance** of the circuit, or more briefly, the **inductance** of the circuit (see Sec. 298).

294. General Law for the Electromotive Forces of Electromagnetic Induction (EXP. DET. REL.).—In every case of electromagnetic induction we have found that the value of the voltage impulse generated or induced in a circuit is equal to the decrement in the flux-linkage of the circuit.

$$\int e\, dt = -\Delta\Lambda. \qquad (442)$$

This relation holds true regardless of the manner and rate at which the flux changes or of the terminal values of the flux-

[2] This algebraic form of this statement presupposes that the arrow direction for e.m.f. (around the coil) and the arrow direction for flux (through the coil) are related by the right-hand screw convention of Sec. 225.

linkage. Therefore, the time-rate at which the value of the voltage-impulse is increasing, namely, the induced electromotive force in the circuit, must at every instant be equal to the rate of decrease of the flux-linkage of the circuit.

$$e \text{ (volts)} = -\frac{d\Lambda}{dt}\frac{\text{(webers)}}{\text{(seconds)}}. \tag{443}$$

This fundamental relation may be expressed as follows:

294a. GENERAL LAW FOR INDUCED ELECTROMOTIVE FORCES

In all cases in which the flux-linkage of a circuit varies, whether by reason of a change in the current in a neighboring circuit, or by reason of the relative motion of the circuit and a steady magnetic field, or by reason of the motion of a portion of the circuit in the magnetic field set up by the current in the circuit itself, or finally by reason of the variation of the current and magnetic field of the circuit itself, the algebraic value of the electromotive force induced in the circuit is equal to the time rate of decrease of the flux-linkage of the circuit. The arrow direction for voltage (around the circuit) is related to the arrow direction for flux (through the circuit) by the right-hand screw convention of Sec. 225.

The electromotive force induced in a circuit is the line-integral taken once around the circuit, of the electromotive intensity electromagnetically generated or induced in the wire. If F represents the induced electromotive intensity at any point along the wire and (F, l) represents the angle between the vector F and the direction of the wire at the point, the general law may be written in the form

$$\int F \cos\ (F,l)dl \text{ (volts)} = -\frac{d\Lambda}{dt}\frac{\text{(webers)}}{\text{(seconds)}}. \tag{444}$$

The general law when written in this form is frequently called the **electromotive force law of circuitation.**

295. General Electromotive Force Law in Vector Notation.—Imagine a one-turn circuit bounding a small plane area which is perpendicular to the B vectors passing through it. For the e.m.f. in this single-turn circuit, Eq. (444) may be written

$$\int F \cos\ (F,l)dl = -\frac{d\Phi}{dt}.$$

Dividing both members of this equation by the area a enclosed by the boundary around which the line integral of the intensity is taken

$$\frac{\int F \cos\ (F,l)\ dl}{a} = -\frac{1}{a}\frac{d\Phi}{dt} = -\frac{d}{dt}\frac{\Phi}{a}.$$

If the area a is infinitesimally small, the right member represents the rate of decrease of the flux density B at a point, and by definition (see Sec. 250) the left member is the curl of the vector F at the point.

Whence
$$\text{curl } \mathbf{F} = -\frac{d\mathbf{B}}{dt}. \tag{445}$$

296. Switching Method of Measuring Magnetic Flux Density. We originally defined and measured flux density by means of mechanical force. In the last chapter (Sec. 281) it was found possible to measure and define B in terms of the voltage-impulse induced in a flip coil. We have now found that the creation of a magnetic field induces a voltage-impulse in any circuit which is equal to the decrease in the flux-linkage of the circuit. It follows that the flux densities at points in a field which may be switched off and on may be obtained from measurements of the voltage-impulses induced in small coils when the field is created. The derived formula for computing the flux densities from these measurements is

$$B(\text{webers per sq. cm.}) = -\frac{\int e\,dt}{a}\frac{(\text{volt-seconds})}{(\text{cm.})} \tag{446}$$

This formula may be worded as follows:

296a. *The magnetic flux density at a point in a magnetic field is a vector quantity whose magnitude is equal to the voltage-impulse induced (during the creation of the field) in a one-turn test coil bounding a plane area of* 1 *square centimeter, centered at the point. The test coil is to be so held that the voltage-impulse is a maximum. The B vector is directed along the normal to the area in a direction opposite to that in which a right-hand screw would advance if rotated in the direction of the induced e.m.f. around the coil.*

297. Faraday's Laws for the Electromotive Forces of Self- and Mutual Inductance.—Faraday's laws of electromagnetic induction as embodied in Eqs. 440 and 438 are put in a more convenient form for application to circuit equations if they express the instantaneous values of the induced electromotive forces rather than the integrated values. By equating the derivatives of both sides of Eq. (440) and then of Eq. (438) the laws expressed in Secs. 293a and 292a assume the following alternative forms.

297a. LAW OF SELF-INDUCTANCE.[3]—**If the current in a circuit varies, an electromotive force is induced in the circuit which is directly proportional to the rate of change of the current; from Lenz's law this e.m.f. is in such a direction as to oppose the change taking place in the current.**

$$e \text{ (volts)} = -L\frac{di}{dt}\left(\text{henries}\ \frac{\text{amperes}}{\text{seconds}}\right). \tag{447}$$

297b. LAW OF MUTUAL INDUCTANCE.[3]—**If the current in a primary circuit varies, an electromotive force e_2 is induced in any secondary circuit; this e.m.f. is directly proportional to the rate of change of the current in the primary circuit, and from Lenz's law, is in such a direction as to oppose the change of magnetic flux threading the secondary.**

$$e_2 \text{ (volts)} = -M\frac{di_1}{dt}\left(\text{henries}, \frac{\text{amperes}}{\text{seconds}}\right). \tag{448}$$

298. Mutual and Self-inductances (DEFINITIONS).—As previously stated, the proportionality constant L appearing in Eq. (440) or its equivalent Eq. (447) is called the coefficient of self-induction, or, in briefer fashion, simply the **self-inductance** of the circuit, while the constant M appearing in Eqs. (438) and (448) is called the coefficient of mutual induction or the **mutual inductance** of the specified circuits.

Since in the equations for the electromotive forces of self-inductance, the induced e.m.f. e and $\frac{di}{dt}$ always have opposite algebraic signs, the negative sign has been arbitrarily written into Eqs. (440) and (447) in order that the constant L may always be a positive quantity.

In the equations for the electromotive forces of mutual induction we are dealing with the e.m.f. in the secondary and the current in a primary. Since the arrow directions in these two circuits are arbitrarily chosen, no fixed relation will exist in all cases between the algebraic signs of e_2 and $\frac{di_1}{dt}$. By chance the arrow directions in the two circuits may have been so chosen that the two quantities have like signs, or by chance they may have unlike signs. Therefore, we may write Eqs. (438) and (448) with a $\pm$ sign on the right side, or we may write the equations with a fixed sign and then assign to the coefficient M a positive

[3] These statements may be very inexact if the field of the conductors contains ferromagnetic materials.

or a negative sign, as may be required correctly to express the relations in any specified case. This latter plan will be followed, and the negative sign will be arbitrarily used in order that the equation and definition for mutual inductance may be similar in form to those for self-inductance.

Equations (447) and (448) when rewritten in a form explicitly to define L and M become

$$L \text{ (henries)} = -\frac{e_1}{\frac{di_1}{dt}} \frac{\text{(volts)}}{\text{(amp. per sec.)}} \quad \text{(defining } L\text{)} \qquad (447a)$$

$$M \text{ (henries)} = -\frac{e_2}{\frac{di_1}{dt}} \frac{\text{(volts)}}{\text{(amp. per sec.)}} \quad \text{(defining } M\text{)} \qquad (448a)$$

thus yielding the following definitions.

298a. Self-inductance (DEFINITION).—*The positive constant which for a given circuit is equal to the e.m.f. of self-induction divided by the time rate of decrease of current is called the "self-inductance" of that circuit.*

298b. Mutual Inductance (DEFINITION).—*The constant (positive, negative, or zero) which for a given pair of circuits is equal to the e.m.f. of mutual induction in the arrow direction in one circuit divided by the time rate of decrease of the current in the arrow direction in the other circuit, is called the "mutual inductance" of the two circuits.*

The e.m.f. induced in a primary is written in terms of current and self-inductance in Eq. (447) and in terms of flux-linkage in Eq. (443)

$$e_1 = -L\frac{di_1}{dt} \qquad (447) \qquad\qquad e_1 = -\frac{d\Lambda_1}{dt}. \qquad (443)$$

Whence

$$L\frac{di_1}{dt} = \frac{d\Lambda_1}{dt}.$$

or

$$L \text{ (henries)} = \frac{d\Lambda_1}{di_1} = \frac{\Lambda_1}{I_1} \frac{\text{(webers)}}{\text{(amperes)}}. \qquad (449)$$

This **derived** general formula for the value of the self-inductance in terms of flux-linkage may be stated in the following words:

298c. Derived Formula for Computing Self-inductance.—*The self-inductance of a circuit is equal to the rate of increase of the flux-linkage of the circuit with increase in current, or it is equal to the flux-linkage of the circuit due to unit current in the circuit.*

In like manner, we may obtain the formula

$$M \text{ (henries)} = \frac{d\Lambda_2}{di_1} = \frac{\Lambda_2 \text{ (webers)}}{I_1 \text{ (amperes)}}, \tag{450}$$

which may be thus stated.

298d. Derived Formula for Computing Mutual Inductance.—*The algebraic value of the mutual inductance of two circuits is equal to the algebraic value of the flux-linkage in the arrow direction in the secondary due to unit current in the arrow direction in the primary.*[4]

298e. Unit of Inductance (Definition).—*The unit of inductance (whether self- or mutual) is termed the "henry." The mutual inductance of two circuits is 1 henry if a change in the primary current at the rate of 1 ampere per second induces an e.m.f. of 1 volt in the secondary. A circuit has a self-inductance of 1 henry if a change in the current at the rate of 1 ampere per second gives rise to an induced e.m.f. of 1 volt, or if a current of 1 ampere causes a flux-linkage of 1 weber.*

$$\text{henry} = \frac{\text{volt-sec.}}{\text{amperes}} = \frac{\text{volts}}{\text{amp. per sec.}} = \frac{\text{webers}}{\text{amperes}}. \tag{451}$$

298f. Mutual Inductance in Both Directions the Same.—In Fig. 248, let twisted lead wires be used from the battery to the

[4] It may be well to restate the conventions which must be taken into consideration in finding the sign of M. These conventions are:

1. A given direction around each circuit is arbitrarily selected as the arrow direction.
2. Currents are designated as positive quantities when they flow in the arrow direction.
3. The arrow directions for flux through the circuits are related to the arrow directions around the circuits by the right-hand screw convention.

To determine the sign of M, we calculate the flux-linkage of circuit 2 with the field set up by unit positive current in circuit 1. M has the same algebraic value as this flux-linkage.

The circuits customarily treated in texts are so elementary that the necessity for clear-cut conventions as to signs is not recognized, and M is rarely regarded as anything but a positive quantity.

coil P, and from the coil S to the galvanometer. Let the two coils be mounted in any two fixed positions, and let the proportionality constant M be determined for the circuit when connected as shown in the figure, that is, with the battery connected to P and the galvanometer connected to S. Then let the connections of P and S be interchanged; that is, let the twisted leads be shifted to connect the battery in series with S and the galvanometer in series with the coil P. Let the proportionality constant M be determined for this connection of the coils. The values are found to be identical.

From this we conclude that the proportionality constant M has the same value in the two equations

$$\int e_2 dt = -M\Delta i_1$$

$$\int e_1 dt = -M\Delta i_2$$

$$\frac{\int e_2 dt}{\Delta i_1} = \frac{\int e_1 dt}{\Delta i_2} = M \tag{452}$$

The voltage-impulse induced in any circuit A per unit increment in the current in circuit B is equal to the voltage-impulse induced in circuit B per unit increment in the current in circuit A.

We will now illustrate the application of the constant, self-inductance, in predicting the currents which flow in electric circuits, and will then discuss the method of calculating the values of L and M.

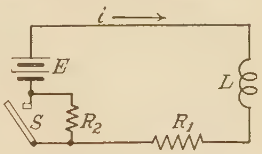

FIG. 252.—Circuit with R and L in series.

299. The Rise and Fall of the Current in an Inductive Circuit (DEDUCTION).—Let us consider the manner in which the current in an inductive circuit changes from one value to another when the constants of the circuit are changed by some switching operation. The circuit to be considered is shown in Fig. 252.

Let E represent the constant voltage of the battery in the arrow direction.
- i " the instantaneous value of the current in the arrow direction.
- L " the inductance of the circuit in henries.
- R_1 " the resistance of the circuit including the battery but excluding R_2.
- R_2 " the resistance which may be inserted or removed from the circuit by opening or closing the switch S.

We know that, after the switch S has been closed for some time (1 second, or less), the current in the circuit has the value E/R_1, and after the switch has been open for some time, the current is $E/(R_1 + R_2)$. That is, we know the value of the current immediately before and after the lapse of some time from the opening or closing of S, or at the two **boundaries** of the time interval during which the circuit remains undisturbed. The question is—How does the current pass from one value to another? instantaneously, or gradually? If gradually, what equation expresses the **march** of the current from one value to another?

After marking the arrow on the circuit for convenience in specifying directions, we may write the expressions (in terms of the current) for the three electromotive forces of the circuit in the arrow direction, namely,

E for the constant e.m.f. of the battery.

$-Ri$ for the e.m.f. of resistance. R stands for $R_1 + R_2$ if the switch is open, and for R_1 if the switch is closed.

$-L\frac{di}{dt}$ for the e.m.f. of self-inductance in the arrow direction.

From Kirchhoff's electromotive force law

$$E - Ri - L\frac{di}{dt} = 0. \tag{453}$$

The values assumed by the current must satisfy this differential equation at every instant of time. The integral of this equation may be recognized upon separating the variables.

Dividing through by L and transposing,

$$\frac{di}{dt} = -\frac{R}{L}\left(-\frac{E}{R} + i\right)$$

or

$$\frac{di}{-\frac{E}{R} + i} = -\frac{R}{L}\,dt.$$

Integrating,

$$\log\left(-\frac{E}{R} + i\right) = -\frac{Rt}{L} + C,$$

or

$$-\frac{E}{R} + i = \epsilon^{-\frac{Rt}{L} + C} = K\epsilon^{-\frac{Rt}{L}}$$

or

$$i = \frac{E}{R} + K\epsilon^{-\frac{Rt}{L}} \tag{454}$$

299a. Assignment of a Value to the Integration Constant.—Equation (454) is the general solution of the differential equation. To make this solution portray the current in a specific circuit after some specific switching operation, the **boundary conditions** must be stated, and that value must be assigned to integration constant K which will cause the equation to satisfy

the specified boundary conditions. We now specify the boundary conditions in the following manner.

Let t be measured from the instant that the change is made in the constants of the circuit by the switching operation.

Let I_0 represent the **initial value** of the current, that is, the value of the current at the instant of (or before) the switching operation, or when $t = 0$. If the switching operation is the opening of S, and if S had been closed for some time, $I_0 = E/R_1$. If the operation is the closing of S, $I_0 = E/(R_1 + R_2)$.

For the instant $t = 0$, at which $i = I_0$, Eq. (454) reduces to

$$i_{(t=0)} = \frac{E}{R} + K = I_o.$$

Therefore,
$$K = I_o - \frac{E}{R}.$$

In Eq. (454), R represents the resistance of the circuit **after** the switching operation. For values of t which are large in comparison with L/R, Eq. (454) reduces to $i = E/R$. That is to say, E/R is the **ultimate value** of the current in the circuit for its condition after the switching operation. Letting I_u represent the **ultimate value** of the current E/R

$$K = I_o - I_u$$

and the final form for the solution is

$$i = I_u + (I_o - I_u)\epsilon^{-\frac{Rt}{L}}. \tag{455}$$

For $t = 0$, this reduces to $i = I_o$, and
for $t = \propto$, this reduces to $i = I_u$. Therefore, Eq. (455) satisfies not only the differential equation of the circuit, but also the specified boundary conditions.

299b. Time Constant and Damping Constant.—The initial and final values, I_o and I_u, are connected by the exponential curve

$$i = (I_o - I_u)\epsilon^{-\frac{Rt}{L}}.$$

After a lapse of time equal to L/R seconds from the moment of the switching operation, this reduces to

$$i_1 = (I_o - I_u)\epsilon^{-1}.$$

That is, all but $(1/\epsilon)$th of the change in current takes place in the first L/R seconds. The ratio L/R is called the **time constant** of the circuit. It is a good index, or criterion, of the length of time taken for the current to change from the initial to the final value. The reciprocal R/L of the time constant is called the **damping constant** of the circuit.

299c. The Start of a Current.—To make the general solution Eq. (455) represent the case of closing a circuit previously open, we assign to R_2

the value infinity. Whence $I_o = 0$ and the equation for the start of a current from zero is

$$i = \frac{E}{R_1}\left[1 - \epsilon^{-\frac{R_1 t}{L}}\right]. \tag{456}$$

This equation is plotted in Fig. 253 for a circuit in which $E = 110$ volts, $R_1 = 20$ ohms, $L = 2$ henries.

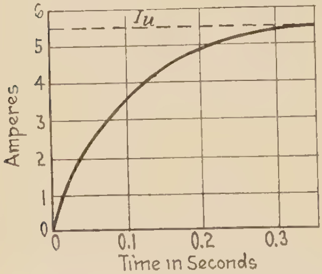

Fig. 253.—Rise of current in an inductive circuit.

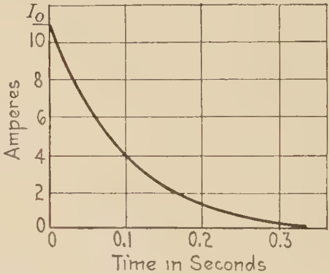

Fig. 255.—Decrease of the current in an inductive circuit.

299d. The Interruption of a Current.—The voltage e_s impressed upon the resistance R_2 or between the separating contacts of the switch S upon opening the switch is

$$e_s = R_2 i = R_2\left[I_u + (I_o - I_u)\epsilon^{-\frac{Rt}{L}}\right]$$

$$e_s = R_2\left[\frac{E}{R_1 + R_2} + \left(\frac{E}{R_1} - \frac{E}{R_1 + R_2}\right)\epsilon^{-\frac{(R_1 + R_2)t}{L}}\right]$$

$$e_s = E\left[\frac{R_2}{R_1 + R_2} + \frac{R_2^2}{R_1(R_1 + R_2)}\epsilon^{-\frac{(R_1 + R_2)t}{L}}\right].$$

For the instant immediately after the opening of the switch, that is, for t substantially equal to zero, this equation reduces to

$$e_{s(t=0)} = \frac{R_2}{R_1} E. \tag{457}$$

The physical significance of this is that the current in an inductive circuit never changes instantaneously in value; upon opening S, the current previously flowing through the switch, namely E/R_1, is diverted to the resistance R_2 and hence causes an e.m.f. of $(R_2/R_1)\,E$ volts.

If R_2 is greatly in excess of R_1, a voltage greatly in excess of the battery voltage is impressed upon R_2. For example, suppose we are opening the 110-volt, 20-ohm circuit of the previous case and that at the time a person is grasping the wires on each side of the switch, one in each hand. In this case R_2 will be the resistance from hand to hand through the body—a resistance of the order of 2000 ohms. The equation shows that **if an arc does not** form between the opening contacts of the switch, then at the first instant

after the opening of the contacts, the voltage impressed between the person's hands is $\frac{R_2}{R_1} E$ or 11,000 volts. As a matter of fact, an arc does form and the voltage may not rise to values in excess of 500 volts. The reason why it is impossible to open an inductive circuit without an arc at the separating contacts is now evident. If we assume a non-arcing interruption, we have the case in which R_2 is infinite, and for this case the formula shows that an infinite voltage would be impressed between the opening contacts

299e. Discharge of an Inductance.—Suppose the connections of Fig. 252 are slightly modified by connecting the resistance R_2 across both the battery and the switch as shown in Fig. 254. If R_2 is a resistance with substantially no inductance, its time constant is extremely short—one-millionth of a second or less. If the resistance R_2 is of the same order or less than R_1, and if the switch S is snapped open, the arc which starts when the contacts first separate may be suppressed very rapidly indeed, since a voltage drop in the arc equal to several times the battery voltage will be sufficient to reverse the direction of the current in R_2 in a few millionths of a second. That is, such a voltage will divert the current of the inductive branch from its previous path through the switch and battery to the bypass R_2 in a few millionths of a second. This means that the arc at the switch will be suppressed before the current in the inductive branch has decreased appreciably.

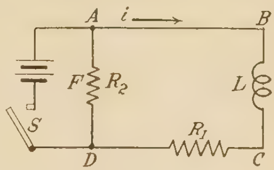

FIG. 254.

Upon the opening of the switch S, as soon as the arc between the opening contact has been suppressed there remains the closed circuit $ABCDFA$ with a current flowing in it. The general Eq. (455) applies to this circuit. For all practical purposes, time may be measured from the moment the contacts of the switch start to separate, and I_o may be taken as equal to E/R_1. Strictly speaking, t should be measured from the instant of suppression, and I_o represents the value of the current in the inductive branch at that instant—a value slightly less than E/R_1. Since the switching operation leaves the closed circuit with no battery in it, $I_u = 0$.

The current accordingly dies out gradually after the opening of the switch according to the equation

$$i = I_o\epsilon^{-\frac{Rt}{L}} = \frac{E}{R_1}\epsilon^{-\frac{(R_1+R_2)t}{L}}. \tag{458}$$

The curve showing the dying out of the current for the case in which $E = 110$ volts, $R_1 = R_2 = 10$ ohms and $L_2 = 2$ henries has been plotted in Fig. 255.

The energy expended in the resistance of the circuit as the current decreases to zero is given by the formula,

$$W = \int_0^\infty i^2 R dt.$$

Substituting the value of i from (458)

$$W = RI_o^2 \int_0^\infty \epsilon^{-\frac{2Rt}{L}} dt,$$

$$W = RI_o^2 \left[-\frac{L}{2R} \epsilon^{-\frac{2Rt}{L}} \right]_0^\infty$$

$$W \text{ (joules)} = \frac{LI_o^2}{2} \text{ (henries, amperes).} \qquad (459)$$

We say that this energy is derived from the magnetic field of the current, in which it was stored during the building up of the current.

300. Energy Required to Establish a Current in an Inductive Circuit (DEDUCTION).—The energy delivered by the generator or battery (in excess of that dissipated as i^2R heating loss in the conductors) during the establishment of a current in a circuit of inductance L may be thus computed: Assume the resistance of the circuit to be zero, and let i represent the value of the current at any instant.

If the current increases by the amount di in the interval of time dt, the e.m.f. e induced in the circuit is $e = -L\frac{di}{dt}$. During this interval the source of power must, therefore, deliver a voltage $-e$ with the current i passing through the source. The energy dW supplied by the source of power during the interval dt is

$$dW = -eidt = L\frac{di}{dt}idt = Lidi.$$

The total energy deliverd by the source of power in building up the current from zero to the value I is

$$W = \int_0^I Lidi$$

$$W \text{ (joules)} = \frac{1}{2}LI^2 \text{ (henries-amperes).} \qquad (459)$$

We have seen that when the current decreases to zero, a voltage is induced in the circuit in the opposite direction, and the same amount of energy is expended in the resistance of the circuit. The energy $\frac{1}{2}LI^2$ furnished by the source of power during the establishment of the current I is, therefore, stored and not dissipated. It is said to be stored in the magnetic field as energy of

an electrokinetic form: It is sometimes imagined to be stored in "concealed" motions established in the field.

Since the inductance L of a circuit is equal to the flux-linkage per unit current, or

$$L = \frac{\Lambda}{I} \tag{449}$$

this value of L may be substituted in Eq. (459), and the expression for the energy stored in the magnetic field of a current I may be written in the form

$$W \text{ (joules)} = \frac{1}{2}\Lambda I \text{ (webers-amperes)}. \tag{460}$$

300a. ENERGY ASSOCIATED WITH A SINGLE CIRCUIT (DEDUCTION).—**The energy stored in the magnetic field of a circuit carrying a current I is equal to one-half the product of the flux-linkage of the circuit times the current.**

301. Energy Stored in the Magnetic Field of Two Circuits Having Mutual Inductance (DEDUCTION).—The energy stored in the field when two circuits numbered 1 and 2 carry the currents I_1 and I_2, respectively, may be computed in the following manner.

Assumed Conditions.—Let both circuits be kept fixed in space, and assume that the resistance of each circuit is zero.

Initial State.—Let the current be zero in each circuit.

Step 1.—Let circuit 2 be kept open, and let the current in circuit 1 be brought up to the value I_1. The energy delivered to circuit 1 by its source of power is $\frac{1}{2}L_1I_1^2$.

Step 2.—Let circuit 2 be closed and let the current in it be brought up to the value I_2. During this step, let the current in circuit 1 be kept constant at the value I_1 by impressing sufficient voltage in circuit 1 to neutralize the voltage induced in it by reason of the increasing current in circuit 2.

The voltage induced in 1 during the interval dt is

$$e_1 = M\frac{di_2}{dt}.$$

Therefore, the energy supplied by the source in circuit 1 during this interval dt is

$$dw = (-e_1)I_1dt = MI_1di_2.$$

The total energy W supplied by the source in 1 while the current i_2 in circuit 2 is building up to the value I_2 is

$$W = \int_0^{I_2} MI_1 di_2 = MI_1I_2.$$

In addition, the source in circuit 2 has delivered to the circuit the energy $\frac{1}{2}L_2I_2^2$.

The total energy delivered by the sources of power in both circuits, or the energy stored in the magnetic field is

$$W \text{ (joules)} = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2. \tag{461}$$

By writing this equation in the form

$$W = \frac{1}{2}(L_1I_1 + MI_2)I_1 + \frac{1}{2}(L_2I_2 + MI_1)I_2$$

and then substituting for L and M their values in terms of flux-linkage, namely,

$$\begin{array}{rcl} L_1I_1 & = & \text{flux-linkage of circuit 1 due to } I_1 \\ MI_2 & = & \text{flux-linkage of circuit 1 due to } I_2 \\ \hline (L_1I_1 + MI_2) & = \Lambda_1 = & \text{total flux-linkage of circuit 1} \end{array}$$

this equation takes the form

$$W \text{ (joules)} = \frac{1}{2}[I_1(\Lambda_{1,1} + \Lambda_{1,2}) + I_2(\Lambda_{2,1} + \Lambda_{2,2})] \tag{462a}$$

$$W \text{ (joules)} = \frac{1}{2}[I_1\Sigma n\Lambda_{1,n} + I_2\Sigma n\Lambda_{2,n} + \ . \ . \ . \ ] \tag{462}$$

$$W \text{ (joules)} = \frac{1}{2}(\Lambda_1I_1 + \Lambda_2I_2) \text{ (webers-amperes)} \tag{462b}$$

in which Λ_1 and Λ_2 represent the **total** flux-linkage of each circuit.

302. Mechanical Force between Two Circuits Carrying Currents (DEDUCTION).—Let the mutual inductance of two circuits carrying the currents I_1 and I_2 be represented by M. Imagine that one of the circuits is now displaced from its former position by the infinitesimal amount dx, and imagine that during this displacement the currents are maintained constant at their initial values I_1 and I_2. The displacement is to be a pure displacement without rotation, or without alteration in the form of the circuit. (The distance dx is measured in the direction of translation, and is always taken as a positive quantity.)

Imagine the displacement of the circuit to cause no change in the self-inductance of the circuit, but to cause an increase in the mutual inductance by the amount dM. The energy stored before the displacement was

$$\frac{1}{2}L_1I_1^2 + MI_1I_2 + \frac{1}{2}L_2I_2^2.$$

The energy stored after the displacement is

$$\frac{1}{2}L_1I_1^2 + (M + dM)I_1I_2 + \frac{1}{2}L_2I_2^2.$$

The increase in the stored energy is I_1I_2dM.

During the displacement of the circuit, a voltage is induced in each circuit, whose time integral in circuits 1 and 2 is I_2dM and I_1dM, respectively. Therefore, the energy supplied to each circuit by its source of power in order to maintain the current constant is I_1I_2dM. The total energy supplied by the sources during the displacement is $2I_1I_2dM$, while the increase in the stored energy is only I_1I_2dM; therefore the difference, or I_1I_2dM, has been expended in doing mechanical work during the displacement of the circuit.

If f represents the **component-of-the-force** tending to move the displaced circuit in the direction of the displacement dx **from the initial to the final position,** the mechanical work done by the moving circuit is $f(dx)$.

Therefore, $f(dx) = I_1I_2dM$,

or $$f \text{ (dyne-sevens)} = I_1I_2\frac{dM}{dx} \text{(amperes, cm., henries)}. \qquad (463)$$

Since

$$I_2 \frac{dM}{dx} = \frac{d\Lambda_1}{dx}$$

this equation for the force may be written

$$f \text{ (dyne-sevens)} = I_1\frac{d\Lambda_1}{dx} \text{ (amperes, centimeters, webers).}$$

This equation is identical with the Eq. (436) derived in Sec. 285 from a direct consideration of the mechanical force acting on the wires of the moving circuit. As stated in Sec. 286, the treatment therein given in terms of mechanical force alone was incomplete, in that it failed to reveal the increase in the energy stored in the field.

303. Determination of the Inductances.—There are two methods of determining the mutual and self-inductances of circuits.

The first method is the **experimental** method of measuring the voltage-impulses caused by measured decrements in the

currents. The inductances are then found by substituting these measurements in the **defining equations** of M and L, namely,

$$M \text{ (henries)} = \frac{\int e_2 dt}{-\Delta i_1}, \frac{\text{(volt-seconds)}}{\text{(amperes)}},\tag{438a}$$

$$L_1 \text{ (henries)} = \frac{\int e_1 dt}{-\Delta i_1} \frac{\text{(volt-seconds)}}{\text{(amperes)}}.\tag{440a}$$

The second method is to **calculate** the flux-linkage of the circuits with the field set up by an assumed current I_1 in one of the circuits. The inductances are then found by substituting the calculated values of the flux-linkages in the **derived general formulas** for M and L, namely,

$$M \text{ (henries)} = \frac{\Lambda_2}{I_1} \frac{\text{(webers)}}{\text{(amperes)}},\tag{450}$$

$$L \text{ (henries)} = \frac{\Lambda_1}{I_1} \frac{\text{(webers)}}{\text{(amperes)}}.\tag{449}$$

In the first method, the inductance is determined by measurement. It is always possible to determine the inductance by measurement; in fact, for all but a few circuits of simple geometrical forms, this is the only **practical** method of determining the inductance.

In the second method, the inductance is determined solely by **calculation.** Hypothetically, it is always possible to calculate the inductances of circuits, **provided the field is free of concealed current systems,** that is, free of ferromagnetic material and of conducting masses in which eddy currents may be set up. Practically speaking, we shall find that it is not feasible to calculate the inductance except for circuits of a few simple geometrical patterns.

We proceed to consider the method of **calculating** the inductance under three headings:

Calculation for core fields.

Calculation by surface integration.

Calculation by circuit integration.

304. Calculation of the Inductances of Core Fields.—In many pieces of electrical apparatus, the magnetic field is confined largely, and in some cases almost entirely, to a **core** over which the circuit is wound. In all transformers, the important field lies within the closed circuit of the iron core.

In motors and generators, the magnetic field lies mainly in the iron core on which the coils are wound and in the narrow air gaps in this core. The ideal core field is obtained in the case in which the circular ring of rectangular cross-section of Fig. 211 (Sec. 251a) is entirely overwound with the circuit setting up the field, with the turns uniformly spaced around the ring. There is substantially no magnetic field outside of the overwound ring.

If the circuit causing the field has N_1 turns and carries the current I, the magnetomotive force around any line (circle) of magnetic intensity within the ring is

$$\mathfrak{F} = N_1 I.$$

The magnetic intensity in the circle of mean circumference l is,

$$H = \frac{\mathfrak{F}}{2\pi r} = \frac{N_1 I}{l}.$$

If the radial width w of the cross-sectional area of the ring is small in comparison with the radius r, the above value of H is very closely equal to the mean value of H at all points of the cross-sectional area wh.
Whence

$$B = \mu H = \frac{\mu N_1 I}{l}.$$

and the flux over the area a bounded by each turn is

$$\Phi = \int B \cos(B, n) da = \mu \frac{N_1 I a.}{l}$$

in which $a = wh$.
The flux-linkage of the N_1 turns of the circuit is

$$\Lambda_1 = N_1 \Phi = \frac{\mu N_1^2 I a}{l};$$

whence, the self-inductance of the circuit is

$$L = \frac{\Lambda_1}{I} = \frac{\mu N_1^2 a}{l}. \tag{464}$$

If a secondary circuit containing N_2 turns is wound on the same core the flux-linkage of the secondary with the field of the primary is

$$\Lambda_2 = N_2 \Phi = \mu \frac{N_1 N_2 I a}{l}.$$

Whence, the mutual inductance of the two coils is

$$M = \frac{\Lambda_2}{I} = \frac{\mu N_1 N_2 a}{l}. \tag{465}$$

305. Calculation of the Inductances by Surface Integration.—The steps involved in the calculation of the mutual inductance of two circuits, p and s, are as follows:

Step 1.—Designate a direction around each circuit to be known as the arrow direction around the circuit. The arrow

directions through the circuits are related to the arrow directions around by the right-hand screw convention.

Step 2.—Select that particular surface of which s is the boundary or contour, for which it appears that it will be the easiest to calculate the flux.

Step 3.—Divide the selected surface into small areas or patches, over each of which the flux density may be regarded as substantially uniform.

Step 4.—Assume a current I flowing in circuit p, and calculate the magnitude and direction of the flux density B at a point P near the center of each area.

Step 5.—Compute the magnetic flux over each small area or patch from the known value of B at the point P near its center.

Step 6.—Compute the flux Φ over the entire surface by taking the sum of the fluxes of all the patches.

Step 7.—Compute M by substituting in $M = \Phi_2/I_1$.

It is at once evident that while it is **hypothetically possible** by arithmetical methods to compute the mutual inductance of any two circuits, yet it will not be **feasible** to carry out the computation except for those few types of circuits of simple geometrical pattern to which the methods of the calculus may be applied. The labor involved in the arithmetical calculation for circuits of involved contours is prohibitive.

306. Mutual Inductance of Two Parallel Circuits (Deduction).—The calculation by surface integration is illustrated in the following derivation of the formula for the mutual inductance of two very long, two-wire circuits which are stretched parallel to each other. The two circuits may be a power circuit and a telephone circuit strung along opposite sides of a roadway, or two telephone circuits strung on the same poles. As illustrated in Fig. 256, all four wires are in the same plane.

Step 1.—Let the directions indicated by the arrows be selected as the arrow directions around the circuits. Then the arrow directions through the circuits are shown by the arrows through the circuits.

Step 2.—Let us compute the flux over the plane area bounded by wires 3 and 4, which is the result of the current I in circuit 1–2.

Steps 3 and 4.—Let the area be divided up into strips of length l and of width dx, as shown by the cross-hatched strip on the diagram. Consider first the flux due to the current I in wire 1. Since this is a long, straight wire, the lines of magnetic intensity are circles concentric with the wire.

The magnetomotive force around any of these circles is

$$\mathfrak{F} = I \text{ ampere-turns.}$$

The magnetic intensity in the circle passing through the strip dx at distance x from the wire 1 is

$$H \text{ (amp.-turns per cm.)} = \frac{\mathfrak{F}}{2\pi x} = \frac{I}{2\pi x}.$$

Whence
$$B \text{ (webers per sq. cm.)} = \mu H = \frac{\mu I}{2\pi x}.$$

Step 5.—Since the flux density vectors are normal to the strip, the flux over the strip of length l and width dx is

$$d\Phi = \frac{\mu I l}{2\pi x}\,dx.$$

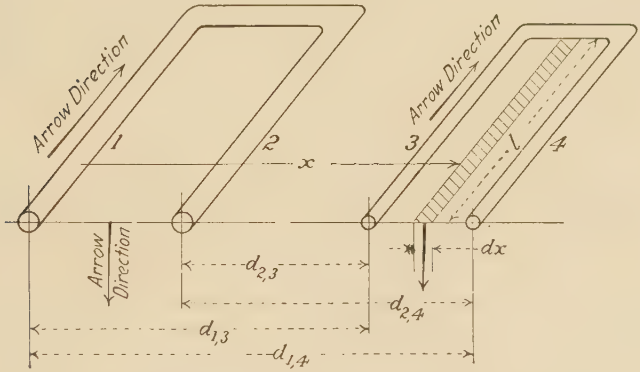

FIG. 256.—Two circuits with parallel wires.

Step 6.—If the wires 3 and 4 are of extremely small diameter, the flux-linkage of a current in the circuit 3–4 with the field due to the current I in in wire 1 will be the summation of the flux over all such strips, namely,

$$\Phi_1 = \int_{d_{1,3}}^{d_{1,4}} \frac{\mu I l}{2\pi}\frac{dx}{x} = \frac{\mu I l}{2\pi}\Big[\log x\Big]_{d_{1,3}}^{d_{1,4}},$$

$$\Phi_1 = \frac{\mu I l}{2\pi}\log\frac{d_{1,4}}{d_{1,3}}, \tag{466}$$

in which, $d_{1,3}$ and $d_{1,4}$ represent the distances from the axis of wire 1 to the axes of wires 3 and 4, respectively. The flux-linkage of a current in circuit 3–4 with the field of the current in wire 2 is

$$\Phi_2 = -\frac{\mu I l}{2\pi}\log\frac{d_{2,4}}{d_{2,3}}.$$

The flux-linkage with the field set up by the current in wires 1 and 2 is

$$\Phi = \Phi_1 + \Phi_2 = -\frac{\mu I l}{2\pi}\log\frac{d_{2,4}\,d_{1,3}}{d_{2,3}\,d_{1,4}}.$$

If the length of the circuits is great in comparison with the distances $d_{1,2}$ and $d_{3,4}$ between the wires, the flux due to the current in the short bridging conductors at the two ends of the circuit 1–2 may be ignored, since it would reduce the flux-linkage given above by an extremely small fraction of 1 per cent.

Step 7.—Therefore the mutual inductance between the two circuits is

$$M \text{ (henries)} = \frac{\Phi}{I} = -\frac{\mu l}{2\pi} \log \frac{d_{2,4}\, d_{1,3}}{d_{2,3}\, d_{1,4}}. \tag{467}$$

306a. Uncertainties in the Application of Circuit Theory to Conductors of Large Cross-section.—We have just noted the necessity of specifying conductors of small cross-section in the secondary circuit if definite results are to be obtained in the calculation of flux-linkage. The same necessity has been with us many times before but has been passed over without notice. The definition of e.m.f. is in terms of work done along a **line.** The "e.m.f. along a conductor" which has a large cross-section may be a very indefinite quantity. Similarly, the flux linking with a closed **line** is a definite quantity, but the flux linking with a conductor of large cross-section is usually indefinite to some extent. Usually, the uncertainties are proportionately small and may be neglected, but they always exist when the definitions in terms of mathematical lines are applied to conductors of large diameter.

The definition of the inductances in terms of e.m.f. and the derived formulas in terms of flux-linkage, Eqs. (449) and (450), are both indefinite when applied to conductors of large cross-section. It, therefore, is necessary to frame a definition for the term "inductance" as applied to such circuits. This we do as follows:

306b. *By the self-inductance of a circuit consisting of conductors of large diameter is meant the value which must be assigned to the constant L to make the following formula express the stored energy associated with the circuit.*

$$W \text{ (joules)} = \frac{1}{2} L I^2, \tag{459}$$

in which the value of the stored energy is to be obtained by dividing the large "tube of flow" into elementary tubes, or filaments, of flow and then applying Eq. (462) *to obtain the energy stored in this system of coupled circuits (see Sec.* 301).

$$W \text{ (joules)} = \frac{1}{2} \Sigma di\, \Sigma^n \Lambda_{1,n}. \tag{462c}$$

$$W \text{ (joules)} = \frac{1}{2} \int di \int d\Lambda. \tag{462d}$$

306c.—*By the mutual inductance of two circuits consisting of conductors of large cross-sectional area is meant the value which must be assigned to the constant M to make the following formula express the stored energy associated with the two circuits:*

$$W \text{ (joules)} = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2 \qquad (461)$$

in which W, L_1, and *L_2 are to be evaluated as outlined in Sec.* 306*b*.

306d. Mutual Inductance for Wires of Large Diameter.—In the derivation of the above formula for the mutual inductance of parallel circuits we have avoided the treatment of the flux which links with only a portion of the current in circuit 3–4 by assuming that the wires were of infinitesimal diameter. Wires of large diameter are represented in cross-section in Fig.

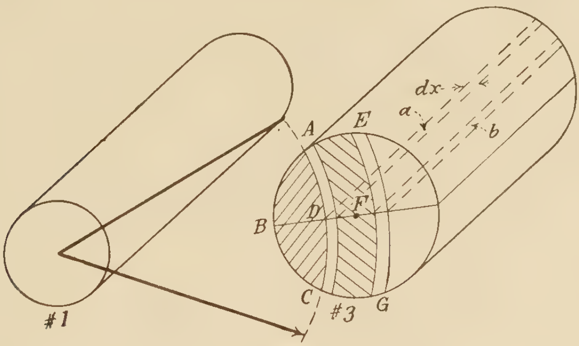

FIG. 257.—Enlargement of wire.

257. The currents are assumed to be uniformly distributed over the cross-section of the wire. The following examination shows that Eq. (466) applies with considerable exactness to this case, provided the shortest distance $d_{2,3}$ between the two circuits is great (10 times) in comparison with the diameter of the conductors.

Consider the flux caused by the current in wire 1 over any two strips a and b lying within the wire 3 and at the same distance from its axis. The fluxes Φ_a and Φ_b over the strips a and b link with the currents I' and I'' of the cross-sectional areas $ABCD$ and $EABCGF$, respectively. The energy stored in the field by reason of the partial linkage of the fluxes of the strips a and b with the current in circuit 3–4 is

$$W = \Sigma I\Lambda = I'\Phi_a + I''\Phi_b.$$

But the flux Φ_a differs very little from (being slightly greater than) the flux Φ_b, and the sum of the currents I' and I'' differs very little from (being slightly less than) the total current I_3 in wire 3. Therefore the above expression may be written

$$W = \Phi_b(I' + I'') = \Phi_b I_3 \text{ (approximately).}$$

That is, the energy stored by reason of the partial linkage of the current with the flux over the pair of strips a and b is equal to the energy which would be stored if the flux over the strip b were linked with the entire current in the wire 3. By pairing, in this manner, all strips from the axis of the wire 3 to its circumference, it is seen that the total energy stored by reason of the partial linkage with wire 3 is equal to the energy which would be stored if the entire current in wire 3 were linked with the flux from the axis of the wire to its far surface (the flux over a strip of width equal to the radius of the wire). But this is equal to the energy which would be stored by reason of the flux over this strip if all the current in wire 3 were concentrated in a filament of infinitesimal radius along the axis of the wire. In like manner, we may see that the energy stored by reason of the partial linkage of the current in wire 4 is equal to that which would be stored if the current in this wire were concentrated at the axis of the wire. Therefore Eq. (467) gives a value of mutual inductance, which to a fair degree of precision is consistent with the equation for the energy stored in the circuit, provided that the length of the circuit is great in comparison with the separation of the circuits and that the separation $d_{1,3}$ is, in turn, great in comparison with the radius of the wires.

In deriving the above formula, we have used an illustration in which the conductors of the two circuits all lie in the same plane. By sketching another figure, it may be readily seen that the formula still applies when the planes of the circuits make any angle whatsoever with each other, provided the four wires are all parallel, and that $d_{1,2}$, $d_{1,3}$, etc. represent the distances between the axes of the wires.

307. Self-inductance of a Long Circuit of Two Parallel Wires.—The circuit is shown in Fig. 258. The circuit may be a two-wire telephone or power

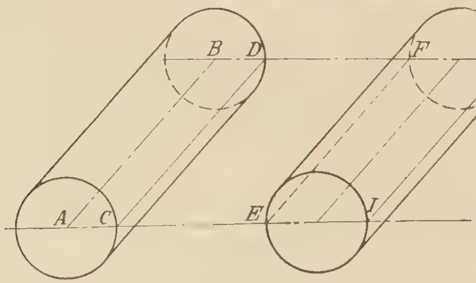

Fig. 258.—Circuit of two parallel wires.

circuit in which the length of the circuit is 100 or more times as great as the separation of the wires, and the separation is, in turn, 20 or more times the diameter of the wire.

The flux set up by the current in wire 1 may be divided into three portions:

1. The flux over the area $CDFE$ which links with all the current.
2. The flux over the area $EFHI$ which is partially linked with the current.
3. The flux over the area $ABDC$ which is partially linked with the current.

A review of the steps and arguments used in the above derivation of the formula for the mutual inductance will lead at once to the conclusion that the first two items contribute the following term to the formula for the self-inductance

$$L = \frac{\mu}{4\pi} 2l \log \frac{d}{r}. \tag{468}$$

Let us now consider the term contributed by the third item. If the current I_1 in wire 1 is uniformly distributed over the cross-section of the wire, the cylindrical shell of radius x encircles a current equal to $\pi x^2 I_1/(\pi r)^2$. Therefore the magnetomotive force around this cylinder is

$$\mathfrak{F} = \frac{\pi x^2 I_1}{\pi r^2} = \frac{x^2}{r^2} I_1.$$

The flux density over the strip of width dx is

$$B = \frac{\mu_w x^2 I_1}{2\pi x r^2}.$$

The flux over this strip is

$$d\Phi = \frac{\mu_w I_1 l x\, dx}{2\pi r^2}.$$

This flux links with the current $x^2 I_1/r^2$; therefore, the energy stored because of the linkage of this flux with this current is

$$dW\left(= \frac{1}{2}\Phi I\right) = \frac{\mu_w I^2{}_1 l x^3 dx}{4\pi r^4}.$$

The total energy of the field due to the partial linkage of the flux within the wire 1 is

$$W = \int_0^r \frac{\mu_w I_1{}^2 l}{4\pi r^4} x^3\, dx = \frac{\mu_w}{16\pi} I^2{}_1 l.$$

Since L is related to W by the formula $W = LI^2/2$, item 3 contributes to the formula for the inductance the term

$$L = \frac{\mu_w l}{8\pi}. \tag{469}$$

in which μ_w represents the permeability of the wire. Hence the complete formula for the self-inductance computed for the field due to the current in one wire alone is

$$L\ (\text{henries}) = \frac{l}{4\pi}\left(\frac{\mu_w}{2} + 2\mu \log \frac{d}{r}\right). \tag{470}$$

For the two wires of the circuit the inductance will be double the above value, or the inductance of the long circuit with the outgoing and return conductors stretched parallel to each other will be given by the above formula provided l is taken to represent the total length of wire in the circuit.

308. Calculation of the Inductances by Circuit Integration.—The fundamental general method of calculating the mutual and self-inductance is the method involving the integration of the flux density over the surface bounded by the coils. This method when applied to the simplest core field leads to the simple Eqs. (464) and (465). We now proceed to show that this general method may be applied to circuits of any contour in such a manner as to convert the integrating operation from a surface integration to an integration around the circuit.

Let P and S (Fig. 259) be any two circuits, and let us proceed to calculate the mutual inductance of the two circuits by applying the general method in the following manner.

Let the directions indicated by the arrows be selected as the arrow directions around the two circuits. Let AC denote an elementary length dp of circuit P. Imagine the surface area bounded by circuit S to be cut into strips by a great number of planes all drawn perpendicular to AC. Let EF and HI represent two consecutive planes whose distance apart is w. Let ds_1 and ds_2 represent the elements of the circuit S included between these two planes. Let r_1 and r_2 represent the distance from dp to these elements, and let (dp, r_1) and (dp, r_2) represent the angles between the arrow direction along AC and the radii r_1 and r_2.

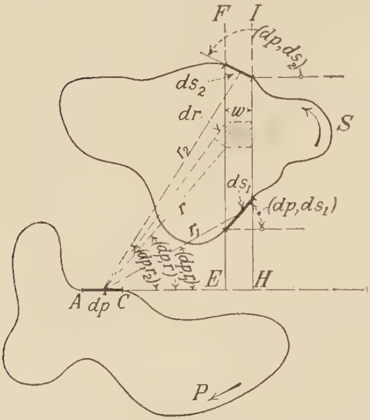

FIG. 259.—Inductance by circuit integration.

Now let us determine the magnetic flux over a strip of the surface bounded by ds_1, ds_2 and the two planes. The lines of magnetic intensity due to dp are circles about dp as an axis. Therefore if r_1 and ds_1 are turned about dp as an axis until r_1 comes into the plane determined by dp and r_2, ds_1 will, during the rotation, move along lines of magnetic intensity and will cut no tubes of magnetic flux. Therefore the magnetic flux over the strip of the original surface included between ds_1, ds_2, and the two planes is equal to the flux over the plane area enclosed between the two planes, ds_2 and ds_1 in its revolved position. Since the flux density is normal to this plane area, the flux over the surface may be readily calculated as follows.

The flux density at the cross-hatched element of area due to the current I in dp is

$$B = \frac{\mu I dp}{4\pi} \frac{\sin (dp, r)}{r^2}.$$

The area of the element is

$$\frac{w(dr)}{\sin (dp, r)}.$$

Thus the flux over the strip is

$$d\Phi = \frac{\mu w I dp}{4\pi} \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{\mu w I dp}{4\pi} \left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Now if the arrow directions of ds_1 and ds_2 make the angles (dp, ds_1) and (dp, ds_2) with the arrow direction along dp, $w = ds_1 \cos (dp, ds_1) = -ds_2 \cos (dp, ds_2)$.

Whence

$$d\Phi = \frac{\mu I dp}{4\pi} \left[\frac{ds_1 \cos (dp, ds_1)}{r_1} + \frac{ds_2 \cos (dp, ds_2)}{r_2}\right].$$

Therefore the expression for the flux over the entire surface bounded by the coil S, due to the current I in the element dp, is

$$d\Phi = \frac{\mu I dp}{4\pi}\int^{OS}\frac{\cos(dp, ds)\,ds}{r},$$

and the total flux due to the current in the entire circuit P is

$$\Phi = \frac{\mu I}{4\pi}\int^{OP} dp \int^{OS}\frac{\cos(dp, ds)\,ds}{r}.$$

From this it follows that the expression for the mutual inductance between the circuits is

$$M \text{ (henries)} = \frac{\mu}{4\pi}\int^{OP} dp \int^{OS}\frac{\cos(dp, ds)ds}{r}. \tag{471}$$

It also follows that the same double integration taken around a single circuit will give the self-inductance of the circuit. Whence

$$L = \frac{\mu}{4\pi}\int^{OP} dp \int^{OP}\frac{\cos(dp, dp')dp'}{r}. \tag{472}$$

309. Electromagnetically Induced Electromotive Intensities Resulting from the Acceleration of Electricity (DEDUCTION).—We may use the formula which gives the mutual inductance of two circuits by circuit integration to derive a formula for the electromotive intensity which is electromagnetically induced in conductors when the electricity in other circuits is accelerated. Thus

$$e_2 = -M\frac{di_1}{dt}.$$

and

$$M = \frac{\mu}{4\pi}\int^{OP} dp \int^{OS}\frac{\cos(dp, ds)ds}{r},$$

and

$$i_1 = qv,$$

in which q is the quantity of electricity per unit length in circuit 1, v is its velocity of drift through the wire, and a is the acceleration. Therefore we may write

$$e_2 = -\frac{\mu}{4\pi}\int^{OP} qdp\,\frac{dv}{dt}\int^{OS}\frac{\cos(dp, ds)ds}{r},$$

$$e_2 = \int^{OP} dp \int^{OS} -\frac{\mu a q}{4\pi r}\cos(dp, ds)ds. \tag{473}$$

Let us tentatively take the view that the acceleration of the electrons in each elementary length of the primary gives rise to

an electromotive intensity $F_m \times dp$ in each elementary length of the secondary, in which F_m is some function of the acceleration, the distance between the elements, etc. Then the expression for the e.m.f. induced in the secondary becomes

$$e_2 = \int^{OP} dp \int^{OS} F_m \cos (F_m, ds) ds. \tag{474}$$

Upon comparing Eqs. (473) and (474) we see that the value of F_m is

$$F_m \text{ (volts per centimeter)} = \frac{\mu q \mathbf{a}}{4\pi r} \frac{\text{(coulombs)}}{\text{(seconds)}}. \tag{475}$$

and that the direction of F_m is parallel to dp in a direction opposite to the direction of acceleration of the charge.

These results are expressed in the following statement.

309a. *When the quantity of electricity q is accelerated, an electromotive intensity whose algebraic value is expressed by* Eq. (475) *may be supposed to be induced in all neighboring bodies. The direction of the induced intensity at any point may be taken as parallel but opposite to the vector* a *representing the acceleration of the charge q in the direction of motion.*

We shall find that Eq. 475 does not express either the value or the direction of the total electromotive intensity which acts on the electricity at a point in a wire when the quantity q in a wire at a distant point is accelerated. It expresses only that component of the intensity which contributes to the double integration around the two circuits. Other terms in the complete expression for the intensities caused by the acceleration cancel out in the double integration. See Chap. XV.

310. Exercises.

1. When a current of 3.5 amperes was suddenly started in a given circuit A, a voltage impulse of 0.0082 volt-second was measured in circuit B. Later a current of 12 amperes was suddenly started in circuit B. What voltage impulse was produced in circuit A? When the current in B was changing at the rate of 230 amperes per second, what e.m.f. was induced in A?

2. A given circuit has a self-inductance of 0.05 henry. What is the algebraic value of the e.m.f. of self-induction in a given direction when the current in that direction is increasing at the rate of 530 amperes per second.

3. A coil whose self-inductance is 0.01 henry has 300 turns of fine wire all bounding practically the same surface S. What is the flux across this surface due to a current of 3 amperes flowing in the coil?

4. A wooden circular ring is uniformly overwound with a coil of 800 turns of fine wire. The inductance is 1 millihenry. Another coil of 1200 turns is wound uniformly over the first, the area bounded by the turns being practically the same as for the first coil.

a. What is the self-inductance of the second coil?

b. What is the mutual inductance between the coils?

c. What is the self-inductance of the coil formed by connecting the two windings in series, (1) so as to trace around the core in the same direction; (2) so as to trace around the core in opposite directions?

5. A current of 3 amperes exists in a coil having a self-inductance of 0.004 henry. If all the energy stored in the magnetic field can be expended in the spark when the circuit is opened, how many calories of heat will be developed?

6. The 800-turn winding of exercise 4 carries a current of 3 amperes and the 1200-turn winding carries a current of 1.5 amperes. What is the total energy stored in the magnetic field (*a*) when the currents are directed around the core in the same direction; (*b*) when the currents are directed around the core in opposite directions? In either case, what energy is available as heat when the 1200-turn circuit is opened and the current in the other winding held constant?

7. A switch is closed connecting a 6-volt battery to a circuit of 0.002 henry self-inductance and a resistance of 1.5 ohms. Draw to scale a current-time curve showing the growth of the current from zero value toward the ultimate value.

8. A long air-core solenoid has a total of 3000 turns in a length of 50 centimeters. Inside this winding another winding is placed near the center of the solenoid, the second winding having 75 turns and each turn bounding an area of 20 square centimeters. Calculate the mutual inductance between the two windings.

9. Two flat coils are arranged so that they are parallel and one above the other. The lower coil is stationary and the upper one is so mounted and counterbalanced that it can move up and down. When the upper coil is 30 centimeters above the lower coil, the mutual inductance of the two coils is 1 henry. When this coil is 32.5 centimeters above the lower, the mutual inductance of the two coils is decreased 0.5 per cent. When the currents in the two coils are respectively 30 and 6 amperes and are in opposite directions, what is the average force in pounds exerted by one coil on the other during the motion of the upper coil from its first to its second position, and what is the direction of this force?

10. A coil has an inductance of 1 henry and a resistance of 1.8 ohms. How long after the switch is closed, connecting this coil to a 40-volt battery, will it be before the current reaches a value of 18 amperes if the resistance of the battery and leads is 0.2 ohm? What is the "time constant" of the coil? Of the circuit?

11. What is the initial rate of increase of the current in exercise 10? How would this initial rate of increase be affected if an additional resistance were added to the circuit?

12. If the e.m.f. applied to the circuit of exercise 10 is suddenly removed, show that the initial rate of current decrease is independent of the circuit resistance.

13. Two coils are wound so closely side by side that the magnetic leakage between them is negligible. The first has 150 turns and 5 ohms resistance, and the second has 450 turns and 25 ohms resistance. The latter is open circuited. What e.m.f. will be induced in the second when the current in the first reaches 5 amperes, with 40 volts across the first?

14. The field resistance of a shunt motor is 100 ohms. An ammeter is placed in the field circuit and a 150-scale voltmeter (resistance 13,300 ohms) across its terminals. Both instruments are between the field terminals and the double-pole switch, the ammeter being the nearer to the field coil. The voltage across the switch is 115 volts. Assume that the switch is opened so quickly that no arc forms at the blades.

a. What will the ammeter read the instant after the switch is opened?

b. What will be the difference of potential across the voltmeter at this instant?

c. Will either instrument be deflected in the reverse direction?

d. Supposing that the voltmeter had been removed and that just at the instant the switch was opened the hands made contact with the two blades of the switch, what would be the voltage across the body if it had a resistance of 5000 ohms?

15. A two-wire power-transmission circuit parallels a two-wire telephone circuit on the opposite side of the road for a distance of l centimeters. If the power wires are numbered 1 and 2 and the telephone wires 3 and 4, and the distances d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, and d_{34} are given, deduce the expression for the mutual inductance between the two circuits?

16. If the power circuit carries a 60-cycle alternating current whose peak value is 100 amperes (equation is $i = 100 \sin 2\pi 60t$) and if the distances are $d_{13} = 30$ feet, $d_{14} = 31$ feet, $d_{23} = 25$ feet, $d_{24} = 26$ feet, compute the expression for the voltage induced in the telephone circuit if the length of the parallel is 1 mile.

Plot, on the same sheet, curves showing the variation in time of the current in the power circuit and the induced voltage in the telephone circuit.

17. A 60-cycle sine current the values of which are expressed by the equation $i = 100 \sin 2\pi 60t$ flows in a line 1000 feet in length which consists of two copper conductors (diameter 0.325 inch) stretched parallel to each other with a distance of 1 foot between centers.

Plot the following curves for an interval of time equal to one period:

a. The current curve.

b. The curve showing the e.m.f. of resistance.

c. The curve showing the e.m.f. of inductance.

d. The curve showing the voltage which must be impressed upon the circuit by the generator to cause the above current to flow.

18. Plot the corresponding curves for the case in which an 800-cycle current flows in the line of exercise 17.

CHAPTER XIV

MAGNETIC CIRCUITS WITH CONCEALED CURRENTS

311. The Magnetic Circuit.—In appliances such as transformers, generators, and motors, the magnetic field is set up mainly in an iron **core,** and in the narrow air gaps in this core. Over this core the coils are wound. Any magnetic field outside of the core and the air gaps is called a **leakage** field. The leakage field, although generally unavoidable, is not an essential feature of the appliance and is frequently objectionable. The core with its air gaps is called a **magnetic circuit** because the tubes of magnetic flux (save the leakage tubes) before returning into themselves make a circuit of the core.

312. The Homogeneous Ring Core.—As stated in Sec. 251 the simplest magnetic circuit is a core field in the shape of a ring of homogeneous material and of uniform cross-section which is entirely overwound with a field coil, the turns of which are uniformly spaced around the ring. Such a core is illustrated in Fig. 261. No magnetic field is found outside of the overwound core, save a very weak leakage field quite close to the individual turns of wire and the field of the single turn or sheet which represents the advance of the turns along the length of the annulus (see footnote 8, Sec. 251a). Within the core, the tubes of magnetic flux are circular filaments of uniform cross-section centered about the axis of the core. All tubes are subject to the same magnetomotive force, namely, NI ampere-turns. From the circular symmetry, the flux over any radial cross-section of the core is equal to the flux over any other radial cross-section.

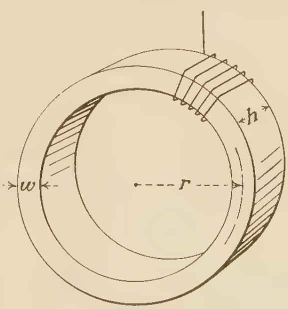

FIG. 261.—Ring core.

Let us review the calculation of the self-inductance of the coil wound on such a core. If the circuit causing the field has N_1 turns and carries the current I, the magnetomotive force $\mathfrak{F}$ around any tube of magnetic flux within the ring is

$$\mathfrak{F} \text{ (ampere-turns)} = N_1 I.$$

The magnetic intensity H in the cylindrical shell of radius x, thickness dx, and height h is

$$H \text{ (ampere-turns per cm.} = \frac{N_1 I}{2\pi x}.$$

The flux density B in this shell is

$$B \text{ (webers per sq. cm.)} = \frac{\mu N_1 I}{2\pi x}.$$

The flux $d\Phi$ over the cross-sectional area $h\ dx$ is

$$d\Phi \text{ (webers)} = \frac{\mu N_1 I h\ (dx)}{2\pi x}.$$

The total flux over the cross-section of the core is

$$\Phi \text{ (webers)} = \int_{r_1}^{r_2} \frac{\mu N_1 I h}{2\pi} \frac{dx}{x} = \frac{\mu N_1 I h}{2\pi} \log \frac{r_2}{r_1} \qquad (481)$$

$$= \frac{\mu N_1 I h}{2\pi} \log \left(1 + \frac{w}{r_1}\right).$$

If the radial width w of the core is small in comparison with its mean radius r (if w/r is less than 0.2), the Naperian logarithm of $1 + w/r_1$ is equal to w/r to within 0.3 of 1 per cent. Therefore, for cores in which the radial width is so small in comparison with the mean radius r that the flux density is substantially uniform, the expression for the flux may be written in the form

$$\Phi \text{ (webers)} = \frac{\mu N_1 I h w}{2\pi r} = \frac{\mu N_1 I a}{l} \text{ (approximately)} \qquad (482)$$

in which a represents the cross-sectional area of the core; $a = hw$. l represents the mean length of the core; $l = 2\pi r$.

The flux-linkage of the N_1 turns of the coil is

$$\Lambda_1 \text{ (weber-turns)} = N_1 \Phi = \frac{\mu N_1^2 I a}{l}. \qquad (483)$$

Whence, the self-inductance of the coil is

$$L \text{ (henries)} = \frac{\Lambda_1}{I} = \frac{\mu N_1^2 a}{l}. \qquad (484)$$

The inductance of the coil is seen to be a constant whose value depends upon the combined proportions of the **coil and the core.** Suppose, however, that instead of taking the ratio of the flux-linkage to the current, we take the ratio of the flux over any cross-sectional area to the magnetomotive force around the magnetic circuit. Representing this ratio by the symbol $\mathcal{P}$, we have

$$\mathcal{P} = \frac{\Phi}{\mathfrak{F}} = \frac{\mu N_1 I a}{l} \div N_1 I = \frac{\mu a}{l}. \tag{485}$$

This ratio $\mathcal{P}$ is seen to be a constant of the magnetic core; it is independent of the number of turns with which the core is wound. This ratio is used so frequently in calculations relating to core fields that the short name **permeance** has been coined for it.

313. PERMEANCE, RELUCTANCE, AND RELUCTIVITY (DEFINITIONS).—**The PERMEANCE (symbol $\mathcal{P}$) either of a complete magnetic circuit, OVER ALL CROSS-SECTIONS OF WHICH THE MAGNETIC FLUX HAS THE SAME VALUE, or of a portion of the circuit from one equimagnetic-potential surface to another, is defined as the ratio of the magnetic flux Φ to the exciting magnetomotive force $\mathfrak{F}$ from surface to surface.**

$$\mathcal{P}\ (\text{webers per ampere-turn}) = \frac{\Phi\ (\text{webers})}{\mathfrak{F}\ (\text{ampere-turns})}\ (\text{definition}). \tag{486}$$

By an equimagnetic-potential surface we mean any surface in a magnetic field which is so drawn with reference to the lines of magnetic intensity that these lines at each point of the surface are perpendicular to the surface at that point.

313a. Unit of Permeance.—*A magnetic circuit has unit permeance if a magnetic flux of* 1 *weber is caused by a m.m.f. of* 1 *ampere-turn. Such a circuit may be said to have a permeance of* 1 *"weber per ampere-turn."*

Since the permeance of a core has been defined by the equation $\mathcal{P} = \Phi/NI$, and the inductance of a coil uniformly wound on this core is given by the formula $L = N\Phi/I$, the following relation exists between the inductance of a coil and the permeance of the core on which it is wound.

313b. *The inductance of a coil of N turns uniformly wound on a core of permeance $\mathcal{P}$ is equal to the product of the permeance of the core and the square of the number of turns in the coil.*

$$L\ (\text{henries}) = N^2\mathcal{P}\ (\text{webers per ampere-turn}). \tag{487}$$

If electric conductors are connected in parallel, it is more convenient to carry on calculations in terms of the conductances of the conductors; if they are connected in series, it is more convenient to use the resistances. Likewise, in magnetic calculations, if cores are in parallel it will be found more convenient to carry on the calculations in terms of the permeances of the cores; if the cores are in series it will be found more convenient to use the reciprocals of the permeances as defined below.

313c. The RELUCTANCE (symbol $\mathcal{R}$) of a portion of a magnetic circuit from one equimagnetic-potential surface to another is defined as the ratio of the exciting m.m.f. to the magnetic flux.

$$\mathcal{R} \text{ (ampere-turns per weber)} = \frac{\mathfrak{F} \text{ (ampere-turns)}}{\Phi \quad \text{(webers)}}. \tag{488}$$

The reluctance of a circuit is the reciprocal of its permeance.

$$\mathcal{R} = \frac{1}{\mathcal{P}}. \tag{489}$$

313d. Unit of Reluctance.—*A magnetic circuit has unit reluctance if a m.m.f. of* 1 *ampere-turn is necessary to set up a flux of* 1 *weber. The unit may be termed the "ampere-turn per weber."*[1]

The **reluctivity** (symbol ν) of a material is defined to be the reciprocal of its permeability.

$$\nu = \frac{1}{\mu} \text{ (definition)}. \tag{490}$$

314. Permeance and Reluctance of a Right Cylinder in Terms of Its Dimensions.—The approximate expression derived in Eq. (485) for the permeance of a ring core, namely, $\mu a/l$, becomes rigorously exact if the mean radius of the ring becomes infinitely great in comparison with its radial width w. But under these conditions, any short length of the core which is included between

[1] In the footnote to Sec. 247, it is pointed out that in the unrationalized E.M. system of units, the unit of magnetomotive force is the gilbert.

$$1 \text{ ampere-turn} = 0.4\pi \text{ gilberts}. \tag{366}$$

In this E.M. system, magnetic flux is expressed in maxwells or eighth-webers. Therefore, the unit of reluctance in the E.M. system is the **gilbert per maxwell.** This unit has been named the **oersted.**

$$1 \text{ ampere-turn per weber} = \frac{4\pi}{10^9} \text{ oersteds (gilberts per maxwell)}. \tag{491}$$

two equipotential plane surfaces may be regarded as a right cylinder. From this it follows that Eq. (485) expresses the relation between the permeance of a right cylinder and its dimensions. This law may be expressed as follows:

314a. **The PERMEANCE between the parallel plane ends of a right cylinder of material, in which the tubes of flux are all parallel to the elements of the cylinder, is equal to the permeability μ of the material times the cross-sectional area a of the cylinder divided by its length l.**

$$\mathcal{P} \text{ (of a cylinder in webers per ampere-turn)} = \frac{\mu a}{l}\cdot \qquad (485)$$

Since the reluctance is the reciprocal of the permeance, we have the following law for computing the reluctance:

314b. **The RELUCTANCE between the parallel plane ends of a right cylinder of material, in which the tubes of flux are all parallel to the elements of the cylinder, is equal to the reluctivity ν of the material times the length l of the cylinder divided by its cross-sectional area a.**

$$\mathcal{R} \text{ (of a cylinder in amperes-turns per weber)} = \frac{\nu l}{a}\cdot \qquad (492)$$

Permeance in the magnetic circuit is seen to be the mathematical analog of **conductance** in the electric circuit, and of **permittance** or capacitance in the dielectric circuit, while **reluctance** is the analog of **resistance** and of **elastance** in the two circuits. The permeance of cores of a few of the simpler geometrical shapes may be computed by the methods which gave the conductance and permittance of conductors and of dielectrics of the same shape. In fact, the formulas for the permeance, conductance, and permittance of the same geometrical shapes are identical save that in the first the permeability appears, in the second the conductivity, and in the third the permittivity.

The permeance may be calculated only for those simple geometrical shapes in which the tubes of flux divide the core into cylinders all in parallel, or in which the equipotential surfaces divide it into cylinders all in series. In the former case, the permeance of the core is found by adding (by the operation of integration) the permeances of all the parallel connected elementary cylinders; in the latter case, the reluctance of the core is found by adding the reluctances of all the series connected elementary cylinders.

The calculation of the permeance and reluctance of cores of non-homogeneous material will follow the study of the properties of the ferromagnetic materials.

315. Localization of the Stored Energy of the Magnetic Field (Deduction).—When energy is stored in the potential form in a compressed gas, or in a bent rod, or in an electrostatic field, or when it is stored in the kinetic form in a rotating body or a moving water column, we think of the energy as being stored or associated in definite proportion with each cubic centimeter of space of the gas, rod, field, etc.

For some purposes it is helpful to think of the electrokinetic energy of an electric current as stored in definite proportions in each cubic centimeter of its magnetic field, and also to think of this energy as associated with the tubes of magnetic flux. For this purpose, the general expression previously derived for the energy of the electric current, namely one-half of the product of the flux-linkage times the current, or

$$W \text{ (joules)} = \frac{\Lambda I}{2} \text{ (weber-turns, amperes)} \tag{460}$$

may be thrown into the following form.

If a tube, over a cross-section of which the flux is Φ, links N times with the current, the flux-linkage Λ is $N\,\Phi$ weber-turns, and the expression for the stored energy is

$$W = \frac{(\Phi N)I}{2} = \frac{\Lambda I}{2}. \tag{460}$$

But we may also look at the matter in this way. If the tube of flux links with the current N times, the magnetomotive force $\mathfrak{F}$ around the tube equals NI ampere-turns. Therefore, by combining N as a factor with the current I, rather than with the flux Φ, the expression for the stored energy may be written

$$W = \frac{\Phi(NI)}{2}$$

$$W \text{ (joules)} = \frac{\Phi\mathfrak{F}}{2} \text{ (webers, ampere-turns)}. \tag{493}$$

315a. (Deduction).—*That is, the energy associated with a tube of magnetic flux is equal to one-half of the product of the flux multiplied by the magnetomotive force around the tube.*

The two viewpoints of the linkage of tubes of magnetic flux with the current which lead to the two forms, Eqs. (460) and (493), for the stored energy are illustrated in Figs. 262 and 263. A closed loop of iron wire, representing the magnetic circuit, is looped five or six times with a conducting circuit of copper wire. Let Φ represent the flux over the cross-section of the iron wire. If the copper wire is pulled taut as in Fig. 262, we ordinarily combine the factor N with the Φ and write the expression for the stored energy

$$W = \frac{(\Phi N)I}{2} = \frac{\Lambda I}{2}. \tag{460}$$

On the other hand, if the iron wire is pulled taut, as in Fig. 263, the expression is generally written in the form

$$W = \frac{\Phi(NI)}{2} = \frac{\Phi\mathcal{F}}{2}. \tag{493}$$

By substituting for Φ or for $\mathcal{F}$ in Eq. (493), their values in terms of the permeance $\mathcal{P}$ or of the reluctance $\mathcal{R}$ of the magnetic circuit, the formula for the energy stored in a core field may be thrown into the forms

$$\left.\begin{aligned} W \text{ (joules)} &= \frac{\Phi^2}{2\mathcal{P}} = \frac{\Phi^2\mathcal{R}}{2} \\ W \text{ (joules)} &= \frac{\mathcal{F}^2}{2\mathcal{R}} = \frac{\mathcal{F}^2\mathcal{P}}{2} \end{aligned}\right\}. \tag{494}$$

To find the energy stored per cubic centimeter of the field, consider a ring core (Fig. 261) in which the radial width is so small in comparison with the

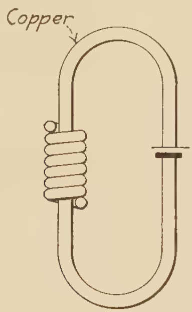

FIG. 262.

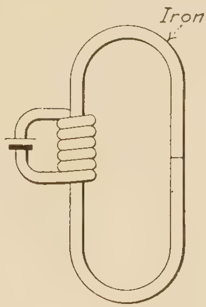

FIG. 263.

FIGS. 262 and 263.—Linkage of tubes of flux and current.

radius, r, that the flux density has substantially the same value at all points of any cross-section of the core. In other words, any short length of the core is for all practical purposes a right cylinder. From the symmetry, the energy stored in any cubic centimeter of the core will equal that stored in any other. Therefore, the energy w stored per cubic centimeter may be found by dividing the total energy stored, namely, $\Phi\mathcal{F}/2$, by the volume of the core, namely, al.

Whence

$$w \text{ (joules per cu. cm.)} = \frac{\Phi\mathcal{F}}{2al},$$

$$w \text{ (joules per cu. cm.)} = \frac{BH}{2} \text{ (webers, ampere-turns, cm).} \tag{495}$$

315b. (DEDUCTION).—*That is, the energy stored per cubic centimeter of the magnetic field is equal to one-half the product of the flux density times the magnetic intensity.*

By substituting for B or for H in the above expression their values in terms of the permeability μ or of the reluctivity ν of the material of the core, the

expression for the energy stored per cubic centimeter may be thrown into the forms

$$w \text{ (joules per cu. cm.)} = \frac{B^2}{2\mu} = \frac{B^2\nu}{2}$$
$$= \frac{H^2}{2\nu} = \frac{H^2\mu}{2}. \tag{496}$$

316. The Relation between the Flux Density and the Magnetic Intensity in Ferromagnetic Substances.—Before entering upon the study of ferromagnetic substances let us recall the sequence in which we have defined the magnetic quantities.

1. The magnetic flux density B at a point was defined as the force per ampere-centimeter upon a test conductor held at the point, in the position for maximum force.

2. In the field set up in free space (or in air) the flux density B at any point was found to be directly proportional to the current causing the field.

3. The line-integral of the flux density taken around any closed path in such a field was found to be directly proportional to the net current crossing any surface bounded by the path.

4. The magnetic intensity at a point was then defined as B/μ, in which the constant μ had such a value assigned to it that the line-integral of B/μ (or of H) around any closed path was equal to the net current crossing any surface bounded by the path of integration.

5. The line-integral of the magnetic intensity from A to B along any path was then termed the magnetomotive force along this portion of the path.

From these definitions it follows that the values of the flux density may be measured, but that the values of the magnetic intensity must be inferred from these measured values of B. Now in fields containing ferromagnetic substances, the flux density at a point is not proportional to the exciting current I, and the line-integral of the flux density around closed paths bears no simple relation to the current crossing the surface bounded by the paths. It is evident that in such fields, the ground for inferring or assigning a value of H is no longer valid. Therefore, it becomes necessary to specify anew the basis upon which we are going to assign values to the magnetic intensity in ferromagnetic substances.

The simplest way to define magnetic intensity in ferromagnetic substances is to agree as follows:[2]

1. To use in defining magnetic intensity a homogeneous ring of the magnetic substance, the ring being uniformly overwound with an exciting coil of N turns. The radial width, w, of the ring is to be small in comparison with its radius r (see Fig. 261).

2. To define the magnetomotive force around any filament of flux in the ring as equal to NI.

3. To define the magnetic intensity in any filament as equal to NI/l, in which l is the length of the filament. $l = 2\pi r$ (approximately).

316a. Experiment 1.—Let a homogeneous ring of iron (Fig. 264) be uniformly overwound with two coils, a primary and a secondary, containing N_1 and N_2 turns of copper wire, respectively. The secondary (or the N_2 coil) is to be wound directly upon the iron ring and is to be connected to the

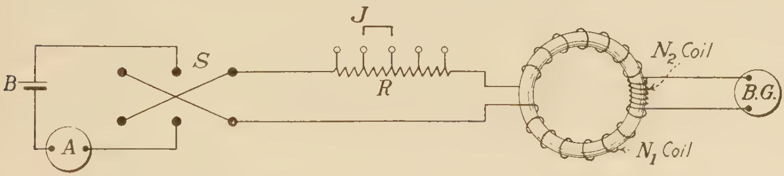

FIG. 264.—Connections for taking magnetization curves.

terminals of a ballistic galvanometer G. The ballistic galvanometer has been calibrated by the method outlined in Sec. 271 to read the values of the voltage-impulses induced in the secondary.

The primary is wound outside the secondary and must be uniformly wound over the the entire ring. The magnetizing current supplied to the primary from the battery B may be read on the ammeter A. This current can be suddenly increased or decreased in small steps, by plugging or withdrawing jumpers J, from the portions of the rheostat R. The current may be reversed by the reversing switch S.

To determine the relation between the flux density B and the magnetic intensity H in the iron ring, the procedure is as follows:

Step-by-step Method.—Starting with an iron core which has never been magnetized (or which has been demagnetized by the method later described), and with all the resistance in the rheostat R, let the switch S be closed and be left closed, say, to the right. Let the throw θ of the ballistic galvanometer and the steady value I of the magnetizing current in the primary be read and

[2] It seems hardly possible to define H in any *usable* way merely by reference to experiments in any involved field made up of non-homogeneous magnetic substances.

recorded. From the throw, the voltage-impulse V induced in the secondary may be read from the calibration curve of the galvanometer. The increment in the flux-linkage of the secondary is equal to this voltage-impulse (sec. 292c).

$$\Delta\Lambda_2 \text{ (weber-turns)} = V\text{(volt-seconds)}. \tag{439}$$

$$\Delta\Phi \text{ (webers)} = \frac{V}{N_2}\text{ (volt-seconds)}. \tag{497}$$

But if the radial width w of the ring is small in comparison with its radius r, the flux density is substantially the same at all points of the cross-sectional area a.

$$\Delta B \text{ (webers per sq. cm.)} = \frac{V}{N_2a}\,\frac{\text{(volt-seconds)}}{\text{(cm.)}}. \tag{498}$$

This is the increment in the flux density (from the zero value) caused by one increment in the magnetic intensity from zero by the amount

$$\Delta H \text{ (ampere-turns per cm.)} = \frac{N_1(\Delta I)}{2\pi r}. \tag{499}$$

Let the resistance of the primary be now reduced by a small amount by plugging a jumper around a portion of the rheostat. Let the throw θ_2 of the galvanometer and the value I_2 of the primary current be recorded. From

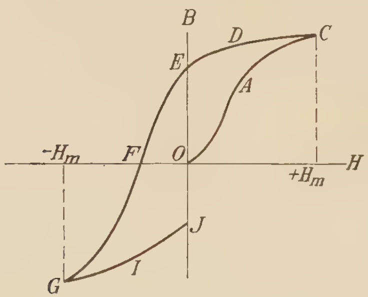

FIG. 265.—Initial B-H characteristic for virgin iron.

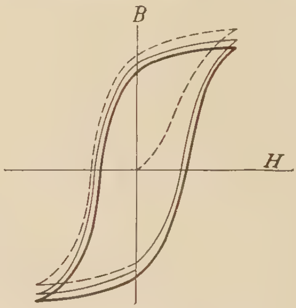

FIG. 266.—Successive B-H characteristics.

these readings, by means of the relations above, the second increase in the flux density corresponding to the second increase in magnetic intensity may be calculated. In like manner a third pair of corresponding increments in B and H may be obtained and so on. The flux densities corresponding to the successive magnetic intensities may then be found by summing up the successive increments in flux density.

By proceeding, step by step, in this manner the flux densities and the corresponding magnetic intensities up to any desired value H_m may be determined and plotted, as in the curve OAC of Fig. 265. In a similar manner, after this maximum value has been reached, the flux densities corresponding to successively smaller values of the magnetic intensity may be determined by making step-by-step increases in the resistance in the primary, finally opening the reversing switch S. These values are plotted

in the curve *CDE* of Fig. 265. The reversing switch may then be closed in the opposite direction, and by a repetition of the steps of the above half cycle, the flux densities corresponding to successive values of magnetic intensity increasing up to H_m in the **negative** direction, and then decreasing to zero, may be determined. These values are plotted in the curves *EFG* for rising, and *GIJ* for falling values of *H*.

The magnetizing current has now been carried through a complete cycle of values; starting from zero, rising to a maximum value I_m, decreasing to zero, reversing, and rising to the same maximum value I_m but in the reverse direction, and then decreasing to zero. The corresponding values of flux density and magnetic intensity have been plotted in the so-called *B-H* **characteristic** for this cycle. At the end of the cycle the iron is not in its original state, since the flux density is not zero, but has a certain residual value, represented by *OJ*. If the current is again reversed and is carried through the original cycle of values, a second *B-H* cycle is traversed which starts from this residual value of the flux density. The first cycle of values is shown by the dotted curve of Fig. 266 and the second cycle by the light, full-line curve. The second curve falls (on the diagram) below the first. A third cycle would fall very slightly below the second, and so on. Eventually, if the current is repeatedly carried through the same cycle of values, the *B-H* curve becomes a closed loop which is repeatedly traversed. This loop (shown by the heavy lines) is symmetrical with respect to the origin of coordinates. This loop may be called the ultimate or **steady-state loop** for the cycle having a peak magnetic intensity of H_m.

316b. Magnetic Hysteresis, Retentivity, and Coercive Force. The feature of this *B–H* loop is that the flux density *B* corresponding to a given value of the magnetic intensity while *H* is increasing is less than that value of the flux density which corresponds to the same value of magnetic intensity while *H* is decreasing. That is, the ascending and descending branches do not coincide, but differ in a way which may be described by saying that there is a tendency at each stage for the preceding condition of the flux density to persist, or to be **retained.** If the values assumed by *B* and *H* (or by *B* and *i*) while the above cycle is traversed are plotted to rectangular coordinates against time, the *B* curve lags behind the *H* curve. See the curves for current and flux density in Fig. 268 of experiment 3. As a name for a relation of this kind between two related variables, Professor Ewing coined the term **hysteresis** (from the Greek, "to be behind").[3] Quoting Ewing: "When there are two quantities *M* and *N* such that cyclic variations of *N* cause cyclic variations

[3] For a further discussion, see Sec. 322*a*.

of M, then if the changes of M lag behind those of N, we may say that there is **hysteresis** in the relation of M to N." The phenomenon exhibited by the relations between B and H in the B-H loop is accordingly called the phenomenon of **magnetic hysteresis,** and the loop is generally referred to as the **hysteresis** loop.

A feature of the phenomenon of magnetic hysteresis is that when the magnetizing current and the magnetic intensity have decreased from the tip value along CDE to zero, the flux density is not zero but has a large value (represented by OE) called the **residual flux density.**

Residual flux density (in webers per sq. cm.) = OE (Fig. 265).

The ratio of this residual flux density to the tip value from which the flux density has decreased is called the "retentivity" of the sample.

$$\text{Retentivity} = \frac{\text{residual flux density}}{\text{tip flux density}} = \frac{OE}{H_mC}.$$

The retentivity of some samples is as high as 92 per cent.

To reduce the flux density from the residual value to zero, the magnetizing current must be reversed and the magnetic intensity must be increased to the value OF in the reverse direction. *The value to which the magnetic intensity must be raised in the reverse direction in order to reduce the flux density from the residual value to zero, is called the "coercive force" of the material.*

Coercive force (in amp.-turns per cm.) = OF (Fig. 265).

For industrial purposes, two classes of ferromagnetic materials are desired. For appliances carrying alternating fluxes (as in transformers, generators, and motors), the area of the hysteresis loop, or the retentivity and coercive force of the iron, must be small. Energy is dissipated in the material when the loop is traversed, and the energy dissipated per cycle is proportional to the area of the loop. On the other hand, for the permanent magnets in measuring instruments and in magnetos, high retentivity and high coercive force are most desirable characteristics.

316c. Step-by-step Method with the Grassot Fluxmeter.—The advantage of the step-by-step method just described is that it gives such a clear conception of the shape of the B-H loops. The method, however, yields values

for the flux density in samples of very soft iron which are in error if the sample is even slightly jarred during the readings. In this case, the error arises from the fact that when the magnetic intensity is either increased or decreased to a given value, the flux density does not immediately assume a fixed value corresponding to that magnetic intensity. The change in the magnetic intensity is accompanied by the main change in the flux density followed by a slight **creep** of the flux density in the same direction as the main change. The throw of the galvanometer reads the main change in flux density, but the method takes no account of the slight drift which occurs between readings. The drift and the corresponding error are prohibitively large if the iron ring is constantly jarred during the experiment.

The Grassot fluxmeter is a galvanometer designed to give a reading which will be proportional to the total change in the flux-linkage of the secondary (or of any exploring coil), regardless of the rate at which the change has taken place. The fluxmeter is a sensitive moving coil galvanometer having a very light coil suspended in the strong uniform field in the narrow air gaps between the poles of permanent magnets, as illustrated in Fig. 267. The moving coil C is suspended by a silk fiber, the upper end of which is attached to a spring R to minimize the effect of shock. The current from the exploring coil is led in and out of the coil C by means of thin, silver strips (s and s'). The feature of the galvanometer is that the torsional forces exerted upon the coil by the silk suspension and the silver-lead wires and the damping forces of the wind friction are made exceedingly small. On the other hand, the electromagnetic damping forces on the current which is induced in the coil when it moves in its magnetic field are made so great that the coil remains in any position to which it is moved by the current resulting from a change in the flux-linkage of the exploring coil to which it is permanently connected. The movable coil carries a pointer which moves over a divided scale.

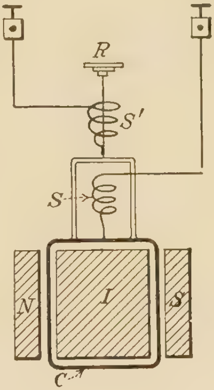

FIG. 267.—Movable coil of Grassot fluxmeter.

It may be shown by experiment or from the electromechanical equations applying to the moving system that a change in the flux-linkage of the exploring coil to which the instrument is connected will produce a permanent deflection of the pointer which is directly proportional to the increment in the flux-linkage of the exploring coil, and is independent of the rate at which the increment has occurred. Therefore, if such an instrument is substituted for the ballistic galvanometer in the above experiment, and if it is left connected to the secondary coil, it serves to detect and to read any drift in the flux in the core.

316d. Experiment 2. Step-from-the-tip Method.—A steady-state B-H loop which is free from the error due to drift may be obtained by taking each

reading from the current and flux density corresponding to the magnetic intensity, H_m at the tip of the desired steady-state loop. The current is reduced by successively greater steps, each time from the value I_m. Eventually the reversing switch is used, and, in the final reading, the current is reduced from I_m to $-I_m$. From the throw of the galvanometer, the decrease in flux density corresponding to each decrease in current may be calculated. This decrease takes place from the peak value B_m. This peak value is equal

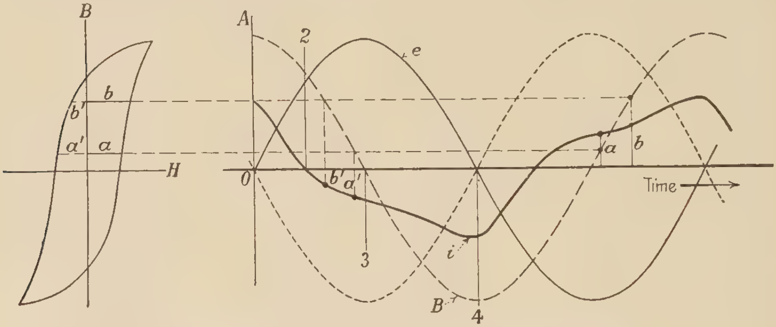

Fig. 268.—Oscillograms of e.m.f. and exciting current.

to one-half of the change in flux-density which occurs when the current I_m is reversed. Between each reading, the magnetizing current must be carried about six times over the cycle from $+I_m$ through $-I_m$, and back to $+I_m$, by manipulating the reversing switch. This is to insure that the flux density starts from the peak value B_m for each reading. The steady-state loop obtained by the step-by-step method does not differ greatly from the more correct loop obtained by this method.

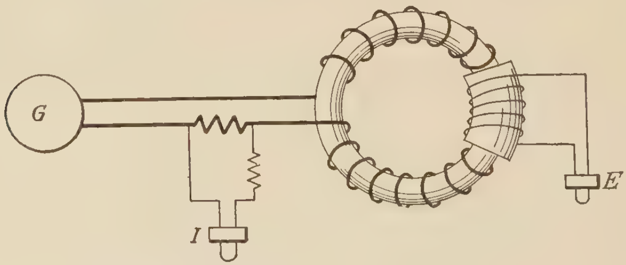

Fig. 269.—Connections for taking oscillograms.

316e. Experiment 3. Oscillograph Method.—The steady-state B-H loop may also be obtained from a photographic oscillograph record showing the wave forms of the electromotive force which is induced in the secondary and of the magnetizing current, which flows in the primary coil on the iron ring when an alternating electromotive force having a sine wave form is impressed upon the primary. Such a record is shown in Fig. 268. The curve marked

e is given by an oscillograph vibrator E (Fig. 269) connected in series with the secondary; its ordinates are directly proportional to the instantaneous values of the electromotive force induced in the secondary. The curve marked i is given by a vibrator I connected in shunt to a small non-inductive resistance in the primary circuit. Its ordinates are directly proportional to the instantaneous values of the magnetizing current. The induced e.m.f. is seen to vary in a sinusoidal manner.[4] Measuring time from the instant indicated by the heavy ordinate OA, the equation of the induced e.m.f. is

$$e \text{ (volts)} = E_m \sin 2\pi ft, \tag{500}$$

in which, f is the frequency (number of cycles per second) of the alternating current.

E_m is peak value of the e.m.f. induced in the secondary. This value is to be obtained by calibrating the vibrator and scaling the maximum ordinates of the e curve.

Now the instantaneous value of the induced electromotive force is equal to the rate of decrease of the flux-linkage.

$$e \text{ (volts)} = \frac{-d\Lambda \text{ (weber-turns)}}{dt \quad \text{(seconds)}}$$

$$\Lambda = -\int e\,dt$$

$$= \int E_m \sin 2\pi ft\,dt$$

$$= \frac{E_m}{2\pi f}\cos 2\pi ft$$

$$\Phi\text{(webers)} = \frac{E_m}{2\pi f N_2}\cos 2\pi ft$$

and

$$B \text{ (webers per sq. cm.)} = \frac{E_m}{2\pi f N_2 a}\cos 2\pi ft. \tag{501}$$

The flux density in the ring is seen to vary in a sinusoidal manner. A curve showing the instantaneous values of the flux densities will therefore be a sine curve 90 degrees in advance of the curve showing the induced e.m.f. Such a curve has been plotted in Fig. 268, as curve B. By erecting ordinates at many points along the time axis for one cycle, the corresponding values of the flux density and of the magnetizing current may be read off, and the corresponding magnetic intensities may be calculated from the relation in Eq. (499).[5]

$$H = \frac{N_1 i}{2\pi r}. \tag{499}$$

[4] The resistance of the primary coil must be so low that the Ri electromotive force of the primary is negligibly small in comparison with the electromotive force of inductance; otherwise the induced secondary voltage and the inducing flux will be distorted from the pure sinusoidal form.

[5] The secondary supplies the current necessary to actuate the voltage vibrator. The magnetomotive force of this current must be taken into account. This statement presupposes that the magnetomotive force of this

If the corresponding values of B and H thus obtained are plotted to rectangular coordinates, they yield a steady-state B-H loop similar to that shown in Fig. 266. Let us follow through one cycle to note the features of the curve. The maximum values of the magnetizing current and of the flux density coincide in time with the moment of zero e.m.f. (see the intercepts of the ordinate OA). These values are the tip values of the B-H loop. When the magnetizing current has decreased to zero, the flux density still has a large residual value (see ordinate 2). To bring the flux density to zero, the current must be reversed and must attain the negative value indicated by the intercept on ordinate 3. The negative tip values of B and H are indicated by the intercepts on the ordinate 4, erected at the next moment of zero e.m.f.

316f. Experiment 4.—The steady-state B-H loops corresponding to higher and higher tip values of flux density and of magnetic intensity may be obtained from oscillograph records taken with sinusoidal voltages of higher and higher value impressed on the primary coil. The features of a set of loops obtained in this manner for a given ring sample are shown in Fig. 270.

316g. Experiment 5. Alternating B-H Characteristic by the Method of Reversals.—The dotted curve drawn through the tips of these loops of

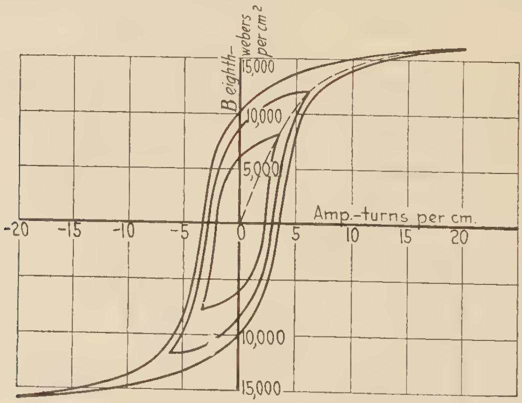

FIG. 270.—Steady-state B-H hysteresis loops.
Ordinary "Electrical Sheets." $w = 2000B^{1.6}$

Fig. 270 gives information which is of importance in the design of transformers. It shows the peak values of the magnetic intensities (ampere-turns per centimeter) necessary to cause the flux densities to alternate

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current has been reduced to a value which is negligibly small in comparison with the m.m.f. of the primary current. It also presupposes that the eddy currents in the iron have been kept so small by laminating the ring that their m.m.f. may be neglected.

between definite values. It will be called the **alternating B-H characteristic of the substance.**

The points for plotting this characteristic may be obtained more expeditiously by the following **method of reversals.** With the connections shown in Fig. 264, the rheostat is set to give the current necessary for the highest magnetic intensity desired. The current is reversed five or six times to establish a steady-state loop and then the ballistic galvanometer is connected to the secondary and its throw is noted for a final reversal of the current. From this throw the change in flux density may be computed. This change is from B_m to $-B_m$ or it equals $2B_m$. The magnetizing current is then reduced to some lower value, and is reversed a number of times to establish a new loop. The throw caused by a final reversal of this current is then observed and the corresponding value of B_m is computed, and so on. These flux densities if plotted against the magnetic intensities (as computed from the magnetizing currents) will give the locus of the tips of the B–H loops for the sample tested.

317. Demagnetization by Reversals.—The B-H curves show that a sample of iron which has been magnetized is, in general, left in a magnetized condition. Even though the magnetizing current may have been interrupted at such a point as to leave the flux density in the iron at zero, the iron may be magnetically **biased.** For example, suppose that on the ascending branch AC of the hysteresis loop of Fig. 271, the current is interrupted at such a point C that as the current drops to zero, the flux density drops along the curve CO to zero. Such a sample would exert no force on a piece of soft iron and would thus seem to be in a neutral state; but its condition is quite different from that of a virgin piece of iron, or from that of a piece which has been made neutral by the process of **demagnetizing by reversals.** Such pieces show no directional difference, but if the rising characteristic of the above sample is taken by the step-by-step method, the curve OP is obtained if the applied magnetic intensities are positive, and the curve ON if they are negative. That is, the iron retains latent traces of the magnetic changes it has passed through, which cause it to show a lack of directional symmetry. This lack of directional symmetry is very strikingly and persistently shown by iron which has been strongly magnetized and left with a high residual flux, and then subsequently subjected to alternating magnetic intensities of small or moderate amplitude.

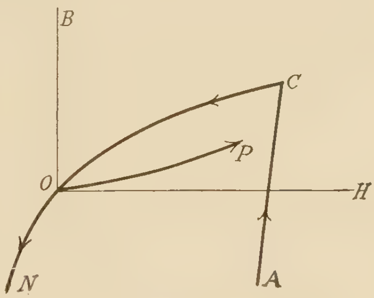

FIG. 271.—Effect of biased condition.

By the following **method of reversals** a biased piece of iron may be demagnetized or rendered magnetically neutral. Place the sample in a field of very high magnetic intensity and then, while continually reversing

the direction of the field relative to the sample, slowly reduce the magnetic intensities to zero. This may be accomplished as follows:

a. Pass a large alternating current through a winding on the sample, or through a large solenoidal coil in which the sample is placed, and slowly reduce the alternating current to zero.

b. Or use a direct current and rapidly reverse the direction of the current by a reversing switch, while slowly reducing the current to zero.

c. Or place the sample in the strong field of a direct current solenoid, and while gradually removing the sample from the field of the solenoid rapidly rotate the sample in the field.

318. Flux Density Attributed to the Molecular Currents.—We conceive that the difference between the value of the flux density in a **material** core under the influence of a given magnetizing current and the value of the flux density which would exist in free space if the core were removed, is due to **concealed molecular** currents in the material. It is found to be very helpful to distinguish between that portion of the flux density at a given point which is attributed to the **obvious** magnetizing current in the exciting winding and the portion attributed to the **concealed** molecular currents. That is, we find it helpful to write:

$$B = B_m + B_o \tag{502}$$
$$B = B_m + \mu_o H_o,$$

in which B is the actual or total flux density.

B_m is the flux density due to the **concealed** molecular currents.
B_o is the flux density due to the **obvious** magnetizing currents.
H_o is the magnetic intensity due to the obvious magnetizing currents.
μ_o is the permeability of free space.

318a. *By the flux density attributed to the "molecular curents," B_m, at any point in a substance is meant the difference between the actual magnetic flux density B at the point and the magnetic flux density which would exist at the point if the substance were replaced by free space under the same magnetic intensity. In ferromagnetic*

materials this quantity is frequently called the "metallic" flux density.[6]

$$B_m = B - \mu_o H_o. \qquad (502a)$$

There is, apparently, no limit to the values which may be reached by the magnetic flux density B in any material (save the experimental limit which is set by the heating of the conductors carrying the magnetizing current). On the other hand, no matter how great the exciting magnetic intensity, the metallic flux density B_m in any given ferromagnetic material can not be made to exceed a certain limiting value. This value is called the **saturation value** B_s for the metallic flux density, in the specified material.

The saturation values for various substances are given in the following table.

318b. Saturation Values B_s of the Flux Densities from Molecular Currents.

Material	B_s in eighth-webers per sq. cm.
Iron-cobalt alloy ($FeCO_2$)	25,800
Pure iron	22,600
Electrical steels	20,000
Cast iron	15,000
Cobalt	17,000?
Nickel	7,000
Heusler alloy ($AlMnCu_2$)	7,000

319. *B-H* Characteristics.—A given ferromagnetic material has an infinite number of **rising** and **falling** *B-H* characteristics. The shape of these characteristics depends upon the tip values between which the magnetic intensity rises and falls. From this it follows that the expression "the *B-H* characteristic" of a given material is a very indefinite expression, unless it is qualified in such a manner as to indicate the purpose for which (or conditions under which) the characteristic was obtained. To deter-

[6] The quantity J defined by the following equation is called the **intensity of magnetization** of a material.

$$J = \frac{B_m}{4\pi} = \frac{B - \mu_o H}{4\pi}. \qquad (503)$$

mine the merits of a material for use in permanent magnets, its falling characteristic from very high magnetic intensities should be taken. If a material is to carry an alternating flux, the hysteresis loops connecting the flux densities between which it is to alternate are of interest, since the area of these loops determines the hysteresis loss.

In Fig. 272 have been sketched the relative positions of three characteristics of a given sample. These characteristics all pass through the origin. It is important to distinguish between them. They may be called:

a. The rising *B-H* characteristic after demagnetization by reversals (step-by-step method).

b. The stable *B-H* characteristic.

c. The alternating *B-H* characteristic.

The method of demagnetizing a sample by reversals and of taking its **rising** *B-H* **characteristic** by the step-by-step method has been discussed. It may be noted that the **virgin rising characteristic** taken on the sample before it has been magnetized at all will not necessarily coincide with the above characteristic.

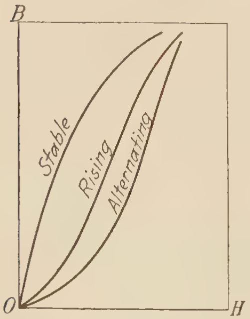

FIG. 272.—Important *B-H* characteristics.

The **stable B-H characteristic** is obtained by holding the magnetizing current constant at each step, while the iron is either thoroughly jarred or is subjected to a superimposed cyclic intensity obtained by passing an alternating current through an auxiliary magnetizing winding.

The alternating *B-H* characteristic is the curve drawn through the tips of the steady-state *B-H* loops. It may be obtained from oscillograph records, or by the method of reversals as explained under experiments 4 and 5. Since this characteristic shows the ratios of the tip values of *B* to *H* for the magnetizing cycles of alternating currents it is the most important of the three. It is the characteristic referred to in subsequent discussions unless otherwise noted.

The typical alternating *B-H* characteristics of a number of the ferromagnetic materials have been plotted in Fig. 273. "The

electric steel sheets" (a low carbon steel) and the 3.5 per cent silicon steel sheets are steels especially made and annealed for use with alternating fluxes.

The magnetic properties of a given material are greatly affected by the heat treatment and the mechanical working which the specimen has received. The effect of a slow annealing

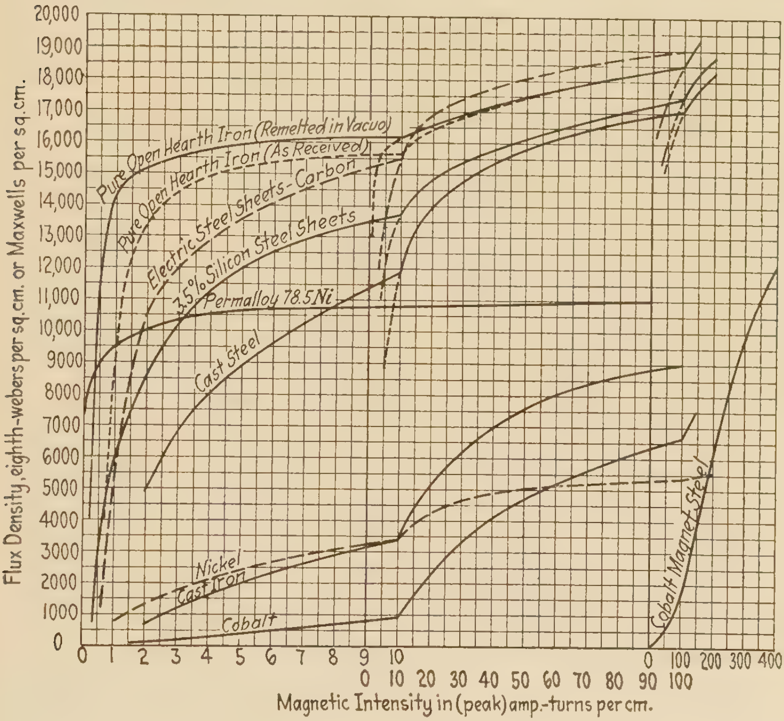

FIG. 273.—Alternating *B-H* characteristics.

from a red heat is to increase the permeability and to decrease the width of the *B-H* loop. The position of the *B-H* curves of impure iron and of the different steels depends greatly upon the nature and amount of the impurities and of the alloying ingredients.

Effect of Temperature.—The characteristics of Fig. 273 are all for materials at room temperature. The effect of raising the temperature of the material is to increase the ratio of B to H

for the lower magnetic intensities, until the so-called **critical temperature,** or the point of **recalescence,** of the specimen is reached. If the temperature is raised above this point, all ferromagnetic materials become paramagnetic. The critical temperature varies from 690 to 870° for different specimens of iron and steel.

320. Permeability of Ferromagnetic Materials.—We have seen that a given flux density in a ferromagnetic ring may be accompanied by any one of an infinite number of positive and negative magnetic intensities between two limiting values. If, then, we wish to call the ratio of B to H the permeability of the ferromagnetic material, we must recognize that in this case the permeability is not a constant of the material. It is not even expressible as a function of B or of H. It is a function of the preceding magnetic history of the sample. Accordingly, the concept of permeability applied to a ferromagnetic material may be worse than a useless notion, except in so far as we attach a precise meaning to it by specifying the experimental procedure by which the values of B and H are to be obtained.

Unless otherwise stated, we shall always mean by the permeability of a ferromagnetic material, the ratio of B to H as taken from the alternating B-H characteristic for a specified tip intensity or tip flux density.

In our discussions and curves of the properties of ferromagnetic materials, we will use the relative permeability μ_r rather than the absolute permeability of the materials.

320a. *By the relative permeability μ_r of a ferromagnetic material we mean the ratio of its permeability (as obtained from the alternating B-H characteristic) to the permeability μ_o of free space ($\mu_o =$* 1.257×10^{-8}).

$$\mu_r \text{ (a numeric)} = \frac{B}{H} \div \mu_o = \frac{B}{\mu_o H} \text{ (definition).} \tag{504}$$

The relative permeability of a material is equal to its absolute permeability as expressed in the maxwell-gilbert-centimeter units of the E.M. system of units. Therefore, the relative permeability of materials may be taken directly from existing tables and curves in which the data is in E.M. units.

From the alternating B-H characteristics shown in Fig. 273, the relative permeabilities of the materials have been computed and the results have been plotted against magnetic intensity in the *relative permeability curves* of Fig. 274.

It will be seen that the relative permeability of sheet steels is small at weak magnetic intensities, that it may rise to values as high as 5,000 at magnetic intensities of the order of 0.5 amperes

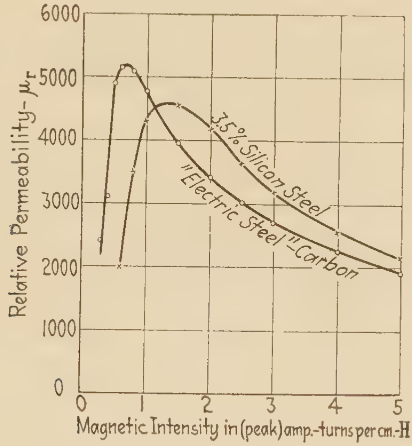

FIG. 274.—Permeability characteristics of steel

turn per centimeter, and that it again decreases and approaches unity as a limit for magnetic intensities in excess of 10,000 ampere-turns per centimeter.

320b. Permeability at Low Magnetic Intensities.—For magnetic intensities less than 0.3 ampere-turn per centimeter, the relative permeability of a certain ring of annealed soft iron was found to be given by the empirical formula

$$\mu_r = 183 + 1730\ H \text{ (for } H < 0.3 \text{ ampere-turn per centimeter).} \quad (505)$$

In striking contrast to this comparatively low permeability of ordinary steels at low magnetic intensities, **permalloy**, an alloy of 78.2 per cent nickel and 21.3 iron with small amounts of other metals (as impurities), has a relative permeability of 10,000 at 0.002 ampere-turn per centimeter. This increases to a maximum of 90,000 at about 0.04 ampere-turn per centimeter.

320c. Permeability at Intermediate Magnetic Intensities.—For the range of magnetic intensities from 0.2 to 1 ampere-turn per centimeter, no simple empirical relations exist between the relative permeability and the magnetic

intensity in the commercial steels. However, it has been found that for wide ranges of magnetic intensity, above 1 ampere-turn per centimeter, for the annealed steels, and above 40 ampere-turns per centimeter for hardened steels, the relative permeabilities can be expressed with considerable exactness by empirical formulas of the form[7]

$$\mu_r = \frac{1}{a + \sigma H} + 1. \tag{506}$$

Since B or $(B_m + B_o) = \mu_r\ (\mu_o H)$ (502 and 504)

$$B = \frac{\mu_o H}{a + \sigma H} + \mu_0 H, \tag{507}$$

and since $B_o = \mu_o H$, we see that the flux densities added by the concealed molecular currents may be calculated from the empirical formula

$$B_m = \frac{\mu_o H}{a + \sigma H}. \tag{508}$$

We see that in Eq. (506), the term $1/(a + \sigma H)$ expresses the relative permeability due to the flux attributed to the **concealed** molecular currents, and the second term (1), expresses the relative permeability due to the flux attributed to the **obvious** exciting current. For all magnetic intensities encountered in engineering practice, the second term is less than 1 per cent of the first and may be dropped. The first term is sometimes called the **relative metallic permeability,** μ_{rm}.[8]

$$\mu_{rm} \text{ (a numeric)} = \frac{B - \mu_o H}{\mu_o H} \text{ (defining } \mu_{rm}\text{)}. \tag{509}$$

For large values of H, the values of the relative metallic permeability may be calculated from the empirical relation,

$$\mu_{rm} = \frac{1}{a + \sigma H}. \tag{510}$$

It may be noted that for very large values of the magnetic intensity Eq. (508) may be written

$$B_m = \frac{\mu_o H}{a + \sigma H} = \frac{\mu_o H}{\sigma H} = \frac{\mu_o}{\sigma}. \tag{514}$$

[7] KENNELLY, A. E.: *Magnetic Reluctance,* Trans. Am. Inst. Elec. Eng., 1891, Vol. VIII, p. 485.

BALL, J. D.: *The Reluctivity of Silicon Steel as a Linear Function of the Magnetizing Force,* Gen. Elec. Rev., 1913, Vol. XVI, p. 750.

[8] In the treatment of magnetic materials from the magnetic pole concept, the term **the magnetic susceptibility** K **of the material** is used. The susceptibility is defined as the ratio of the intensity of magnetization J to the magnetic intensity H.

$$K = \frac{J}{H} = \frac{B - B_o}{4\pi H} = \frac{B_m}{4\pi H}. \tag{511}$$

It will be seen that the susceptibility bears the following relations to the relative permeabilities.

$$\mu_{rm} = 4\pi K. \tag{512}$$

$$\mu_r = \mu_{rm} + 1 = 1 + 4\pi K. \tag{513}$$

That is, the indicated saturation value of the metallic flux density is μ_o/σ. Since σ indicates the saturation value, it has been called the **saturation coefficient.** The coefficient a has been called the **coefficient of magnetic hardness,** since a large value for a indicates that the material is magnetically **hard,** or does not become strongly magnetic under small intensities. The values of a and σ are given for different materials in the following table.[9]

320d. Permeability and Reluctivity Constants.

Material	Limiting values of H, ampere-turns per centimeter	a	σ	μ_o/σ eighth-webers
2.5 per cent silicon steel, annealed.	1– 40	1.0×10^{-4}	7.85×10^{-5}	16,200
2.5 per cent silicon steel, annealed.	40–160	9.4×10^{-4}	6.24×10^{-5}	20,200
3.5 per cent silicon steel, annealed.	1– 40	1.3×10^{-4}	7.84×10^{-5}	16,000
3.5 per cent silicon steel, annealed.	40–160	9.2×10^{-4}	6.34×10^{-5}	19,850
Carbon steel sheets, annealed.	2– 10	1.1×10^{-4}	7.0×10^{-5}	18,000
Nickel.	8-300	1.6×10^{-3}	2.05×10^{-4}	6,150
Cobalt (cast).	25–300	8.7×10^{-3}	1.13×10^{-4}	11,100

320e. Permeability at Extremely High Magnetic Intensities.—The substitution of any of the values from this table in Eq. (507) will show the justification of the following rule for calculating the magnetic intensity necessary to set up a flux density which is greater than the metallic saturation value of that material.

The magnetic field intensities required to set up in iron or steel a magnetic flux density exceeding the metallic saturation value by more than 1000 *eighth-*

[9] If the absolute permeability of a material is expressed not in practical units, but in E.M. units, in which B is expressed in maxwells per square centimeters, and H in gilberts per centimeter, its value may also be calculated by an empirical formula similar to Eq. (506)

$$\mu \text{ (in E.M. units)} = \frac{1}{a_1 + \sigma_1 H} + 1. \tag{506a}$$

The relation of the numerical values of the a and σ in Eq. (506) to the a_1 and σ_1 in Eq. (506a) is as follows:

$$a = a_1$$
$$\sigma = 0.4\pi\sigma_1.$$

webers may be found approximately by dividing the difference between the required flux density and the saturation value B_m of the material by the permeability of free space.

$$H \text{ (amp.-turns per cm.)} = \frac{B - B_m}{\mu_0} \text{ (webers per sq. cm.)} \tag{515}$$

321. Relative Reluctivity of Ferromagnetic Materials.—The relative reluctivity of a ferromagnetic material is defined to mean the reciprocal of its relative permeability as defined in the previous section.

$$\nu_r \text{ (a numeric)} = \frac{\mu_0 H}{B}. \tag{516}$$

The relative reluctivities of the materials whose alternating B-H characteristics are given in Fig. 273 have been plotted against the magnetic intensities in Fig. 275. It will be noted that the curves each consist of two portions.

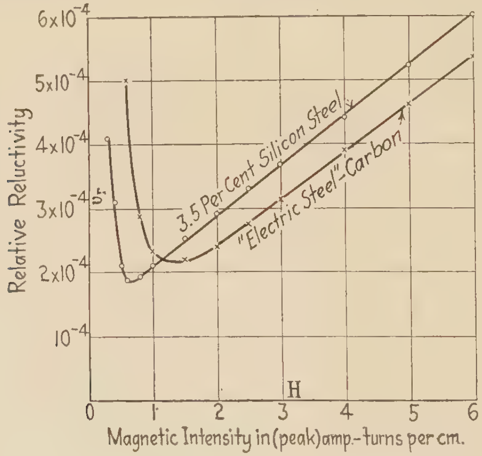

FIG. 275.—Reluctivity characteristics of steel.

For low values of H, the reluctivity decreases in value as H increases. For higher values of H, the reluctivity increases as H increases, and the relation between ν_r and H is seen to approximate a straight line. These straight lines are those portions of the curve for which the corresponding empirical equation for the permeability is Eq. (506), and for which the empirical equation for the relative reluctivity will be

$$\nu_r = \frac{1}{\mu_r} = \frac{1}{\frac{1}{a + \sigma H} + 1} \tag{517}$$

For soft steels and for values of H below 100, the second term in the denominator is negligibly small and the equation may be written

$$\nu_r = a + \sigma H \text{ (approximately)}. \tag{517a}$$

If the curves were continued to magnetic intensities as high as 5000 ampere-turns per centimeter, they would depart (as indicated by 517) from the straight-line relation, becoming concave downward. For very high values of H, the relative reluctivity approaches unity as a limit.

In reluctivity curves covering the values of H above 100 ampere-turns per centimeter, it is customary to plot, not the relative reluctivity (as defined by Eq. 516), but the **relative metallic reluctivity** ν_{rm} as defined below.

$$\nu_{rm} \text{ (a numeric) } = \frac{\mu_o H}{B - \mu_o H}. \tag{518}$$

For large values of H, the value of ν_{rm} may be calculated from the empirical relation

$$\nu_{rm} = a + \sigma H. \tag{519}$$

322. Hysteresis Loss in Ferromagnetic Materials.—Thus far, our discussion of the properties of ferromagnetic materials has dealt almost entirely with the relation between the values of the flux density and magnetic intensity as obtained from the **tips** of steady-state B-H loops. We now turn to a consideration of equal importance. When the flux density in a ferromagnetic substance is carried through the cycle of values shown by the hysteresis loop, energy is found to be expended or dissipated in the form of heat in the ferromagnetic material. The experimental evidence of this is that when the material is repeatedly carried through the cyclic process by a current which rapidly alternates (at a frequency of 25 cycles per second, or higher) the temperature of the material rises above that of its surroundings. Since this expenditure of energy is accounted for by the mechanism used to account for magnetic hysteresis, it is termed the loss of energy due to hysteresis, or briefly, the **hysteresis loss.**

We proceed to compute the energy which is delivered by the generator to the magnetizing coil of the ring core of Fig. 261 while the hysteresis loop is being traversed.

Let l represent the mean length of the core.
a represent the cross-sectional area of the core.
N represent the number of turns in the magnetizing coil.

For the purpose of simpifying the calculations, let us assume the resistance of the winding to be zero. Let Fig. 276 represent the hysteresis loop which is traversed. If the magnetic flux density increases from the value B_1 to the value $(B_1 + dB)$ in

the interval of time dt, the e.m.f. induced in the winding by the changing flux is

$$e \text{ (volts)} = -aN\frac{dB}{dt}. \tag{443a}$$

If H_1 is the magnetic intensity corresponding to the flux density B_1, the current in the winding must be

$$i = \frac{H_1 l}{N}.$$

Since the e.m.f. of the generator must be equal and opposite to the e.m.f. induced in the magnetizing winding by the changing flux, the energy dW delivered by the generator to the exciting winding in the interval of time dt is

$$dW = (-ei\,dt) = aN\frac{dB}{dt}\frac{H_1 l}{N}dt = H_1 al(dB). \tag{520}$$

Since the time dt taken for the flux to increase by the amount dB does not appear in this expression, it follows that the energy supplied by the generator when the flux so increases is independent of the time taken. The energy is seen to be equal to the volume of the core, al, times the area of the small cross-hatched figure.

If the hysteresis loop is traversed in the interval of time p, the energy delivered to the exciting winding, while the loop is being traversed, is

$$W = \int_0^p ei\,(dt) = al\int^{1\text{ cycle}} HdB.$$

The energy w expended per cubic centimeter of the core in 1 cycle is

$$w \text{ (joules per cu. cm. per cycle)} = \int^{1\text{ cycle}} HdB \text{ (amp.-turns webers, centimeters)} \tag{521}$$

An inspection of Fig. 276 will show that while the flux density increases from the zero value (point A in the loop), to the tip value (point C), both H and dB are positive quantities and the value of the integral between these limits (or of the energy supplied by the generator) is represented by the area $OACDEO$. As the portion CE of the loop is traversed, H is positive but dB

is negative, and energy to the amount represented by the area *CDE* will be **returned** from the field of the coil to the generator. As the portion *EFG* is traversed, both H and dB are negative quantities, and the energy supplied by the generator will be represented by the area *EFGIJOE*. As *GJ* is traversed, H is a negative quantity, but dB is positive and energy to the amount represented by the area *GIJ* will be returned to the generator. Finally in traversing the portion *JA*, both H and dB are positive quantities, and the generator will supply energy to an amount represented by the area *OAJ*. Upon summing all these items, it is seen that the value of the integral $\int HdB$ taken over 1 cycle, or the net amount of energy supplied by the generator, is equal to the area *ACEFGJA* included within the hysteresis loop. That is to say:

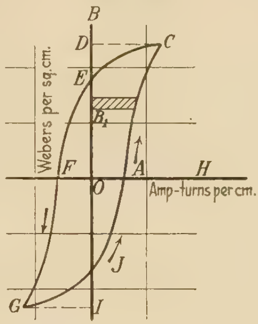

FIG. 276.—Hysteresis loop.

322a. *The hysteresis loss in joules per cubic centimeter of core is equal to the area (expressed in webers per sq. cm.* $\times$ *amp.-turns per cm.) enclosed by the hysteresis loop.*

It has always been assumed that the energy expended by reason of hysteresis (to the amount expressed by Eq. 521) all appears as heat energy in the ferromagnetic core. While attempts have been made to confirm this assumption by calorimetric determinations of the rise in temperature of the core, the agreement between predicted and observed results has not been very satisfactory. The difficulty arises partly from the fact that when the hysteresis loop is traversed rapidly, electromotive forces are induced in the core by the changing flux, and eddy currents flow in the core. It is difficult to effect a precise separation of the losses due to eddy currents from those due to hysteresis. There is, however, no reason to suppose that the dissipated energy is converted into any other form than heat energy in the core.

For the purpose of obtaining a clearer insight into the energy relations presented in the above calculations, one may assume that the flux density varies in time in a sinusoidal manner. For this case it is a simple matter to calculate and to plot the curves showing: (*a*) the flux density; (*b*) the magnetizing current; and (*c*) the necessary generator e.m.f. Such curves are shown on the oscillogram of Fig. 268, experiment 3. The necessary generator e.m.f. is a sine curve 90 degrees in advance of the curve showing the flux densities. By taking the product of the instantaneous values of

current and generator voltage, and plotting the results, a curve showing the power supplied by the generator is obtained. The net area under this curve for 1 cycle represents the net amount of energy supplied by the generator. The portions of this power curve which correspond to the different portions of the hysteresis loop may be readily identified (see Fig. 277).

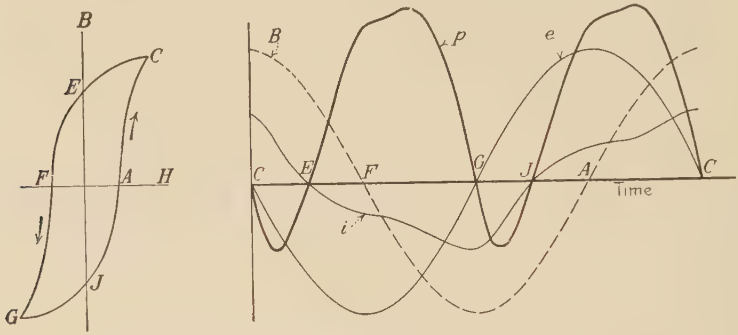

FIG. 277.—Power curve for hysteresis cycle.

322b. Loss per Cycle is Unaffected by the Time Taken to Traverse the Loop.—The statement to the effect that the curve showing the values of the magnetic flux densities lags behind the curve of magnetic intensities is frequently (and erroneously) interpreted to mean that the flux density corresponding to a given value of the magnetic intensity lags in time behind the intensity, or is attained later in time than the intensity; and that if the magnetic intensity is carried through the cycle of values with great rapidity, the magnetic flux density will not be able to follow the variations in the intensity and will be substantially unvarying. This notion is erroneous. The magnetic intensity H and the magnetic flux density B corresponding to it for a given cycle of values occur simultaneously in time. Between frequencies of 25 cycles and 200,000 cycles per second, the speed with which the magnetic intensity is carried through the cycle of values apparently has no influence upon the shape and size of the hysteresis loop. That is to say, the loss per cycle is apparently the same at all frequencies between 25 and 200,000 cycles per second.

323. Empirical Relation between the Hysteresis Loss per Cycle and the Maximum Value Attained by the Magnetic Flux Density (EXP. DET. REL.).—An examination of the family of hysteresis loops shown in Fig. 270 indicates that the greater the tip value of the flux density the greater the hysteresis loss per cycle. From extensive experimental determinations of the hysteresis losses at different tip flux densities, Steinmetz deduced the following empirical relation.

323a. *For symmetrical hysteresis loops, the hysteresis loss per cycle per cubic centimeter of a given sample varies approximately as the* 1.6 *power of the maximum (tip) flux density of the cycle.*

$$W \text{ (joules per cu. cm. per cycle)} = \eta B_m^{1.6} \text{ (webers per sq.cm.)} \quad (522)$$

in which, B_m is the maximum value of the flux density expressed in webers per square centimeter.

The exponent 1.6 is an empirically determined constant. The coefficient η is called the hysteresis coefficient of the material. Its value is determined by experiment for any given material.

Equation (522) expresses the losses quite accurately in many different magnetic materials over a limited range of flux densities. In the electrical steels it holds between the limits B_m = 1000 and B_m = 12,000 eighth-webers per square centimeter. At flux densities below 1000 and above 12,000 the hysteresis loss is greater than would be calculated from the constants given below.

The values of the hysteresis coefficient η for a few materials are as follows:[10]

Material	Hysteresis coefficient η
Hardened tungsten steel	50,000
Gray cast iron	8,000
Electrical steel sheets (carbon)	760
Average silicon steel sheets	500
Best silicon steel sheets	400

Let us compute the rate at which the hysteresis loss will cause the temperature of average silicon steel to rise. Assume that the steel is carried through a cycle in which the tip value of the flux density is 10,000 eighth-webers per square centimeter at the rate of 60 cycles per second.

Computing the hysteresis loss from Eq. (522)

$$W = 500\ (10^{-4})^{1.6}$$

it is found to be 1.99×10^{-4} joules per cubic centimeter per cycle.

[10] Steinmetz' formula is commonly written in the form,

$$W \text{ (ergs per cu. cm. per cycle)} = \eta_1 B_m^{1.6} \text{ (maxwells per sq. cm.)} \quad (522a)$$

The relation between the value of η in Eq. (522) and η_1 in Eq. (522*a*) is

$$\eta = 631{,}000\eta_1. \quad (523)$$

Since the energy required to raise 1 cubic centimeter of steel 1 degree Centigrade is 3.55 joules, the energy expended in 1 cycle will cause a rise of 5.6×10^{-5} degrees. If none of the heat were carried away from the steel by conduction, the temperature of the steel would increase at the rate of $60 \times 60 \times 5.6 \times 10^{-5}$ or 0.2 degree Centigrade per minute.

323b. Unsymmetrical Hysteresis Loops.—If a continuous current and an alternating current flow simultaneously in the magnetizing winding, the B-H loop is an unsymmetrical loop which is traversed around a mean flux density differing from zero. Under these conditions the loss per cycle for a given amplitude of variation in flux density is greater than if the variation had taken place around zero flux density. Ball[11] has found the following empirical formula to express the loss for unsymmetrical loops with fair accuracy.

$$W \text{ (joules per cu. cm per cycle)} = \left[\eta + a\left(\frac{B_1 + B_2}{2}\right)^{1.9}\right]\left[\frac{B_2 - B_1}{2}\right]^{1.6}, \tag{524}$$

in which B_1 and B_2 represent the algebraic values of the flux densities (expressed in webers per square centimeter) at the tips of the hysteresis loops. For values of $(B_2 - B_1)$ between 1000 and 4000 eighth-webers per square centimeter and for values of $(B_1 + B_2)/2$ up to 10,000 eighth-webers per square centimeter, the coefficients η and a were found to have the following values:

Material	η	a
2.5 per cent annealed silicon steel................	663	3.2×10^{10}
Annealed low-carbon steel........................	670	3.44×10^{10}

Annealing.—The hysteresis coefficient of a steel of given composition varies greatly with the manner in which the steel has been worked and annealed. By careful annealing, the hysteresis coefficient can be reduced to one-half or less of the unannealed value. The sheet steels used in all equipment in which the flux alternates rapidly are annealed after the punching operation by slowly cooling them from a red heat.

Magnetic Aging.—When the low carbon steel sheets are continuously used at temperatures above 100°C., the hysteresis coefficient gradually increases and has been known to rise to twice its original value. This is known as magnetic **aging**. The silicon steels are practically free from this defect.

324. Hysteresis Loss Due to Rotation in a Magnetic Field.—If a ferromagnetic material (as an armature) is mounted upon an axis and is rotated in a magnetic field, or if the material is fixed but the magnetic field is caused

[11] BALL, J. D.: *Investigation of the Magnetic Laws for Steel and Other Materials*, J. Franklin Inst., 1916, p. 459.

to rotate with respect to it, the flux density vector at any point in the material remains of constant magnitude but rotates in direction. Because of this rotation, the **component** of the flux density normal to any plane cyclically alternates between two equal and opposite values, B_1 and $-B_1$. The loss per cycle per cubic centimeter of material is assumed to be the same for this kind of a cycle as for the cycle in which the flux-density vector remains fixed in direction but alternates in value between the two limits B_1 and $-B_1$. In this case the energy dissipated in the material is derived from the agent driving the rotating member, and not from the agent supplying the magnetizing current.

325. Diamagnetic and Paramagnetic Substances.—When a bar of iron is freely suspended in a magnetic field, it assumes a position with its length parallel to the flux density vectors. In a non-uniform field, a sphere of iron is attracted towards the regions of high flux density. On the other hand, a bar of bismuth, if freely suspended between the poles of a strong electromagnet, sets itself at right angles to the B vectors. A sphere of bismuth is drawn from regions of high to regions of low flux density. Although the force of repulsion between bismuth and a compass was reported in 1778 and again in1827, it was not until 1845 that Faraday discovered that many substances, such as antimony, copper, lead, glass, water, mercury, etc., possess magnetic properties similar to bismuth. Faraday called this group of substances, **diamagnetic substances.**[12]

Faraday followed his discovery of the diamagnetic properties of substances by the discovery that many other substances, previously supposed to possess no magnetic properties, when placed in the intense field of a strong electromagnet are acted upon by feeble forces of the same type as the forces upon iron. Among these substances are platinum, aluminum, a tube of oxygen, manganese, and the small group of ferromagnetic substances, iron, nickel, and cobalt. These substances when in bar form all set themselves with their lengths **parallel** to the flux density vectors. Apparently this circumstance led Faraday to designate such substances as **paramagnetic** substances.

[12] The term **diamagnetic** was first used by Faraday in his demonstrations of the rotation of a beam of polarized light during its passage through a substance like water or glass in a strong magnetic field. He used the term to designate substances which, unlike iron, **do not** become magnetic, but through or across (dia) which magnetic effects may act. (See FARADAY: *Experimental Researches*, Vol. III, Secs. 2149, 2270, and 2790.)

The permeabilities of all diamagnetic and of all paramagnetic substances, save the ferromagnetic group, differ so little from that of free space that the ballistic method is unsuited to the measurement of their values. Their values may be measured by a more sensitive method which utilizes the forces which the substances experience when placed in the field of a strong electromagnet. By measurements of this kind, the relative permeabilities of most substances, save the ferromagnetic, are found to differ from unity by only a few parts in a million, the relative permeabilities of the diamagnetic substances being slightly less, and those of the paramagnetic being slightly greater than unity. This is shown in the following table:

325a. Relative Permeabilities (Free space = 1).

Diamagnetic	
Bismuth	$1 - 170 \times 10^{-6}$
Antimony	$1 - 65 \times 10^{-6}$
Copper	$1 - 10 \times 10^{-6}$
Silver	$1 - 19 \times 10^{-6}$
Water	$1 - 9 \times 10^{-6}$
Quartz	$1 - 5.5 \times 10^{-6}$

Feebly paramagnetic	
Air	$1 + 0.4 \times 10^{-6}$
Oxygen at −182°C	$1 + 0.004$
Aluminum	$1 + 22 \times 10^{-6}$
Platinum	$1 + 360 \times 10^{-6}$

Strongly paramagnetic (ferromagnetic)	
Cobalt	270
Nickel	500
Soft iron (low H)	183
Special silicon steel (maximum)	66,000
Commercial steel (maximum)	6,000
Heusler alloy (maximum)	500

The permeabilities of the diamagnetic and of feebly paramagnetic substances seem to be constants whose values are inde-

pendent of the value of the magnetic intensity. These substances show no hysteresis effects. All the ferromagnetic substances become feebly paramagnetic if heated above their critical temperatures.

326. The Electron Theory of Magnetism.—Shortly after the discovery in 1820 that the effects obtained from permanent magnets could all be obtained from currents flowing in copper coils, Ampere advanced the hypothesis that the phenomena of magnetism are to be accounted for in terms of the properties of electric currents. He proposed to account for the properties of ferromagnetic substances by assuming that the molecules of these substances contain electric currents permanently circulating in resistanceless paths within the molecule. To the currents in these molecular circuits, he proposed to attribute magnetic properties identical with the properties which the currents in copper wire in free space had been found to have. In other words, the flux density at any point, whether in free space outside a ferromagnetic core or within the core itself, is to be regarded as the flux density which would be set up in **free space** by the combined action of the **obvious** magnetizing current in the winding of the core and the **concealed** molecular currents of the core. The magnitude of the mechanical force upon any part of the structure of winding and core is to be arrived at from a consideration of the forces upon the obvious and concealed currents in a field in which the flux densities have been computed (as outlined above) as for free space.

The electron theory of magnetism is an extension of Ampere's hypothesis. It embodies the above features of the hypothesis, and in addition presents a picture of the nature of the molecular currents and of the structure of the atom. We propose to sketch in a qualitative way the manner in which the properties of diamagnetic and paramagnetic materials and the features of the B-H loop are accounted for by this electron theory.

An electron of charge q revolving in a circular orbit at a frequency of f revolutions per second is equivalent to a current of qf amperes flowing around the orbit. The atoms of higher atomic weights are conceived to have many electrons rotating in such orbits. Imagine a substance whose atoms have the planes

of the electronic orbits so distributed that the resultant magnetic effect of each atom at distant points or the resultant magnetic moment of each atom is zero. If a magnetic field is established in such a substance, it exercises no directive effect upon the atoms of the substance. The substance, however, will not be magnetically inert. During the building up of the current in the magnetizing winding, the electrons in the substance are subject to forces which accelerate those circling in the opposite direction to the electrons in the winding, and retard those circling in the same direction (see Sec. 309). The frequency of revolution of the electrons circling in one direction is raised, and that of those circling in the opposite direction is decreased. The widening of the characteristic lines of the spectrum of an incandescent substance when it is placed in an intense magnetic field (the Zeeman effect) is experimental evidence of these effects upon the characteristic frequency. The result is that the atoms are no longer neutral but they exert a magnetic intensity in a direction opposite to that of the current in the magnetizing winding. The flux densities in the regions occupied by such a substance will be less than they would be in free space. In other words, the permeability of the substance is less than unity, or it is a **diamagnetic** substance. When the external field through the electronic orbits vanishes, the electrons resume their normal frequencies.

Let us now imagine a substance in which the resultant magnetic moment of the atom is not zero, by reason of the fact that it contains one or more electronic orbits whose magnetic effect is not neutralized by oppositely directed orbits. If a magnetic field is established in such a substance, the field will exert a force upon the atoms tending to turn them so that the planes of the uncompensated orbits are perpendicular to the B vectors, with the electrons circling in the same direction as the electrons in the magnetizing winding. The atoms may be assumed to turn somewhat from their previous random distribution. Their uncompensated currents will give rise to higher flux densities in the substance than in free space. Such a substance is **paramagnetic,** and if the increase in flux density is pronounced it is said to be **ferromagnetic.**

If the atoms turned freely and then remained in alignment, the slightest magnetizing currents would be sufficient to produce the

saturation value of the metallic flux density. But feebly paramagnetic substances do not saturate at all, and ferromagnetic only under extremely high intensities. In a paramagnetic gas the intense forces between atoms during collision may be assumed to reorient the atoms and thus permit only a partial alignment directly proportional to the magnetic intensity. In a ferromagnetic solid, the atoms may be assumed in turning to encounter constraints arising from chance stable groupings of these magnetic atoms. Such a system of internal forces would account for the features of the permeability μ-H curve; namely, low permeabilities at intensities which are insufficient to break up the stable groupings, a maximum permeability at intermediate intensities (under which the groupings become unstable and start to break up), low permeability with the approach to saturation at high intensities, and a residual flux density due to stable atomic groupings formed while the atoms were aligned. The atomic oscillations which occur when the atomic groupings become unstable and break up, may be supposed to lead to increased molecular velocities of thermal agitation. This accounts for the dissipation of energy when the hysteresis loop is traversed.

As a first indication of the plausibility of the hypothesis which accounts for the flux densities in iron in terms of atomic currents, let us calculate the atomic currents and frequencies necessary to furnish the saturation value of the metallic flux density. Taking the saturation value of the metallic flux density in iron to be 24,500 eighth-webers per square centimeter, the magnetic intensity necessary to set up such a flux density in free space is found to be

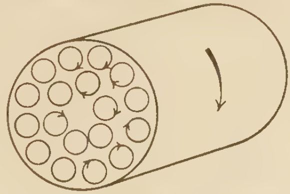

FIG. 278.—Concealed currents.

$$H = \frac{B_s}{\mu_o} = \frac{24{,}500}{10^8}\ \frac{10^8}{1.257} = 19{,}500 \text{ ampere-turns per cm.}$$

That is, a long, air-core solenoid, or a ring coil with an air core, would require the astonishing magnetizing current of 19,500 ampere-turns per centimeter of length in order to establish a flux density equal to that contributed by the atomic currents.

A portion of the core 1 square centimeter in cross-sectional area and 1 centimeter long has been illustrated in Fig. 278. The atomic or mesh currents illustrated in this figure must be equivalent in magnetic effect to a

current sheet of 19,500 amperes circulating around this cubic centimeter of iron. That is, the sum of the magnetic moments (see Sec. 232*a*) of the uncompensated atomic currents in 1 cubic centimeter of iron must equal

$$M = I \times \text{area} = 19{,}500 \times 1 \text{ (ampere-turn, centimeter).}$$

But since a gram-atom of iron weighs 55.8 grams and contains 6.06×10^{23} atoms, the cubic centimeter of iron (of density 7.78) contains 8.45×10^{22} atoms. Therefore, the magnetic moment m of the iron atom must be

$$m = \frac{19{,}500}{8.45 \times 10^{22}} = 2.31 \times 10^{-19} \text{ (ampere-turns, centimeters).} \quad (525)$$

Now the magnetic moment m of a charge q making f revolutions per second in a circular orbit of radius r is

$$m = ia = (qf)(\pi r^2),$$

whence,

$$qf\pi r^2 = 2.31 \times 10^{-19}.$$

Substituting for r the estimated radius of the iron atom 1.5×10^{-8} centimeter, and for q, the charge of a single electron, 1.59×10^{-19} coulombs, the computed value of the frequency of revolution is

$$f = 2.05 \times 10^{15} \text{ revolutions per second.}$$

If the atom contains n uncompensated orbits, or one uncompensated orbit containing n electrons, the frequency would have to be $1/n$th of 2.05×10^{-15}. It will be recalled that the frequencies of vibration which produce the visible portion of the spectrum lie between 4×10^{14} and 7.5×10^{14} vibrations per second. To have the frequency fall between these limits, the iron atom would have to contain three uncompensated orbits.

The saturation values of the metallic flux density have been obtained for a number of substances at temperatures near the absolute zero. From these, the values of the magnetic moments of the atoms have been computed. Upon examining these values, Weiss found that many of them seemed to be integral multiples of 1.85×10^{-20} ampere-turns, centimeters. He advanced the hypothesis that this is the magnetic moment of the ultimate or smallest uncompensated orbit and called it the **magnetron.**

In the next section we make use of the hypothesis of concealed currents in deriving expressions for the force between electromagnets containing iron cores.

327. The Force between Coils and Cores in Terms of Flux Density. (DEDUCTION).—Suppose that the overwound ring core of rectangular cross-section which is illustrated in Fig. 279 has been divided into two parts along the plane AB. Suppose the exciting windings to be wound upon forms of non-magnetic material, and each form to be attached to its half of the core. The core may be of ferromagnetic or of non-magnetic material. If a current is passed through the two windings in such a direction that the magnetomotive forces of the two coils are in the same direction around the core, the two halves of the structure attract; if the m.m.fs. are opposed, the two halves

repel. We proceed to derive an expression for the force between the two halves of the structures, and to put this into a form which will permit us to compute the attractive force between electromagnets in terms of the flux density over the abutting surfaces.

To facilitate the derivation, let the winding consist of a single layer of wire, and imagine that the wire is of square cross-section, as illustrated in Fig. 279. Let the radial width w_1 of the form be small in comparison with the radius r of the ring; let the radial depth t of the layer of wire be small in comparison with the width w_1 of the form. Let there be a total of N turns of wire on the two-part core, each turn carrying the current I (in the same direction around the core).

The force pulling the two halves of the structure together is the resultant of the forces on the currents in the elementary lengths of the conductors and of the core. The force on any short length of wire is given by Eq. (341), namely,

$$f(\text{dyne-sevens}) = BIl \sin (B,l). \tag{341}$$

The directions of the forces on the four segments of a turn are as indicated by the arrows on the illustration. The short segments on the two plane end faces of the core are urged in opposite directions with forces which balance each other. The segments of length $h_1 + t$ along the inner and outer cylindrical surfaces are respectively urged inwardly and outwardly in radial directions. Any force between the two coils is the resultant of these two systems of forces.

The m.m.f. around any filament of flux which lies within the space enclosed by the winding is NI. For filaments of flux which lie within the layer of wire itself the m.m.f. varies from zero to NI ampere-turns. The manner in which the magnetic intensity varies from point to point in the radial plane CB is shown by the broken curve $CDEFGB$.

Case I.—In which the winding is of copper, and the core is of non-magnetic material having the permeability of free space, namely, μ_o.

The average value of the flux density in the layer of wire on the inner face of the core is

$$B = \frac{\mu_o NI}{4\pi\left(r - \frac{w_1 + t}{2}\right)} \text{ (approximately).} \tag{526}$$

From Eq. (341) and (526), the radially directed force on the current I in the length $h_1 + t$ of the segment of a turn on the inner face of the core is

$$f_1 = \frac{\mu_o NI^2(h_1 + t)}{4\pi\left(r - \frac{w_1 + t}{2}\right)}.$$

In like manner, the outwardly directed force on the segment of the turn on the outer face is found to be

$$f_o = \frac{\mu_o NI^2(h_1 + t)}{4\pi\left(r + \frac{w_1 + t}{2}\right)}.$$

The resultant force f_t on a complete turn of wire is directed radially inward toward the axis of the core and has the value

$$f_t = f_1 - f_o = \frac{\mu_o N I^2 (h_1 + t)(w_1 + t)}{4\pi\left[r^2 - \frac{(w_1 + t)^2}{4}\right]}.$$

There are $N/2$ forces of this magnitude uniformly distributed around the semicircumference of each of the two coils and each is directed radially inward. The resultant of the components of these forces perpendicular to

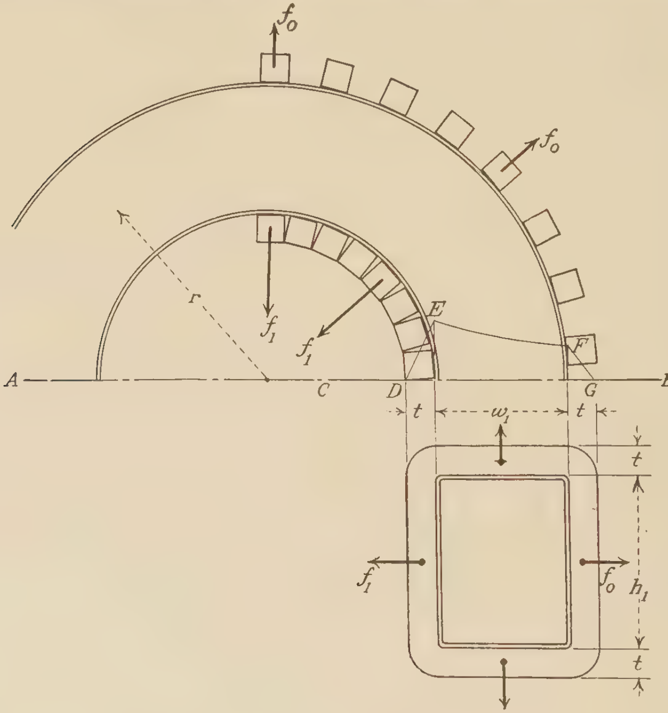

FIG. 279.—Over-wound ring core.

the plane of division AB of the coils is $2/\pi$ times $N/2$ times the radial force on a single turn, or the pull f between the two coils is

$$f = \frac{\mu_o N^2 I^2 (h_1 + t)(w_1 + t)}{4\pi^2\left[r^2 - \frac{(w_1 + t)^2}{4}\right]}. \tag{527}$$

If $(w_1 + t)^2/4$ is negligibly small in comparison with r^2, that is, if the flux density is substantially uniform within the winding form, this may be written without appreciable error in the form

$$f = \frac{\mu_o N^2 I^2}{4\pi^2 r^2}(h_1 + t)(w_1 + t). \tag{528}$$

But $NI/(2\pi r)$ is the mean magnetic intensity within the core, and $(h_1 + t)$ $(w_1 + t)$ is the area bounded by a turn of the coil from center to center of the wire, or it is equal to one-half of the cross-sectional area a_1 of the air gap separating the abutting surfaces. Therefore the expression for the force may be written in the forms

$$f \text{ (dyne-sevens)} = \frac{\mu_o H^2 a_1}{2} \text{ (amp-turns per cm., cm.)}. \tag{529a}$$

$$f \text{ (dyne-sevens)} = \frac{B^2 a_1}{2\mu_o} \begin{matrix}\text{(webers per sq. cm. cm.)}\\ \text{(webers, amp-turns, cm.)}\end{matrix} \tag{529}$$

in which, B represents the flux density over the abutting area.

a_1 represents the cross-sectional area of the air gap separating the abutting surfaces.

This predicted value of the pull between the two halves of the coil is confirmed by experimental measurements of the force.

Equation (529) is a remarkably simple result. From it we may formulate the following simple rule for computing the force.

327a. Force between Abutting Solenoids.—The force of attraction in dyne-sevens between the two halves of a ring-core copper coil with the ends abutting, or between two long air-core solenoidal coils placed end to end, may be computed by assuming a force equal to $B^2/(2\mu_o)$ per square centimeter of area of the air gap between the abutting surfaces.

$$f \text{ (dyne-sevens per sq. cm.)} = \frac{B^2}{2\mu_o} \begin{matrix}\text{(webers per sq. cm.)}\\ \text{(webers, amp.-turns, cm.)}\end{matrix}. \tag{530}$$

(Note that the abutting or cross-sectional area is to be reckoned from center to center of the winding. The abutting area a_1 of two straight solenoids is $(h_1 + t)(w_1 + t)$ and of an endless ring core coil it is twice this.)

If the current in one of the coils is reversed, each elementary length of the coil will now be subject to a force from the field of the other coil equal and opposite to the former force. Therefore the two coils will now repel each other with a force equal to the attractive force which is computed from Eq. 529.

327b. Case II.—In which the winding is of copper, but the core is of iron having the permeability μ_c.

According to the Amperian conception of concealed molecular currents, the permeability of the space occupied by the iron core is the same as that of free space; but the flux densities are greater in the core than in free space, and its apparent permeability μ_c is higher, because the concealed molecular currents in the iron give rise to concealed magnetomotive forces whose magnitude is to the magnetomotive force of the obvious currents in the ratio of $\mu_c - \mu_o$ to μ_o. That is to say, we must conceive that the concealed molecular currents in the core are the equivalent in magnetomotive force of a current sheet having the value

$$I_c = \frac{\mu_c - \mu_o}{\mu_o} NI. \tag{531}$$

This current flows around the core in the same direction as the exciting current. It may be conceived to be in the form of a current sheet flowing in a film of negligible thickness on the surface of the core. The location of the film is indicated in Fig. 280. Obviously in computing the pull between the two halves of the structure the forces on this concealed current must be taken into account. This concealed current does not affect the flux densities within the copper winding. Therefore, the pull between the two halves of the copper winding will be unaffected by the presence of the iron core, and will be as computed for case I. We proceed to compute the forces on the concealed currents, the resultant of which will give the force upon the iron core.

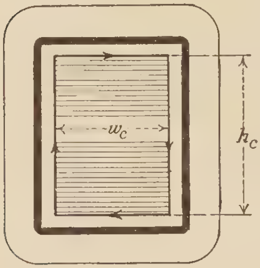

Fig. 280.—Equivalent concealed current sheet.

The average flux density in the current sheet on the inner face of the core is

$$B = \frac{\mu_o}{2\pi\left(r - \frac{w_c}{2}\right)}\left[NI + \frac{(\mu_c - \mu_o)NI}{2\mu_o}\right].$$

The concealed current on the inner face of the core corresponding to one turn of the winding is $(\mu_c - \mu_o)I/\mu_o$. The force on this current is directed radially inward and has the value

$$f_1 = \frac{\mu_o NI^2 h_c}{2\pi\left(r - \frac{w_c}{2}\right)} \frac{\mu_c{}^2 - \mu_o{}^2}{2\mu_o{}^2}$$

In like manner, the outwardly directed force on the corresponding concealed current on the outer face of the core is found to be

$$f_0 = \frac{\mu_o NI^2 h_c}{2\pi\left(r + \frac{w_c}{2}\right)} \frac{\mu_c{}^2 - \mu_o{}^2}{2\mu_o{}^2}$$

The resultant of these two forces is directed radially inward toward the axis of the structure and has the value

$$f_t = f_1 - f_o = \frac{\mu_o NI^2 h_c w_c}{2\pi\left(r^2 - \frac{w_c{}^2}{4}\right)} \frac{\mu_c{}^2 - \mu_o{}^2}{2\mu_o{}^2}. \tag{532}$$

As in case I, there are $N/2$ of these forces distributed around the semicircumference of each of the two halves of the core, and each force is directed radially inward. The resultant pull between the two halves will be N/π times one of these radial forces, or the pull between the two halves of the core is

$$f = \frac{\mu_o N^2 I^2 h_c w_c}{4\pi^2\left(r^2 - \frac{w_c{}^2}{4}\right)} \frac{\mu_c{}^2 - \mu_o{}^2}{\mu_o{}^2}. \tag{533}$$

If $w_c^2/4$ is negligibly small in comparison with r^2—that is, if the width of the core is so small that the flux density is uniform over the cross-section, this may be written

$$f = \frac{N^2I^2}{4\pi^2r^2}\frac{(u_c^2 - \mu_o^2)h_cw_c}{\mu_o}.$$

But $NI/(2\pi r)$ is the magnetic intensity H due to the exciting current. Therefore, the expression for the force of attraction between the two halves of the iron core may be written

$$f \text{ (dyne-sevens)} = \frac{(B_c^2 - B_o^2)a_c}{2\mu_o}, \qquad (534)$$

in which,

$B_c = \frac{\mu_c NI}{(2\pi r)}$, is the flux density in the iron core.

B_o is the flux density which would obtain in the region if the iron core were removed, namely, $\mu_o NI/(2\pi r)$.

μ_o is the permeability of free space, and

a_c is the cross-sectional area of the air gap in the iron core, namely, $2h_cw_c$.

This is the force on one of the half portions of the iron core due to the field of the concealed currents in the other half and of the obvious current in the magnetizing winding. Because of the large value of B_c as compared with B_o this force is a thousand-fold as great as the force between the air-core coils, as expressed by Eq. (529).

If, now, each half of the winding is attached to its half of the iron core, the force necessary to pull the two halves of the iron core structure apart is

$$f \text{ (dyne-sevens)} = \frac{(B_c^2 - B_o^2)}{2\mu_o}a_c + \frac{B_o^2a_1}{2\mu_o} \qquad (535)$$

Although the area a_1 for the winding is necessarily somewhat greater than the area a_c for the core, we may (without appreciable error) write a_c for a_1 in the second term, thereby obtaining

$$f \text{ (dyne-sevens)} = \frac{B_c^2a_c}{2\mu_o}. \qquad (536)$$

From this we may (as for the air-core coil) formulate the following simple rule.

327c. Force between Abutting Iron Cores.—*The force of attraction between the two parts of an iron core structure (each coil being attached to its core), having parallel plane faces separated by an infinitesimal air gap, may be computed by assuming a force equal to $B_c^2/2\mu_o$ per square centimeter of area of the air gap.*

$$f \text{ (dyne-sevens per sq. cm.)} = \frac{B_c^2}{2\mu_o} \text{ (webers, cm.)}. \qquad (537)$$

In iron cores, flux densities as high as 15,000 eighth-webers per square centimeter can readily be obtained in industrial appliances. By substitution in Eq. (537), the force of attraction at this flux density is found to be 0.896 dyne-sevens or 9.15 kilograms or 20.2 pounds (g = 980 centimeters) per square centimeter of surface. This means that a steel shaft 1170 centimeters in length can be lifted by the magnetic pull upon the surface of one

end. The magnetic pull at 20.2 pounds per square centimeter is only one-tenth of 1 per cent as great as the ultimate tensile strength of steel.

If the current in one of the magnetizing coils of the iron ring core is reversed, the two halves of the structure will repel each other, but with a force which is far less than the attractive force between the halves when the magnetizing currents are in the same direction. When the current in one-half is reversed, the concealed atomic currents in that half reverse but rise to lower values in the reverse direction, while the magnitude of concealed currents in the other half also decreases. This is because each half of the core is under the magnetizing influence of both coils, and the effects of these coils no longer aid but oppose. It is impossible to predict the value of the force of repulsion under these conditions.

328. Forces on Solenoids and Bar Magnets. The Concept of Magnetic Poles.—Figure 281 may represent either two solenoids or two magnetized

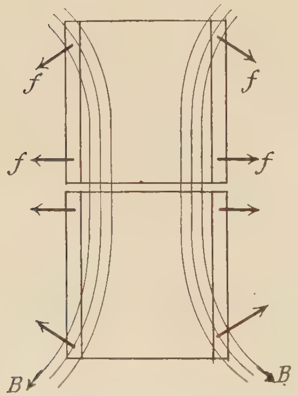

Fig. 281.—Forces on the currents in two abuting-solenoids.

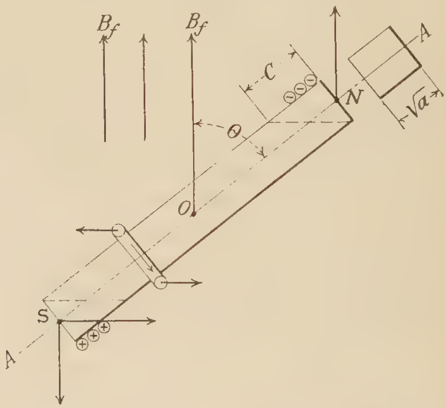

Fig. 282.—The poles of a solenoid.

steel bars placed end to end. The atomic currents in the steel bars would be the (approximate) magnetic equivalent of a current sheet around the circumference of the cylinders. The directions of the lines of magnetic intensity are shown for the case in which the two parts are magnetized in the same direction. The directions of the forces upon the moving electrons in the winding or in the current sheet are shown by the arrows. It is to be noted that although the formulas of computing the force give it in terms of an assumed force per unit area of the abutting surfaces, the actual forces which have an effective component are the forces upon the moving electrons near the outer ends of the two steel bars.

Figure 282 is a plan view of a solenoid of square cross-section. The solenoid is pivoted so that it may turn about an axis which is perpendicular

to the plane of the paper at O. The solenoid is shown with its longitudina axis AA making an angle θ with the B vectors of a uniform horizontal field. Let us predict the resultant of the forces acting upon this solenoid when the winding carries a current I in the direction indicated.

Let B_f represent the flux density of the uniform field.
 N represent the number of turns per centimeter of length of the solenoid.
 a represent the cross-sectional area of the solenoid (center to center of the winding).
 l the length of the solenoid.
 I represent the current per turn in the solenoid.
 θ represent the angle between the B vectors and the axial length AA.

Only a few of the turns of the winding are shown, together with the direction of the forces upon the vertical quarters of the turns. The forces upon the top and bottom quarters of each turn are perpendicular to the plane of the paper in an upward and downward direction, respectively. They produce no turning moment about the vertical axis O and will receive no further consideration.

The force on each vertical quarter of a turn is

$$f = B_f I \sqrt{a}.$$

The twisting moment or torque τ_1 of these two forces about the axis O is

$$\tau_1 = f\sqrt{a}\sin\theta = B_f I a \sin\theta.$$

The total torque of the Nl turns of the solenoid is

$$\tau \text{ (dyne-seven, cm.)} = NlB_f I a \sin\theta. \tag{538}$$

We arrive at the same result by the following argument.

The only forces whose turning moments about O are not neutralized by oppositely directed forces are the forces on the vertical quarter turns in the length c at each end.

$$c = \sqrt{a}\tan\theta.$$

The force on the Nc turns at each end is

$$f = B_f I\sqrt{a}N\sqrt{a}\tan\theta.$$

The arm d of each force is

$$d = \frac{l\cos\theta}{2}.$$

Therefore the total torque of the two sets of forces is

$$\tau \text{ (dyne-seven, cm.)} = NlB_f I a \sin\theta. \tag{538}$$

But we have seen in Sec. 266 that in a long solenoid the magnetic intensity near the center is equal to NI. Therefore, the flux density at the center of the solenoid due to its own current alone would be

$$B = \mu H = \mu_o NI.$$

Therefore, if Φ represents the magnetic flux set up by the current in the solenoid over its central cross-sectional area, we may write

$$\Phi = \mu_o NIa.$$

Substituting this in Eq. (538), it becomes

$$\tau(\text{dyne-seven, cm.}) = \frac{B_f \Phi l \sin\theta}{\mu_o}. \tag{539}$$

But B_f/μ_o is the magnetic intensity H in the uniform field. Therefore, the expression for the torque may also be written in the following form:

$$\tau\ (\text{dyne-seven, cm.}) = H\Phi l \sin\theta. \tag{540}$$

We see that the torque on the solenoid is a maximum when the axis AA of the solenoid is at right angles to the B vectors. The forces tend to turn the bar parallel to the B vectors of the field, with the B vectors inside the solenoid pointing in the same direction as the B vectors of the field. There are two positions of zero torque; the stable position is $\theta = 0$; the unstable position is $\theta = 180$ degrees. The predictions contained in Eq. (539) or (540) are confirmed by experimental measurements of the torque.

The above treatment applies rigorously to a solenoid. Now the resultant current sheet around a permanent bar magnet of square cross-section differs from the current in the solenoid only in this respect: In the bar magnet the current per centimeter length of bar drops off somewhat near the ends of the bar, whereas in the solenoid the ampere-turns of the last centimeter are equal to the ampere-turns of any centimeter of length. Therefore, the above formulas will apply to a bar magnet except that the effective length of the bar magnet will be somewhat less than the distance between the end faces.

In the above treatment we have, from a knowledge of the force upon a current element in a magnetic field, predicted the resultant torque on the solenoid. But suppose we knew nothing whatsoever of this inner mechanism from which the torque results, and suppose we had made measurements of the torque at various angles, and were devising a description of the manner in which the solenoid behaves in a magnetic field. A comparison of the torques which can be predicted from the following statement with the predictions from Eq. (540) will indicate the validity of the statement.

The solenoid acts as though a point N at the center of one end face and a point S at the center of the other end face were acted upon by forces of the magnitude $H\Phi$, in which Φ is a constant for any solenoid (when carrying a given current), and H is the magnetic intensity of the field in which the solenoid is placed. The force upon N is directed in the positive direction and the force upon S in the negative direction **along the lines of magnetic intensity** *of the field.*

If the solenoid is mounted in the earth's magnetic field, the force upon the point N is directed in a northerly direction along the lines of magnetic intensity. Therefore, for descriptive purposes we will agree to call the points N and S the **north-seeking pole** *(or north pole) and the* **south-seeking pole,** *respectively. Furthermore, we will agree to designate the fictitious force which a pole experiences in a field of unit intensity as the* **strength of the pole.**

We may frame the following definitions of the quantities introduced in the above statement.

328a. MAGNETIC POLES (DEFINITION).—**Magnetic poles are the points of application of fictitious forces which would produce the same torque upon a solenoid or bar magnet as the actual distributed forces upon the elements of the structure. (Note that the definition implies that there are always two poles to be taken into account.)**

328b. MAGNETIC POLE STRENGTH (DEFINITION).—**The STRENGTH m of a pole is defined to be the value of the fictitious force upon the pole in a field of unit intensity (1 ampere-turn per centimeter).**

From Eq. (540) it will be seen that the strength of a pole is numerically equal to the magnetic flux Φ (in webers) across the midsection of the solenoid. Therefore we may use the term *weber* as the name for the unit pole.

328c. Unit Magnetic Pole (DEFINITION).—*The strength of a pole is unity, or 1 weber, when it is acted upon by a fictitious force of 1 dyne-seven in a field having an intensity of 1 ampere-turn per centimeter.*[13]

$$m \text{ (webers)} = \frac{f}{H} \frac{\text{(dyne-sevens)}}{\text{(amp-turns per cm.)}} \text{ (definition)} \tag{541}$$

$$m = \Phi \text{ (deduction)} \tag{542}$$

We may reframe the definition of magnetic pole strength (Eq. 541) into the following statement of the force upon a pole in a magnetic field.

328d. Mechanical Force Acting upon a Pole in a Magnetic Field.—*The fictitious mechanical force f exerted by the field upon a magnetic pole whose strength is m webers, the pole being placed at a point where the magnetic intensity is H ampere-turns per centimeter,* is

$$f \text{ (dyne-sevens)} = mH \text{ (webers, ampere-turns per cm.)} \tag{543}$$

329. Mechanical Force between Magnetic Poles.—We have seen (Sec. 266) that the magnetic intensity at a distant point P in the field of a long solenoid may be regarded as made up of two components directed along the two lines from the point P to points (the poles) on the ends of the distant solenoid. Each component of the magnetic intensity is given by the formula

$$H \text{ (amp-turns per cm.)} = \frac{m_1}{4\pi\mu_0 l^2}.$$

in which l is the distance from P to the pole under consideration, and m_1 is equal to the flux over the mid section of the solenoid. We would now call it the pole strength of the solenoid.

If a pole of strength m_2 ($= \Phi_2$) is imagined at the point P it must be imagined to be subject to forces directed toward the unlike pole and away from the like pole of the distant solenoid. The magnitude of these forces will be expressed by the following law.

[13] The unit pole of the E.M. system of units is equal to 1.257×10^{-7} weber unit poles as here defined.

329a. Mechanical Force between Two Poles.—*The fictitious force of repulsion f between two concentrated like magnetic poles of strengths m_1 and m_2 webers, separated by the distance l in an infinitely extended homogeneous medium of permeability μ_0, is*

$$f\,(\text{dyne-sevens}) = \frac{m_1 m_2\,(\text{webers})}{4\pi\mu_0 l^2\ \ (\text{cm.})} \tag{544}$$

330. Exercises.

1. A wooden annulus with a rectangular cross-section is uniformly wound with 1500 turns of wire carrying a current of 5 amperes. The inner and outer radii of the core are 15 and 20 centimeters. The axial thickness is 10 centimeters.

a. Determine the maximum and the minimum values of the magnetic flux density in the cross-section and on the mean circumference.

b. Determine by integration the average value of the magnetic flux density over the cross-section, and compare it with that for the mean circumference. Under what conditions do these two values become practically equal?

2. If the annular core of exercise 1 is made of wood, calculate the reluctance of the magnetic circuit.

a. What current in the winding would be required to establish a flux of 0.004 weber?

b. What percentage increase in current would be necessary to increase the total flux 50 per cent?

3. *a.* Calculate the current necessary to produce the same flux as in exercise 2*a*, if the wooden core were replaced by a cast-iron core.

b. If the current were to be increased 50 per cent, what percentage increase in flux would result?

c. What total flux would be established by a current of 1.8 amperes?

4. Assume that the annulus of exercise 1 is made up of two semicircular parts, the materials of the respective parts being cast iron and cast steel. Assume that the joints have a negligible reluctance.

a. Calculate the exciting current necessary to produce a flux of 0.004 weber if the leakage flux is assumed zero.

b. What difference of magnetic potential will exist between the two joints?

c. Discuss the probability of leakage flux. Explain quantitatively how this leakage flux could be reduced to a minimum.

d. Calculate the total flux that would be established by a current of 2 amperes, the leakage flux being assumed zero.

5. *a.* If an air gap of length 0.4 centimeter is cut in the ring of exercise 4, what current is required to establish the same flux through the circuit? Assume that the iron removed to form the air gap does not appreciably change the reluctance of the remainder of the circuit.

b. Discuss the leakage flux in the above case. Is there any practical way of materially reducing it? Can the calculations carried out in *a* be considered exact?

6. Compute the relative permeabilities of air, electric steel sheets, cast steel, and cast iron at a flux density of 9000 maxwells per square centimeter.

7. A sample of "14-mil" electric sheet steel is cut into long narrow strips which are then laid up to form a closed magnetic circuit. The circuit bounds a rectangle. The average length of the circuit is 200 centimeters. The net cross-section of the steel is 4.8 square centimeters. An exciting coil of 1600 turns is wound as nearly uniformly over the length as can be arranged, and a current of 1.1 amperes is passed through the coil. After reversing the current a number of times, a ballistic galvanometer is connected to a secondary circuit of 50 turns around the core, and a voltage impulse of 0.015 volt-second is observed on a final reversal of the exciting current. This then is repeated for other values of exciting current, and other values of voltage impulse are obtained as given below:

Current.........	0.12	0.25	0.40	0.75	1.10	1.8
Volt-second.....	0.0051	0.0102	0.0121	0.0140	0.0150	0.0161

From these points sketch in the alternating *B-H* characteristic for the material.

8. Two concentric rings are fitted tightly together. The inner ring has a mean diameter of 16 centimeters and a section of 2 by 6 centimeters. The outer ring has a mean diameter of 19 centimeters and a section of 1 by 6 centimeters. The relative permeabilities are 1200 and 2500, respectively. The combination is wound with an exciting winding. Calculate the total permeance and the total reluctance of the magnetic circuit.

9. Two similar coils are to be wound on legs *A* and *B* of Fig. 283 in such a manner that their fluxes will combine through *C*. Calculate the

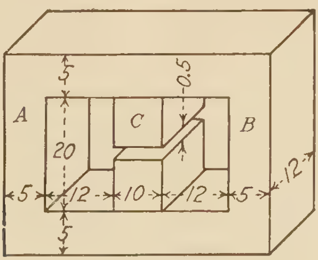

FIG. 283.

approximate number of ampere-turns excitation per coil, if the total flux through the air gap is to be 0.012 weber. Due to the "fringing effect" around the air gap, the average cross-section of the gap is greater than that of the iron surfaces bounding it. Where the air gap is short, it is usually sufficiently accurate to increase each dimension of the section by the length of the gap. Neglect the reluctance of the joints. All dimensions are in centimeters. The core is of cast steel.

10. If the polarity of one of the coils in the preceding exercise is reversed, the total ampere-turns of excitation remaining as determined, what will be the flux in parts A, B, and C?

11. *a.* With only one exciting coil, and that on C, how many ampere-turns must it have to produce the same fluxes as in exercise 9?

b. With similar coils on A, B, and C, how many ampere-turns must each coil have to produce the same results as in exercise 9? Must the three coils have the same number of turns, it being assumed that the coils are electrically connected in series?

12. Make use of the concept of magnetic poles and of the apparent or fictitious force on them to calculate the force exerted on one long air-core solenoid, A, by another, B. On a plane marked off in rectangular coordinates with the centimeter as the unit of length, coil A is placed with its north-seeking pole at (0, 0) and the south-seeking pole at (10, 0). Coil B is placed with the south-seeking pole at (0, 10) and the north-seeking pole at (0, 20). Each coil has 400 turns and carries a current of 3 amperes. Each turn bounds a circle 1 centimeter in diameter.

CHAPTER XV

PROPAGATION OF ELECTRICAL EFFECTS AND FLOW OF ENERGY IN SPACE

341. Purpose.—When a positively charged sphere has been stationary at A for some time, the state of affairs in the surrounding space is indicated by the full lines of Fig. 291. The vectors representing the electric intensities in the surrounding space all point radially outward from the charge and at any point have a value which is inversely proportional to the square of the distance from the center of the sphere. The tubes of electric intensity are truncated cones having their vertices at the center of the sphere and extending off to infinite distances. The full radial lines on the diagram may be taken as the axes of conical tubes of intensity.[1]

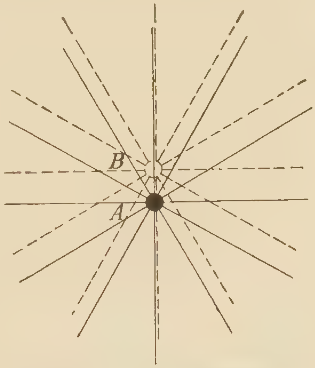

FIG. 291.—Lines of intensity for two positions of a charge.

After the charged sphere has been shifted to B, the final state of affairs is pictured by the dashed radial lines issuing from B. The questions we now propose to consider relate to the march of affairs from one state to the other. Upon the instant that the charged body comes to rest in the position B,

[1] Parenthetically, it may be remarked that among the many things which an electric charge is, it is (when stationary) the state of affairs in space which has been presented in the symbolism of electric intensity vectors or of tubes of electric intensity. The point of view in which the charge at A is regarded as **one thing** and the state of affairs in space as **another thing** may be a useful, workable way of looking at relations for the purpose of, say, an engineer who is concerned with the practical problems of electroplating, but it is a very imperfect view. The state of affairs in space is (one aspect of) the charge at A.

does the state pictured by the dashed lines exist throughout all space? In other words, as the charged body is moved from one position to another is the steady state of affairs corresponding to each intermediate position of the body instantaneously assumed throughout all space at the instant the body arrives at each intermediate position?

Another question of the same type is this: Associated with a fixed coil carrying a continuous current of I_o amperes is a definite distribution of magnetic flux densities throughout all space. Associated with a larger continuous current I_1 is a second distribution of flux densities. As the current increases from one value to the other, what is the **march** in the values of the flux densities? Is the steady-state distribution, corresponding to each intermediate value of the current, instantaneously assumed throughout all space?

This chapter is to deal with the propagation of electrical effects in time and space, and with the radiation of electromagnetic energy from circuits. The discussion naturally begins with the presentation of Maxwell's outstanding contribution to electrical theory, namely, the concept of the **electric displacement current** and its magnetic effects.

342. The Electric Displacement Current.—In the case of an unvarying unidirectional current, the flow necessarily occurs in a closed circuit or loop. Otherwise, the unvarying current would result in the accumulation of infinite charges of unlike sign at the two terminals of the open circuit.

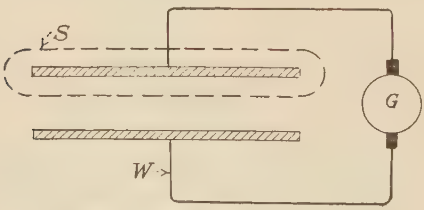

FIG. 292.—Open versus closed circuit.

Figure 292 illustrates a condenser consisting of two extended plates connected by a wire, W. If the wire contains a source of alternating e.m.f., G, an alternating current flows in the wire W, and charges of opposite sign accumulate on the two plates in alternating fashion. Prior to Maxwell's extension of the definition of "electric current," such a circuit was viewed as an **open**

circuit in which there is at any instant a current, not in a closed loop, but only in the wire.

Imagine any closed surface, as S, completely enclosing either of the plates on which the charge accumulates. From Gauss's theorem, the electrostatic flux Ψ in the outward direction over the surface, or the surface-integral of the electrostatic flux density over the surface, is at every instant equal to the charge enclosed by the surface (Secs. 101–102).

$$\Psi \text{ or } \oint^{surface} D \cos (D,n) da = \Sigma q. \tag{117}$$

Taking derivatives $$\frac{d\Psi}{dt} = \frac{dq}{dt}. \tag{571}$$

But dq/dt, the rate at which charge is accumulating on either plate, is equal to the conduction current, i_c, crossing the surface S in the inward direction.

Now Maxwell called the vector quantity D, the **electric displacement** at a point, and he called Ψ, or the surface-integral of D, the **total displacement** over the surface. The time rate of change of the total displacement over the surface he called the **displacement current,** i_d, over the surface.[2]

342a. DISPLACEMENT CURRENT (DEFINITION).—**By the displacement current i_d in a specified direction across a specified surface is meant the**

[2] The germ from which the notion of the displacement current developed is to be found in the writings of Faraday, William Thomson, and others. Thus, Thomson had pointed out the analogy between electric intensity and an elastic displacement in a solid, and Faraday had compared the particles of a dielectric to small spherical shot embedded in an insulating medium and insulated from each other. The motion of the electricity in the shot, when the strength of the field is varied, is equivalent to an electric current. It is perhaps from this that Maxwell obtained the conception that variations of total displacement (electrostatic flux) are to be treated as currents. "But in adopting the idea, Maxwell altogether transformed it; for Faraday's conception of displacement was applicable only to ponderable dielectrics, and was in fact introduced solely in order to explain why the permittivity of such dielectrics is greater than that of free ether; whereas according to Maxwell there is displacement wherever there is an electric intensity, whether material bodies are present or not." (See WHITTAKER, *History of the Theories of Electricity and Magnetism,* pp. 279, 284; also MAXWELL, *Treatise on Electricity and Magnetism,* Arts. 60–62, 111; also Sec. 95 of this text.)

time rate of change of the electrostatic flux Ψ in the specified direction over the surface.

$$i_d(\text{amperes}) = \frac{d\Psi}{dt}\,\frac{(\text{coulombs})}{(\text{sec.})} \tag{572}$$

The displacement current over a surface having been defined in this way, it follows from Eqs. (571) and (572) that the displacement current in the outward direction over any closed surface is exactly equal but opposite in sign to the sum of the conduction plus the convection currents in the same direction over the surface. From Maxwell's point of view, the conduction current in the wire W of Fig. 292 is accompanied by a displacement current in the dielectric separating the plates of the condenser. The direction of the displacement current is along the tubes of electrostatic flux extending from plate to plate. The circuit is not to be viewed as an open circuit, but as a closed circuit consisting of a conducting and a dielectric portion in series. In general, all tubes or filaments of current form closed tubes and the following proposition is true.

342b.—The total current in a specified direction across any closed surface (that is, the sum of the conduction, convection, and displacement currents across the surface) is at every instant of time equal to zero.

$$i_t \text{ over a closed surface (or } i_c + i_v + i_d) = 0. \quad (573$$

Now in all that we have stated up to this point about displacement currents, nothing but definition and deductions from definition has been involved. We now come to the most important feature of Maxwell's conception, the feature which is not a matter of definition or of viewpoint, but which involves a new physical relation, namely the conception of the magnetic effects which are associated with displacement currents.

343. Magnetic Effects of Displacement Currents.—Maxwell postulated that the displacement current in any given tube of flux is accompanied by a magnetic field which is identical with that which would be caused by an equal conduction current flowing in the same channel. This postulate may be stated in various ways. It may take the form that "the magnetic flux densities associated with a short elementary filament of displacement current are to be calculated by means of Ampere's formula."

The magnetomotive force law of circuitation of Sec. 248 when revised to specifically include this postulate reads:

343a. Magnetomotive Force Law of Circuitation.—The magnetomotive force $\mathfrak{F}$ around a closed line in a specified direction (or the line-integral of the magnetic intensity in the specified direction around the line) is equal to the sum of the conduction current plus the convection current plus the displacement current in the arrow direction across any surface which is bounded by the closed line of integration.

$$\mathfrak{F}\text{ (closed line)} = \oint^{\text{line}} H\cos(H, l)dl = i_c + i_v + i_d. \quad (574)$$

(As an exercise in the use of these conceptions, exercises 1 to 4 of Sec. (365) may be solved at this point).[3]

At the time this postulate about the magnetic effects of displacement currents was advanced, and for years afterward, it

[3] As a further illustration of the meaning of these conceptions let us consider the magnetomotive force exerted upon the iron core of the current transformer illustrated in two different positions A and B in Fig. 293.

To find the current passing through and looped with the core we imagine any surface, plane or curved, of which the core is the boundary. The current looping with the iron core at any instant is, then, the net current passing across this surface. Imagine a plane surface of which the core is the boundary. Then, with the current transformer in the position A, substantially the only current which crosses the plane surface is the conduction current in the high-tension lead of the power transformer P. (The secondary circuit of the current transformer is assumed to be open.)

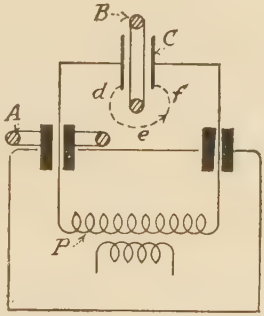

FIG. 293.—Displacement current through a current transformer B.

Suppose, however, the current transformer is shifted to the position B, a position in which the plane surface bounded by the core cuts through the dielectric of the condenser C, and, consequently, a position in which no conduction current crosses the plane surface. In this position the current crossing the plane surface is a displacement current. This displacement current—the rate of change of the electrostatic flux which passes across the plane surface bounded by the core—is less than the conduction current in the transformer lead at A by the displacement which takes place between the leads along paths, as *def*, which do not loop through the iron core. If the leads are short and the condenser plates large, the magnetomotive force exerted upon the core in the position B will be only slightly lower than in the position A.

was not possible to adduce any direct experiments on which to base it. Nevertheless, Maxwell embodied it in the fundamental equations of his system, and by so doing obtained a set of differential equations, the solution of which indicated the propagation of electromagnetic effects in space with the velocity of light.

344. The Field Equations of Maxwell.—The equations, or formulas, which express the fundamental experimentally determined relations between the electric intensity F, the magnetic flux density B, the charges q, and the currents i in the electromagnetic field are often called the field equations of Maxwell.

These relations are:

a. The inverse-square law of force.
b. The principle of the conservation of electricity.
c. The law of circuitation for magnetic flux density.
d. The law of continuity for magnetic flux density.
e. The law of circuitation for induced electromotive intensities.

These relations have all been presented and applied in previous chapters in many diverse forms. For the purpose of emphasizing that these diverse forms are simply different ways of expressing and applying a few fundamental experimental relations, we present in condensed form each relation, together with the definitions and formulas by which it is expressed.

344a. The Inverse-square Law of Force.

Exp. Det. Rel. The force f varies as $\frac{q_1q_2}{l^2}$.

Definition. Quantity of electricity, $q = \sqrt{4\pi pf}\, l$.

Definition. Electric intensity, $\mathbf{F} = \frac{\mathbf{f}}{q_t}$.

Definition. Electric potential, $E = \frac{w}{q_t}$.

Exp. Det. Rel. Effects of charges are linearly superposable.
Deductions. All to be regarded as alternative ways of expressing the inverse-square law in a single dielectric.

a. Gauss's theorem: Surface-integral of F is proportional to volume integral of ρ.

b. $$\oint^{s} F \cos (F, n)da = \frac{1}{p_0}\int^{\text{vol.}} \rho dv.$$

c. $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\rho}{p_0}.$

d. $\text{div.}\, \mathbf{F} = \frac{\rho}{p_0}.$

e. $\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\frac{\rho}{p_0}.$

For a field in two or more dielectrics:

f. $\int^{\text{O}s} F \cos(F, n)da = \frac{1}{p_o}\int^{\text{vol.}} (\rho_o + \rho_c)dv.$

g. $\Psi = \int^{\text{O}s} D \cos(D,n)da = \int^{\text{vol.}} \rho_0 dv.$

Definition. Electrostatic flux density, $\mathbf{D} = p\mathbf{F} = p_r p_o \mathbf{F}.$

Definition. Electrostatic flux, $\Psi = \int D \cos(D,n)da.$

344b. Principle of Conservation of Electricity.

Exp. Det. Rel. Conservation of electricity.

Definition. Conduction current, $i_c = \frac{dq}{dt}.$

Definition. Convection current, $i_v = \frac{dq}{dt} = \int (\rho V) \cos(V,n)da.$

Definition. Displacement current, $i_d = \frac{d\Psi}{dt}.$

Definition. Current density $\mathbf{J} = \frac{i}{a}.$

Deductions. All to be regarded as alternative ways of expressing the principle of the conservation of electricity.

a. Kirchhoff's law. Σi (toward a junction) $= 0.$

b. $\int^{\text{O}s} J_v \cos(J,n)da = -\int^{\text{vol.}} \frac{d\rho}{dt}dv.$

c. $\frac{\partial(\rho V_1)}{\partial x} + \frac{\partial(\rho V_2)}{\partial y} + \frac{\partial(\rho V_3)}{\partial z} = -\frac{d\rho}{dt}.$

d. $\text{div.}\, \mathbf{J}_v = -\frac{d\rho}{dt}.$

e. $\text{div.}\, \mathbf{J}_t$ or $\text{div.}\, (\mathbf{J}_c + \mathbf{J}_v + \mathbf{J}_d) = 0.$

344c. Laws of Circuitation and Continuity for *B*.

Exp. Det. Rel. Force on a current, f varies as $Il \sin\theta.$

Deduction. Force on a moving charge, f varies as $QV \sin\theta.$

Definition. Magnetic flux density, $\mathbf{B} = \frac{f}{Il \sin(B,l)}.$

Deduction. $\mathbf{B} = \frac{f}{QV \sin(V,B)}.$

Exp. Det. Rel. Ampere's formula $dB_c = \frac{\mu I_c \sin(r,l)dl}{4\pi r^2}.$

Rowland's term $dB_v = \dfrac{\mu I_v \sin (r,l)dl}{4\pi r^2}.$

Maxwell's term $dB_d = \dfrac{\mu I_d \sin (r,l)dl}{4\pi r^2}.$

Deductions. All to be regarded as alternative ways of expressing the relations contained in Ampere's formula.

Law of Circuitation for B:

a. M.m.f. around a closed loop is proportional to current threading the loop.

b. $\displaystyle\int^{\bigcirc l} B \cos (B,l)dl = \mu \Sigma i$ (over cap).

c. $\displaystyle\int^{\bigcirc l} B \cos (B,l)dl = \mu \int^{\text{cap}} J \cos (J,n)da.$

d. $$\mathbf{i}\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right) = \mu\left\{\mathbf{i}\left(\rho V_1 + p\frac{\partial F_1}{\partial x}\right) + \mathbf{j}\left(\rho V_2 + p\frac{\partial F_2}{\partial y}\right) + \mathbf{k}\left(\rho V_3 + p\frac{\partial F_3}{\partial z}\right)\right\}.$$

e. $\text{curl } \mathbf{B} = \mu\left(\rho \mathbf{V} + p\dfrac{d\mathbf{F}}{dt}\right).$

Law of Continuity for B:

a. The magnetic flux in the outward direction across any closed surface is zero.

b. $\displaystyle\int^{\bigcirc s} B \cos (B,n)da = 0.$

c. $\dfrac{\partial B_1}{\partial x} + \dfrac{\partial B_2}{\partial y} + \dfrac{\partial B_3}{\partial z} = 0.$

d. $\text{div. } \mathbf{B} = 0.$

Definition. Magnetic flux $\Phi = \displaystyle\int B \cos (B,n)da.$

344d. Law of Circuitation for Induced Electromotive Intensities.

Exp. Det. Rel. The e.m.f. induced in a circuit is equal to the rate of decrease of the flux-linkage of the circuit.

Deductions. Alternative ways of expressing the laws for induced electromotive intensities.

a. $$\int^{\bigcirc \text{ line}} F \cos (F,l)dl = -\frac{d\Lambda}{dt} = -\int^{\text{cap}} \frac{d}{dt} B \cos (B, n)\, da.$$

b. $$\mathbf{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = -\mathbf{i}\frac{\partial B_1}{\partial t} - \mathbf{j}\frac{\partial B_2}{\partial t} - \mathbf{k}\frac{\partial B_3}{\partial t}$$

c. $\text{curl } \mathbf{F} = -\dfrac{d\mathbf{B}}{dt}.$

345. The Visualization of the Circuital Relations.—Before proceeding to solve the field equations, it may be well to obtain a space picture of the circuital relations, and to show by a qualitative physical argument that the insertion of the displacement current term in these relations results in a picture in which the velocity of propagation of effects is finite.

The law of circuitation for B states that a filament of current is accompanied by a magnetic field in which the tubes of flux density are closed loops linked with the closed tube of current. Figure 294 illustrates two of the tubes of magnetic flux density linked with the current in a long, straight wire, and Fig. 295 illustrates some of the B tubes linking with the current in a circular wire hoop.

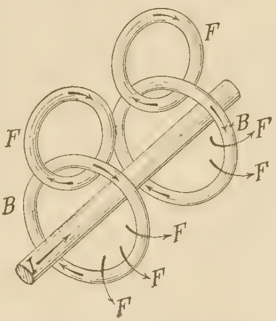

FIG. 294.—Inter-linkage of tubes of current, magnetic flux, and electromotive intensity.

If the current is increasing in value, the values of the magnetic flux over the cross-section of these B tubes is increasing. Now from the law of circuitation for induced electromotive intensities these increasing fluxes in the B tubes are accompanied by a field of electromotive intensities in which the tubes of electromotive intensity are closed loops linking with the B tubes. Two of these F tubes are shown in Fig. 294. From a sufficient study of the figures the conclusion may be drawn that all the F tubes of all the B tubes when combined will give, as a resultant, tubes of electromotive intensity in the form of cylinders enclosing the wire. The electromotive intensity in these cylinders will be parallel to the direction of the current in the wire in a direction opposite to the direction in which the current is increasing. No attempt has been made to show this resultant in Fig. 294, but in Fig. 295 portions of one of the resultant tubes of electromotive intensity have been sketched.

This completes the picture as it existed before Maxwell postulated the magnetic effect of the F tubes. Let us use this picture to discuss the energy relations for the case in which the current in the wire in Fig. 295 is assumed to vary in the manner shown by the curve i in Fig. 296. The current starts from zero,

increases at a uniform rate for the interval ab, remains constant in value for the interval bc and then during the interval cd decreases at a uniform rate to zero. The picture is, that the steady-state values of flux densities corresponding to each value

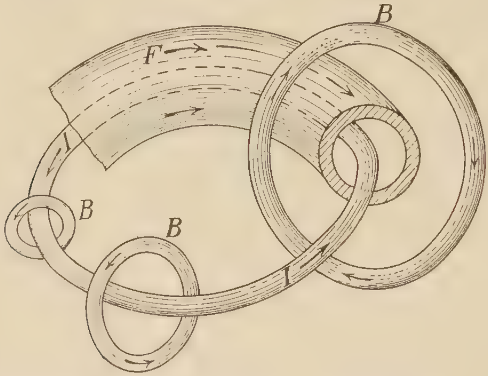

FIG. 295. Inter-linkage of tubes of current, magnetic flux, and electromotive intensity.

of the current are attained throughout all space at the instant the current value is attained. Accordingly, the e.m.f. of inductance will be constant at the value e_1 during the interval ab in which the current increases at a uniform rate, will be zero during

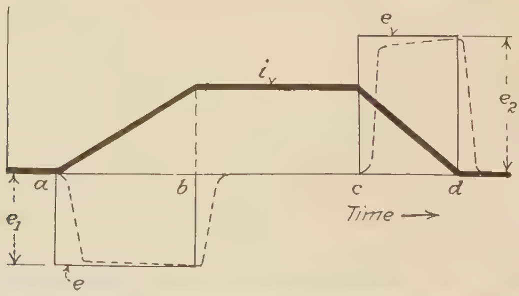

FIG. 296.—Electromotive forces accompanying the rise and the fall of current.

the interval bc, and constant at the value e_2 during the interval cd. The voltage-impulses during the two intervals will be equal and therefore the work done by the e.m.f. of inductance while the current is decreasing will just equal the work done against

this e.m.f. while the current is increasing. That is, the energy expended in establishing the magnetic field is all returned when the field dies out and no energy is lost to the system by radiation. The picture is faulty in several respects, for example, the F values are pictured as springing to full value throughout all space at the instant the current starts to increase. But since the current is infinitesimally small during this initial interval, no energy is delivered during the initial interval and therefore it would not be consistent to associate with these electromotive intensities the customary notion of stored energy to the extent of $p_oF^2/2$ joules per cubic centimeter which, from the study of the energy stored in charged condensers, is associated with electric intensities in the electrostatic field.

Let us now consider the manner in which the postulated magnetomotive force of the displacement current alters the picture. We have just seen that the non-retarded rise of magnetic flux densities throughout all space carries with it the picture of the electromotive intensities throughout space jumping from zero values to finite values at the moment the current starts to increase. Consider an F tube of any depth (say 1 millimeter) adjacent to and ensheathing the copper wire. When the current is increasing the electromotive intensities in this F tube are in a direction opposite to the direction of increase of the current. If the intensities in this tube are imagined to jump from zero to a finite value at the moment the current starts to increase, such a jump implies an infinite displacement current in the tube. (This is postulating that variations of electromotive intensity are to be treated as displacement current densities.)

Consequently, this would imply that, in a B tube encircling the outer skin of this innermost F tube, the magnetomotive force is infinitely great and in a direction opposite to that of the current. This picture is utterly inconsistent. Sufficient study will show that the only consistent picture is obtained by imagining that the magnetic flux density starts to build up with the current only in the film of space immediately around the conductor and that as time passes the region in which B has built up extends farther and farther from the wire. Only by such a process can the opposing magnetomotive force of the displacement currents (lying between the wire and the expanding region

in which the B tubes are being laid down) be kept less than the magnetomotive force of the current in the wire.

Let us consider the energy relations which go with this picture. Let the current rise and fall as in Fig. 296. The e.m.f. of inductance, instead of jumping abruptly to the values e_1 and e_2 when the current starts to rise and starts to fall, varies in the manner shown by the dotted curve. (The difference between the e.m.f. curves for this picture and for the previous picture is extravagantly exaggerated.)

It is evident that the work done by the e.m.f. of inductance while the current is decreasing is less than the work done against it in building up the current. That is, to say, in carrying the current through this cycle of values, energy has been lost from the system by radiation.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF THE FIELD[4]

346. The Given Data.—Let it be supposed that the distribution of the charge which gives rise to the field and the equations of motion of the charge are given. That is, the volume density ρ, the velocity $\mathbf{V}$, and the acceleration $\mathbf{a}$ of the charge are known functions of time and space. The problem is to find the equations which will express for any point in space the values of the electric intensity $\mathbf{F}$ and the flux density $\mathbf{B}$ which these specified motions will cause.

The physical laws relating $\mathbf{F}$ and $\mathbf{B}$ to ρ and $\mathbf{V}$ are given in the form of differential equations which express the mode of space variation of $\mathbf{F}$ and $\mathbf{B}$. By the solution of these equations, is meant those functional relations, $\mathbf{F} = f(t, x, y, z)$ and $\mathbf{B} = \theta(t, x, y, z)$, which will satisfy these differential equations.

The differential equations when expressed in rectangular coordinates are as follows:[5]

[4] This solution is reprinted from a paper entitled *High versus Low Antenna in Radio Telegraphy*, Bennett, Edward, Bulletin 810, Univ. of Wis., Eng. Series, Vol. VIII, No. 4.

[5] Expressed in vector notation these four relations are:

$$\text{div. } \mathbf{F} = \frac{\rho}{p}. \tag{575}$$

$$\text{div. } \mathbf{B} = 0. \tag{576}$$

$$\text{curl } \mathbf{B} = \mu\left(\rho\mathbf{V} + p\,\frac{d\mathbf{F}}{dt}\right). \tag{577}$$

$$\text{curl } \mathbf{F} = -\frac{d\mathbf{B}}{dt}. \tag{578}$$

From the inverse-square law of force,

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\rho}{p}. \tag{575a}$$

From the law of continuity for B,

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0 \tag{576a}$$

From the law of circuitation for B,

$$\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} = \mu\left(\rho V_1 + p\frac{\partial F_1}{\partial t}\right). \tag{577a}$$

$$\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} = \mu\left(\rho V_2 + p\frac{\partial F_2}{\partial t}\right). \tag{577b}$$

$$\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} = \mu\left(\rho V_3 + p\frac{\partial F_3}{\partial t}\right). \tag{577c}$$

From the law of circuitation for F,

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = -\frac{\partial B_1}{\partial t}. \tag{578a}$$

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = -\frac{\partial B_2}{\partial t}. \tag{578b}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -\frac{\partial B_3}{\partial t}. \tag{578c}$$

347. To Obtain a Differential Equation in the Dependent Variable F Alone.—Differentiating Eq. (577a) with respect to t,

$$\frac{\partial^2 B_3}{\partial t\partial y} - \frac{\partial^2 B_2}{\partial t\partial z} = \mu\left(\frac{\partial(\rho V_1)}{\partial t} + \rho\frac{\partial^2 F_1}{\partial t^2}\right). \tag{579}$$

Differentiating Eq. (578c) with respect to y and Eq. (578b) with respect to z and substituting in Eq. (579),

$$-\frac{\partial^2 F_2}{\partial y\partial x} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_3}{\partial z\partial x} = \mu\left(\frac{\partial(\rho V_1)}{\partial t} + p\frac{\partial^2 F_1}{\partial t^2}\right). \tag{580}$$

Differentiating Eq. (575a) with respect to x and adding to Eq. (580),

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} = \mu\left(\frac{\partial(\rho V_1)}{\partial t} + p\frac{\partial^2 F_1}{\partial t^2}\right) + \frac{1}{p}\frac{\partial\rho}{\partial x}. \tag{581}$$

Writing $$\nabla^2 \text{ for } \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{582}$$

and $$s \text{ for } \frac{1}{\sqrt{\mu p}} = 3 \times 10^{10}, \tag{583}$$

Eq. (581) may be written

$$\nabla^2 F_1 - \frac{1}{s^2}\frac{\partial^2 F_1}{\partial t^2} = \frac{1}{p}\left(\frac{\partial\rho}{\partial x} + \frac{1}{s^2}\frac{\partial(\rho V_1)}{\partial t}\right)$$

or $$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)F_1 = \frac{1}{p}\left(\frac{\partial\rho}{\partial x} + \frac{1}{s^2}\frac{\partial(\rho V_1)}{\partial t}\right). \tag{584a}$$

In like manner the following equations are obtained

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)F_2 = \frac{1}{p}\left(\frac{\partial\rho}{\partial y} + \frac{1}{s^2}\frac{\partial(\rho V_2)}{\partial t}\right). \tag{584b}$$

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)F_3 = \frac{1}{p}\left(\frac{\partial\rho}{\partial z} + \frac{1}{s^2}\frac{\partial(\rho V_3)}{\partial t}\right). \tag{584c}$$

348. To Obtain a Differential Equation Involving B Alone.—Differentiating Eq. (578a) with respect to t,

$$\frac{\partial^2 F_3}{\partial t \partial y} - \frac{\partial^2 F_2}{\partial t \partial z} = -\frac{\partial^2 B_1}{\partial t^2}. \qquad (585)$$

Differentiating Eq. (577c) with respect to y and Eq. (577b) with respect to z, and substituting the values so found for

$\frac{\partial^2 F_3}{\partial t \partial y}$ and $\frac{\partial^2 F_2}{\partial t \partial z}$ in Eq. (585), we obtain

$$\frac{1}{\mu p}\left\{\frac{\partial^2 B_2}{\partial x \partial y} - \frac{\partial^2 B_1}{\partial y^2} - \mu\frac{\partial(\rho V_3)}{\partial y} - \frac{\partial^2 B_1}{\partial z^2} + \frac{\partial^2 B_3}{\partial z \partial x} + \mu\frac{\partial(\rho V_2)}{\partial z}\right\} = -\frac{\partial^2 B_1}{\partial t^2}. \qquad (586)$$

Differentiating Eq. (576a) with respect to x and subtracting from Eq. (586), we obtain

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)B_1 = -\mu\left(\frac{\partial}{\partial y}(\rho V_3) - \frac{\partial}{\partial z}(\rho V_2)\right). \qquad (587a)$$

In like manner the following equations may be obtained:

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)B_2 = -\mu\left(\frac{\partial}{\partial z}(\rho V_1) - \frac{\partial}{\partial x}(\rho V_3)\right). \qquad (587b)$$

and

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)B_3 = -\mu\left(\frac{\partial}{\partial x}(\rho V_2) - \frac{\partial}{\partial y}(\rho V_1)\right). \qquad (587c)$$

349. The Dalembertian Operator.—The differential equations for which the solution is desired take the forms:

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)F_1 = \frac{1}{p}\left(\frac{\partial \rho}{\partial x} + \frac{1}{s^2}\frac{\partial}{\partial t}(\rho V_1)\right). \qquad (584a)$$

and

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)B_1 = -\mu\left(\frac{\partial}{\partial y}(\rho V_3) - \frac{\partial}{\partial z}(\rho V_2)\right). \qquad (587c)$$

That is to say, in the solutions or integrated equations, F_1 and B_1 must be such functions of time and of the position of a point that the operation $\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)$ applied to the function will yield a result whose value is determined by the volume density (ρ) and the current density (ρ**V**) at the point.

It is very convenient to have a name for the result of the operation $\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)$ upon a quantity. H. A. Lorentz has suggested[6] that the result of the operation be called the "Dalembertian of the quantity," since d'Alembert was the first to solve the differential wave equation involving the operation $\left(\frac{\partial^2}{\partial x^2} - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)$, which is a special case of $\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)$.

Adopting this suggestion, we may say that the Dalembertians of the components of **F** and **B** are given in Eqs. (584) and (587) in terms of the volume density (ρ) and current density (ρ**V**).

Now in the system of moving charges (ρ) and (ρ**V**) are known functions of time and of the position of a point in space. That is to say, the right-hand

[6] LORENTZ, H. A., *Theory of Electrons*, p. 17.

members of Eqs. (584a) and (587a) are functions of known form, and the problem is to find the functional expression which will give the value of F_1, or of B_1, at any point in space and for any moment of time. If the known form assumed by the right-hand member of Eq. (584a) is represented by the expression $f(t, X, Y, Z)$, we have given

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)F_1 = f(t, X, Y, Z). \tag{588}$$

Or, dropping the subscripts, the problem is:
Given, Dalembertian $\mathbf{F} = f(t, X, Y, Z)$,
to find, the expression for $\mathbf{F}$.
We proceed to demonstrate the following proposition.

350. Proposition That the "Retarded Potentializing Operation" Performed upon the $f(t, X, Y, Z)$ Yields a Function Whose Dalembertian Equals the $f(t, X, Y, Z)$.[7]—"If the Dalembertian of $\mathbf{F}$ equals $f(t, X, Y, Z)$, then the value of $\mathbf{F}$ may be found by an operation which may be called the operation of forming the 'retarded potential' of the $f(t, X, Y, Z)$. The operation of forming the retarded potential of the $f\ (t, X, Y, Z)$ may be symbolized by stating that the value of $\mathbf{F}$ will be given by the equation,

$$\mathbf{F} = -\frac{1}{4\pi}\int^{\text{all space}} \frac{f\Big((t - r/s), X\ ,\ Y, Z,\Big)}{r}\,dv. \tag{589}$$

This expression is to be read as follows: The value of $\mathbf{F}$ at a given instant (t) and for a given point $P \equiv (x, y, z)$ is equal to$(-1/4\pi)$ times the summation obtained

a. By dividing all space into volume elements (dv).

b. Dividing the volume of each element by its distance

$$r = \sqrt{(X - x)^2 + (Y - y)^2 + (Z - z)^2}$$

from the point P.

c. Multiplying this quotient by the value of the $f(t, X, Y, Z)$ at the volume element, *not for the instant* (t), but for the instant of time $(t - r/s)$, or r/s seconds earlier.

d. And finally summing up all the products so obtained."

Before taking up the proof of this proposition it may be noted that if there are no moving charges in the field, the operation of obtaining the retarded potential becomes identically the same as the more familiar operation of obtaining the potentials in the gravitational or in the electrostatic field. If there are no moving charges, the Dalembertian of F degrades to the Laplacian of F.

[7] WHITTAKER, *History of the Theories of Aether and Electricity*, pp. 268, 298.
LORENTZ, H. A., *The Theory of Electrons*, p. 233.
LORENZ, LUDWIG, Phil. Mag., 1867, Vol. XXXIV, p. 287.
RIEMANN, M. B., Phil. Mag., 1867, Vol. XXXIV, p. 368.

351. Proof for Charges at a Distance from the Point.—To show that the value of F defined by Eq. (589) satisfies the differential relation expressed in Eq. (588).

Let $P \equiv (x, y, z)$, (Fig. 297), be any point for which the value of F is desired for the instant (t). In performing the integration indicated in Eq. (589), we are concerned with only those volume elements for which the $f(t, X, Y, Z)$ has a value at the instant $(t - r/s)$. That is, only those volume elements which contain a charge at the instant $(t - r/s)$ contribute to the integral. All the balance of space contributes nothing to the integral. A few of the volume elements presumed to contribute to the integral have been shown in Fig. 297.

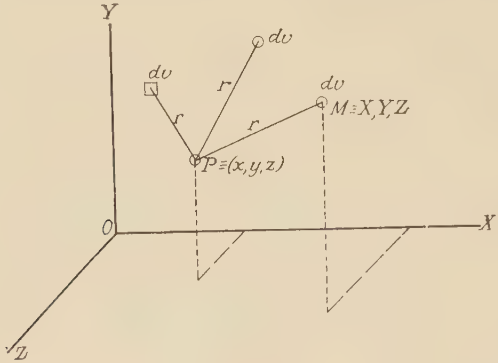

FIG. 297.

Consider first the part of the integral contributed by any volume element whatsoever except the element immediately surrounding the point P.

The part of the integral contributed by the volume element (dv) at $M \equiv (X, Y, Z)$ is

$$F_M = -\frac{1}{4\pi}\frac{f((t - r/s), X, Y, Z)}{r}\,dv$$

$$= -\frac{1}{4\pi}\frac{f\left\{\left(t - \frac{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}{s}\right), X, Y, Z\right\}}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}\,dv. \tag{590}$$

Taking the second derivative of F_M with respect to (t),

$$\frac{\partial^2}{\partial t^2}F_M = -\frac{1}{4\pi}\frac{f''((t - r/s), X, Y, Z)}{r}\,dv. \tag{591}$$

(See the footnote[3] for the meaning of $f''[(t - r/s), X, Y, Z]$.)

[3] In the expression $f((t - r/s), X, Y, Z)$, let $(t - r/s)$ be represented by (u). Then

$$\frac{\partial}{\partial x}f((t - r/s), X, Y, Z) = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}f\,(u, X, Y, Z)$$
$$= \frac{\partial u}{\partial x}f'\,(u, X, Y, Z)$$

and

$$\frac{\partial^2}{\partial x^2}f\,((t - r/s), X, Y, Z) = \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}f'\,(u, X, Y, Z)\right\}$$
$$= \frac{\partial^2 u}{\partial x^2}f'\,(u, X, Y, Z) + \left(\frac{\partial u}{\partial x}\right)^2 f''\,(u, X, Y, Z)$$
$$= \frac{\partial^2 u}{\partial x^2}f'\,((t - r/s)\,X, Y, Z) + \left(\frac{\partial u}{\partial x}\right)^2 f''((t - r/s)\,X, Y, Z).$$

Taking the second derivative of F_M with respect to (x),

$$\frac{\partial^2}{\partial x^2}F_M = -\frac{dv}{4\pi}\left\{f''((t - r/s), X, Y, Z)\frac{(X - x)^2}{s^2r^3}\right.$$
$$+ f'((t - r/s), X, Y, Z)\left(\frac{3(X - x)^2}{sr^4} - \frac{1}{sr^2}\right)$$
$$\left.+ f((t - r/s), X, Y, Z)\left(\frac{3(X - x)^2}{r^5} - \frac{1}{r^3}\right)\right\}.$$

In like manner, the following derivatives are obtained:

$$\frac{\partial^2}{\partial y^2}F_M = -\frac{dv}{4\pi}\left\{f''((t - r/s), X, Y, Z)\frac{(Y - y)^2}{s^2r^3}\right.$$
$$+ f'((t - r/s), X, Y, Z)\left(\frac{3(Y - y)^2}{sr^4} - \frac{1}{sr^2}\right)$$
$$\left.+ f((t - r/s), X\ Y, Z)\left(\frac{3(Y - y)^2}{r^5} - \frac{1}{r^3}\right)\right\}$$

and

$$\frac{\partial^2}{\partial z^2}F_M = -\frac{dv}{4\pi}\left\{f''((t - r/s), X, Y, Z)\frac{(Z - z)^2}{s^2r^3}\right.$$
$$+ f'((t - r/s), X, Y, Z)\left(\frac{3(Z - z)^2}{sr^4} + \frac{1}{sr^2}\right)$$
$$\left.+ f((t - r/s,) X, Y, Z)\left(\frac{3(Z - z)^2}{r^5} - \frac{1}{r^3}\right)\right\}.$$

Adding

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_M = -\frac{1}{4\pi s^2}\frac{f''((t - r/s), X, Y, Z)}{r}dv. \qquad (592)$$

Hence from Eq. (591) and (592),

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)F_M = 0.$$

As the volume element at M represents any volume element save the element surrounding P, it follows that the Dalembertian of all that portion of F which is contributed by volume elements other than the element immediately surrounding P is zero.

There remains to be considered the value contributed to the integral by the volume element N immediately surrounding the point P. Let this element be taken as a small spherical volume of radius R. To establish our proposition, the Dalembertian of the quantity which is contributed to F by this volume element must be demonstrated to equal the value of the $f(t, X, Y, Z)$ at P.

If the $f\ ((t - r/s), X, Y, Z)$ is zero within the sphere N, this portion of space contributes nothing to F. Therefore, if at P the $f\ (t, X, Y, Z)$ is zero, the Dalembertian of F for the point P is zero, and the value obtained for F by the summation expressed by Eq. (589) satisfies the differential relation expressed in Eq. (588).

352. Proof for Charges at the Point.—If the $f\ ((t - r/s), X, Y, Z)$ is not zero within the small sphere N surrounding the point P—in other words, if this sphere contains a charge—the value F_N contributed to F by the values of $f((t - r/s), X, Y, Z)$ within this spherical volume must be determined,

and, as previously stated, to establish our proposition, the Dalembertian of F_N must be shown to equal the value of the $f(t, X, Y, Z)$ at the point P.

The value of F_N is defined by the equation

$$F_N = -\frac{1}{4\pi}\int^{\text{Vol. of the sphere}} \frac{f((t - r/s)\ X, Y, Z)}{r}\,dv.$$

In the first place, it is to be noted that the infinite value assumed by the integrand when r equals zero does not mean that the integral F_N is infinite. This may be demonstrated as follows:

Over the space within a sphere of infinitesimal radius R, the $f((t - r/s), X, Y, Z)$ will, for any given instant of time, have substantially the same values at all points within the sphere. Or, at any rate, the values of the function for the different points within the sphere may be conceived to lie between a maximum value f_1 and a minimum value f_2. By letting the radius of the sphere approach zero, these values may be made to differ by an infinitesimal amount from the value of the function at the center of the sphere, namely, $f(t, X_o, Y_o, Z_o)$.

Consequently, the $f\ ((t - r/s), X, Y, Z)$ may be imagined to have the uniform value $f(t, X_o, Y_o, Z_o)$ at all points within the sphere. The value of F_N may then be computed by dividing up the spherical volume into spherical shells of thickness (dr) and carrying out the indicated integration.

Whence

$$F_N = -\frac{1}{4\pi}\int_0^R \frac{f\,(t, X_o, Y_o, Z_o)\ 4\pi r^2 dr}{r}$$

$$= -\frac{R^2}{2} f(t, X_o, Y_o, Z_o). \tag{593}$$

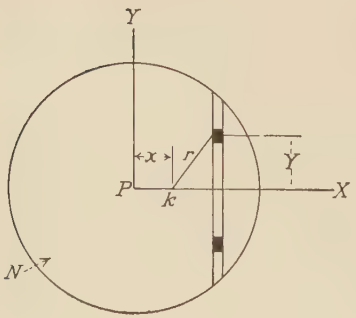

FIG. 298.

The value of F_N is therefore not only finite, but it can be caused to decrease without limit by decreasing the radius of the sphere without limit.

For the purpose of calculating the Dalembertian of the value F_N which is contributed to the retarded potential by the space within the sphere N, let the origin of the system of coordinates be transferred to the center (P) of the sphere. Then let the retarded potential be calculated—not for the point P—but for a point K (Fig. 298), displaced along the X axis by an infinitesimal distance (x) from the center. (This is for the purpose of obtaining the expression for F_N in such a form that the value of $\dfrac{d^2F_N}{dx^2}$ may be calculated.)

Imagine the volume of the sphere N to be made up of plane slices of thickness (dX), and these slices in turn to be constituted of circular rings of radius (Y), as shown in Fig. 298.

The expression for the integral F_N now takes the form:

$$F_N = -\frac{1}{4\pi}\int_{X=-R}^{X=x}\int_{Y=0}^{Y=\sqrt{R^2-X^2}} \frac{f(t,\ X_o,\ Y_o,\ Z_o)\ 2\pi Y dY dX}{\sqrt{(x-X)^2+Y^2}}$$

$$-\frac{1}{4\pi}\int_{X=x}^{X=R}\int_{Y=0}^{Y=\sqrt{R^2-X^2}} \frac{f(t,\ X_o,\ Y_o,\ Z_o)\ 2\pi Y dY dX}{\sqrt{(X-x)^2+Y^2}}. \tag{594}$$

Integrating,

$$F_N = -\left(\frac{R^2}{2} - \frac{x^2}{6}\right)f(t,\ X_o,\ Y_o,\ Z_o). \tag{595}$$

Taking the second derivative of F_N with respect to x and letting R approach zero, the second derivative approaches the following limit:

$$\frac{\partial^2 F_N}{\partial x^2} = \frac{1}{3}f(t,\ X_o,\ Y_o,\ Z_o).$$

In like manner it may be shown that

$$\frac{\partial^2 F_N}{\partial y^2} = \frac{1}{3}f(t,\ X_o,\ Y_o,\ Z_o) \text{ and } \frac{\partial^2 F_N}{\partial z^2} = \frac{1}{3}f(t,\ X_o,\ Y_o,\ Z_o).$$

Taking the second derivative of Eqs. (593) or (594) with respect to (t) and letting R approach zero, the second derivative approaches the following limit:

$$\frac{\partial^2 F_N}{\partial t^2} = 0\,.$$

Therefore

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right) F_N = f(t,\ X_o,\ Y_o,\ Z_o).$$

This establishes the proposition that the values of F determined by Eq. (589) satisfy the differential relation expressed in Eq. (588), or the values of a time and point function F whose Dalembertian is a given time and point function f (t, X, Y, Z) may be found by the operation of forming the retarded potential of the f (t, X, Y, Z).

Hence from Eqs. (584a) and (589)

$$F_1 = -\frac{1}{4\pi}\int \frac{\frac{1}{p}\left[\frac{\partial \rho}{\partial x} + \frac{1}{s^2}\frac{\partial}{\partial t}(\rho V_1)\right]_{(t-r/s)}}{r}\,dv.\text{[9]} \tag{596}$$

Splitting the right-hand member of Eq. (596) into two parts, it may be written

$$F_1 = F_1' + F_1'' = -\frac{1}{4\pi}\int\frac{\frac{1}{p}\left[\frac{\partial\rho}{\partial x}\right]_{(t-r/s)}}{r}\,dv$$

$$-\frac{1}{4\pi}\int\frac{\left[\mu\frac{\partial}{\partial t}(\rho V_1)\right]_{(t-r/s)}}{r}\,dv, \tag{569a}$$

In which, F_1' represents the first and F_1'' the second term. In like manner

$$F_2 = F_2' + F_2'' = -\frac{1}{4\pi}\int\frac{\frac{1}{p}\left[\frac{\partial\rho}{\partial y}\right]_{(t-r/s)}}{r}\,dv - \frac{1}{4\pi}\int\frac{\left[\mu\frac{\partial}{\partial t}(\rho V_2)\right]_{(t-r/s)}}{r}\,dv. \tag{596b}$$

[9] For the meaning of the subscript ($t - r/s$), see the footnote to Eq. (600).

$$F_3 = F_3' + F_3'' = -\frac{1}{4\pi}\int \frac{\frac{1}{p}\left[\frac{\partial \rho}{\partial z}\right]_{(t-r/s)}}{r}dv - \frac{1}{4\pi}\int \frac{\left[\mu\frac{\partial}{\partial t}(\rho V_3)\right]_{(t-r/s)}}{r}dv. \tag{596c}$$

By Eqs. (587) and (589),

$$B_1 = -\frac{\mu}{4\pi}\int \frac{-\left[\frac{\partial}{\partial y}(\rho V_3) - \frac{\partial}{\partial z}(\rho V_2)\right]_{t-(r/s)}}{r}dv. \tag{597a}$$

$$B_2 = -\frac{\mu}{4\pi}\int \frac{-\left[\frac{\partial}{\partial z}(\rho V_1) - \frac{\partial}{\partial x}(\rho V_3)\right]_{(t-r/s)}}{r}dv. \tag{597b}$$

$$B_3 = -\frac{\mu}{4\pi}\int \frac{-\left[\frac{\partial}{\partial x}(\rho V_2) - \frac{\partial}{\partial y}(\rho V_1)\right]_{(t-r/s)}}{r}dv. \tag{597c}$$

That is to say, the components of F and B for any point in space may be found by performing the retarded potentializing operations expressed in Eqs. (596) and (597). The retarded potentializing operations are to be performed upon such **point functions** as $\frac{\partial \rho}{\partial x}$ and $\frac{\partial}{\partial y}(\rho V_3)$.

It is generally much more convenient to calculate F and B from two **auxiliary point functions** which may be derived from Eqs. (596) and (597). We proceed to define these functions.

353. Proof that F and B May Be Calculated from the "Retarded Potential" and the "Retarded Vector Potential."—Let the **retarded scalar potential** E be defined as a scalar point function satisfying the differential relation,

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)E = -\frac{\rho}{p}. \tag{598}$$

Also let the **retarded vector potential** $\mathbf{A}$ be defined as a vector point function satisfying the differential relation,

$$\left(\nabla^2 - \frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu\rho\mathbf{V}. \tag{599}$$

In other words, E and $\mathbf{A}$ are defined as quantities whose Dalembertians are to be equal to the negative of ρ/p and the negative of $\mu\rho\mathbf{V}$ respectively. Therefore the values of E and $\mathbf{A}$ will be found by the operation of forming the retarded potentials of $-\rho/p$ and $-\mu\rho\mathbf{V}$. Thus

$$E = \frac{1}{4\pi}\int \frac{\left[\frac{\rho}{p}\right]_{(t-r/s)}}{r}dv.^{10} \tag{600}$$

$$\mathbf{A} = \frac{1}{4\pi}\int \frac{\left[\mu\rho\mathbf{V}\right]_{(t-r/s)}}{r}dv. \tag{601}$$

[10] The square brackets $[\quad]$ with or without the subscript $(t - r/s)$ will be used hereafter in this chapter to indicate that in finding the retarded potential at a point P for the instant of time (t), any element of volume is to be multiplied by the value of ρ/p or $\mu\rho\mathbf{V}$ in the element at the instant $(t - r/s)$.

The components of the vector **A** will be given by the equations,

$$A_1 = \frac{1}{4\pi}\int \frac{\Big[\mu\rho V_1\Big]_{(t-r/s)}}{r}dv. \tag{601a}$$

$$A_2 = \frac{1}{4\pi}\int \frac{\Big[\mu\rho V_2\Big]_{(t-r/s)}}{r}dv. \tag{601b}$$

$$A_3 = \frac{1}{4\pi}\int \frac{\Big[\mu\rho V_3\Big]_{(t-r/s)}}{r}dv. \tag{601c}$$

It may be shown that

$$F_1' = -\frac{\partial E}{\partial x} \text{ and } F_1'' = -\frac{\partial}{\partial t}A_1 \text{ or } F_1 = -\frac{\partial E}{\partial x} - \frac{\partial}{\partial t}A_1. \tag{602a}$$

$$F_2' = -\frac{\partial E}{\partial y} \text{ and } F_2'' = -\frac{\partial}{\partial t}A_2 \text{ or } F_2 = -\frac{\partial E}{\partial y} - \frac{\partial}{\partial t}A_2. \tag{602b}$$

$$F_3' = -\frac{\partial E}{\partial z} \text{ and } F_3'' = -\frac{\partial}{\partial t}A_3 \text{ or } F_2 = -\frac{\partial E}{\partial z} - \frac{\partial}{\partial t}A_3. \tag{602c}$$

Or in vector notation,

$$\mathbf{F} = -\operatorname{grad} E - \frac{\partial}{\partial t}\mathbf{A} \tag{602}$$

And that

$$B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = X \text{ component of curl of } A. \tag{603a}$$

$$B_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = Y \text{ component of curl of } A. \tag{603b}$$

$$B_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = Z \text{ component of curl of } A. \tag{603c}$$

Or in vector notation,

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{603}$$

The proof of these statements is as follows:

354. To Show That $F_1' = -\frac{\partial E}{\partial x}$.—Let P (Fig. 299), represent any point in space and K a second point displaced from P in a direction parallel to the X axis by the infinitesimal amount (dx). Let the values of the retarded potential E at these two points for a given instant of time be represented by E_k and E_p.

Let the operation of forming the retarded potentials at the point P and K for the instant (t) be visualized as carried out in the following manner:

1. Visualize all space as divided into volume elements, with radii extending from the point P to all those volume elements which contain a charge at the instant $(t - r/s)$. A few of these volume elements and radii have been shown in Fig. 299. The value of E_p is the result of carrying out the summation expressed by Eq. (600) over this system.

2. Now visualize a second set of radii (indicated by the dotted lines in Fig. 299) all issuing from the point K and drawn parallel and equal to the radii issuing from point P. The value E_k is the result of carrying out the summation expressed by Eq. (600) over this system.

How do the summations for E_p and E_k differ? Every radius with its terminal volume element in one system can be matched by a corresponding radius and volume element in the other. The only difference is in the density of the charge ρ in corresponding volume elements. If the volume density in any element belonging to the point P system (as the element C of

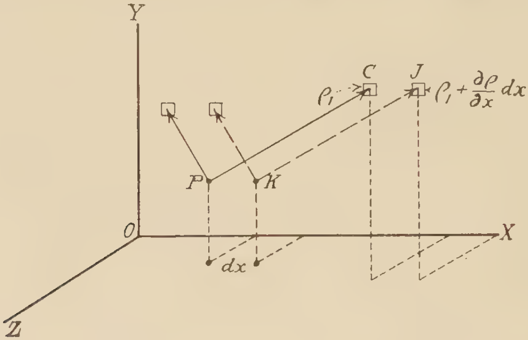

FIG. 299.

Fig. 299) is ρ, the volume density in the corresponding element of the point K system (J of Fig. 299) is $\left(\rho + \frac{\partial \rho}{\partial x} dx\right)$. Therefore the values of E_p and E_k will be given by summations involving identical combinations of (r) and (dv) associated with the volume density ρ in the P system and with $\left(\rho + \frac{\partial \rho}{\partial x} dx\right)$ in the K system.

$$E_k = \frac{1}{4\pi} \int \frac{\frac{1}{p}\left[\rho + \frac{\partial \rho}{\partial x} dx\right]_{(t-r/s)}}{r} dv$$

$$= \frac{1}{4\pi} \int \frac{\frac{1}{p}\left[\rho\right]_{(t-r/s)}}{r} dv + \frac{1}{4\pi} \int \frac{\frac{1}{p}\left[\frac{\partial \rho}{\partial x} dx\right]_{(t-r/s)}}{r} dv$$

and $$E_p = \frac{1}{4\pi} \int \frac{\frac{1}{p}\left[\rho\right]_{(t-r/s)}}{r} dv.$$

Therefore $$\frac{E_k - E_p}{dx} = \frac{1}{4\pi} \int \frac{\frac{1}{p}\left[\frac{\partial \rho}{\partial x}\right]_{(t-r/s)}}{r} dv.$$

But $$\frac{E_k - E_p}{dx} \text{ is } \frac{\partial E}{\partial x}$$

and from Eq. (596a)

$$F_1' = -\frac{1}{4\pi}\int \frac{\frac{1}{p}\left[\frac{\partial \rho}{\partial x}\right]_{(t-r/s)}}{r}\,dv.$$

Therefore

$$F_1' = -\frac{\partial E}{\partial x}.$$

In like manner it may be shown that

$$F_2' = -\frac{\partial E}{\partial y} \quad \text{and} \quad F_3' = -\frac{\partial E}{\partial z}.$$

355. To Show That $F_1'' = -\frac{\partial}{\partial t}A_1$.

From Eq. (601a), the X component A_1 of the retarded vector potential is

$$A_1 = \frac{1}{4\pi}\int \frac{\left[\mu\rho V_1\right]_{(t-r/s)}}{r}\,dv.$$

By visualizing the operations involved in finding the values of A_1 at any point P for two instants of time t_1 and $(t_1 + dt)$, it may be seen that

$$\frac{\partial}{\partial t}A_1 \text{ or } \frac{\partial}{\partial t}\left[\frac{1}{4\pi}\int \frac{\left[\mu\rho V_1\right]_{(t-r/s)}}{r}\,dv\right] = \frac{1}{4\pi}\int \frac{\left[\frac{\partial}{\partial t}\left(\mu\rho V_1\right)\right]_{(t-r/s)}}{r}\,dv.$$

But from Eq. (596a),

$$F_1'' = -\frac{1}{4\pi}\int \frac{\left[\mu\frac{\partial}{\partial t}(\rho V_1)\right]_{(t-r/s)}}{r}\,dv.$$

Therefore

$$F_1'' = -\frac{\partial}{\partial t}A_1.$$

In like manner it may be seen that

$$F_2'' = -\frac{\partial}{\partial t}A_2 \quad \text{and} \quad F_3'' = -\frac{\partial}{\partial t}A_3.$$

356. To Show That $B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}$.

By Eq. (601c)

$$A_3 = \frac{1}{4\pi}\int \frac{\left[\mu\rho V_3\right]_{(t-r/s)}}{r}\,dv.$$

By visualizing the summations involved in finding the values of A_3 at a point P and at a second point K displaced from P in a direction parallel to the Y axis by the infinitesimal amount (dy), it may be seen that

$$\frac{\partial}{\partial y}A_3 = \frac{\mu}{4\pi}\int \frac{\left[\frac{\partial}{\partial y}(\rho V_3)\right]_{(t-r/s)}}{r}\,dv.$$

In a similar way it may be seen that

$$\frac{\partial}{\partial z} A_2 = \frac{\mu}{4\pi} \int \frac{\left[\frac{\partial}{\partial z}(\rho V_2)\right]_{(t-r/s)}}{r} dv.$$

But from Eq. (597*a*),

$$B_1 = \frac{\mu}{4\pi} \int \frac{\left[\frac{\partial}{\partial y}(\rho V_3) - \frac{\partial}{\partial z}(\rho V_2)\right]_{(t-r/s)}}{r} dv.$$

Therefore

$$B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}. \tag{603a}$$

Similar demonstrations may be used to establish Eqs. (603*b*) and (603*c*)

357. To Summarize.—If a system of charges moves in space in a known manner, the electric intensity **F** and the magnetic flux density **B** at any point in space may be derived from the retarded potential E and the retarded vector potential **A** by the equations,

$$\mathbf{F} = -\operatorname{grad} E - \frac{\partial}{\partial t}\mathbf{A}. \tag{602}$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{603}$$

The values of the retarded potentials E and **A** are to be computed by the equations,

$$E = \frac{1}{4\pi} \int \frac{\left[\frac{\rho}{p}\right]_{(t-r/s)}}{r} dv. \tag{600}$$

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\left[\mu\rho\mathbf{V}\right]_{(t-r/s)}}{r} dv. \tag{601}$$

358. The Retarded Integration over a Small Moving Charge. Let us apply these solutions of the field equations to determine the values of **F** and **B** in the field set up by a small moving charge. The discussion will be limited to points whose distance from the charge is 100 or more times as great as the largest linear dimension of the charge.

In Fig. 300, let the point P, whose coordinates are (x, y, z), be the point at which the values of **F** and **B** are desired. Let the coordinates of the moving charge be represented by X, Y, Z.

Let the charge be moving with a velocity **V** which may make any angle with the line from Q to P, and let it be subject to an acceleration, **a**, which makes any angle with the direction of the velocity.

The first step is to calculate the value of the retarded potentials at the point P. Now the **retarded** potentializing operation for the value of E or **A** at P at a given instant t_1 may be regarded as a process of summation which is started at a suitable instant earlier than t_1 in a spherical shell drawn about P far enough away to include all charge, and which closes in on P with the velocity of light, s.

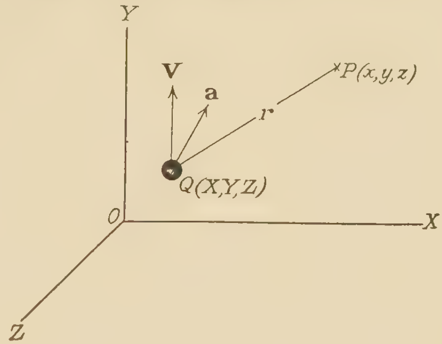

FIG. 300.

If the charges giving rise to the field are stationary, the only portion of space which contributes to the summation is the space occupied by the stationary charge. If the charge is in motion, the integrating process starts at the point of the charge most remote from P, and by the time the process has swept over the charge, the nearer portions have moved to a new position. That is to say, the effective volume of the space which contributes to the integral is different from the volume of space occupied by the charge at any one instant of time.

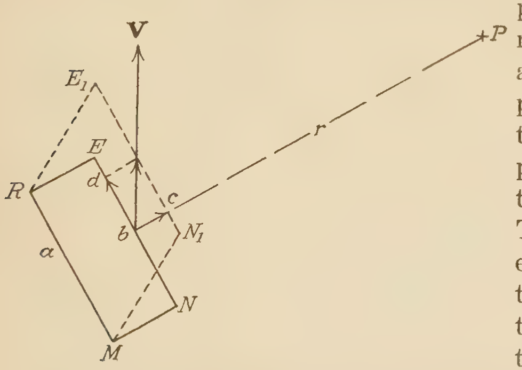

FIG 301.—Shift of charge during retarded summation.

In Fig. 301, $RMNE$ represents any slice, or wafer, of the charge included between two planes drawn perpendicular to the line QP of Fig. 300. (These planes move with the charge). The

integrating process starts at the more remote plane RM and sweeps over the slice to the nearer plane NE with the velocity of light. By the time the process reaches the plane NE of the charge, the plane has shifted to the position N_1E_1. The summation at the remote face is taken at the instant $(t - r_a/s)$ and at the nearer face at the instant $(t - r_c/s)$. The difference in time is $(r_a - r_c)/s$. In this time interval, all points on the nearer face move a distance $bd = V_n\,(r_a - r_c)/s$ in a direction parallel to the plane of the face, and move toward P a distance $bc = V_r\,(r_a - r_c)/s$ in a direction perpendicular to the face. In these expressions, V_n and V_r repesent the components of the velocity $\mathbf{V}$ in a direction normal to the radial line aP, and in a direction parallel to aP toward P, respectively. The former motion is a shear which does not change the volume of the space to be integrated over, while the latter increases the volume of space to be integrated over in the ratio of the distance ac to ab. That is,

$$\frac{\text{Volume of space, charge moving}}{\text{Volume of space, charge stationary}} \text{ or } \frac{(\text{Vol.})_m}{(\text{Vol.})_s} = \frac{ac}{ab} = \frac{ac}{ac - bc}$$

$$\frac{(\text{Vol.})_m}{(\text{Vol.})_s} = \frac{r_a - r_c}{r_a - r_c - (r_a - r_c)\dfrac{V_r}{s}} = \frac{1}{1 - \dfrac{V_r}{s}}. \qquad (604)$$

From this it follows that the retarded potentializing operation for the retarded scalar and vector potentials when carried out over a small charge, q, all parts of which are moving with the same velocity, yields the following expressions for the potentials at a distant point P:

$$E = \left[\frac{q}{4\pi p r\left(1 - \dfrac{V_r}{s}\right)}\right]_{(t-r/s)} \qquad (605)$$

$$\mathbf{A} = \left[\frac{\mu q \mathbf{V}}{4\pi r\left(1 - \dfrac{V_r}{s}\right)}\right]_{(t-r/s)} \qquad (606)$$

in which

r represents the distance from P to the center of the charge.

V_r represents the algebraic value of component of the velocity of the charge in the direction toward P.

[ ], the square brackets, indicate that, to obtain the values of E and $\mathbf{A}$ for the instant t, the values of $\mathbf{V}$, r, and V_r for the instant $t' = (t - r/s)$ must be used.

Since the rate at which the distance r is changing is related to the velocity by the equation,

$$\frac{\partial r}{\partial t'} \text{ (or } \dot{r}) = -V_r, \tag{607}$$

the above equations may also be written in the forms

$$E = \left[\frac{q}{4\pi p r\left(1 + \frac{\dot{r}}{s}\right)}\right]_{(t - r/s)} \tag{605a}$$

$$\mathbf{A} = \left[\frac{\mu q \mathbf{V}}{4\pi r\left(1 + \frac{\dot{r}}{s}\right)}\right]_{(t - r/s)}. \tag{606a}$$

359. The Calculation of F in the Field of a Small Moving Charge.—The retarded scalar and vector potentials at the point P due to the moving charge q are given by Eqs. (605) and (606).

The value and direction of the electric intensity F at P is given by

$$\mathbf{F} = -\text{grad } E - \frac{\partial \mathbf{A}}{\partial t}.$$

359a. To Obtain $-\frac{\partial \mathbf{A}}{\partial t}$.—Equation (606$a$) may be written

$$\mathbf{A} = \frac{\mu q}{4\pi}\left[\mathbf{V}\left(1 + \frac{\dot{r}}{s}\right)^{-1}\left\{(x - X)^2 + (y - Y)^2 + (z - Z)^2\right\}^{-\frac{1}{2}}\right]. \tag{606b}$$

Now in the operation of taking the derivative of $\mathbf{A}$ with respect to time, x, y, and z are constant and the coordinates of the moving charge X, Y, Z, and V are explicitly known, not as functions of the time t at which the influence of q reaches P, but as functions of a time t' which is related to t by the relation

$$t' = t - \frac{r}{s}. \tag{608}$$

Accordingly, the derivative of $\mathbf{A}$ with respect to t will be given by

$$-\frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{A}}{\partial t'}\frac{\partial t'}{\partial t}. \tag{609}$$

From Eqs. (608) and (607)

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{s}\frac{\partial r}{\partial t'}\frac{\partial t'}{\partial t} = 1 - \frac{\dot{r}}{s}\frac{\partial t'}{\partial t}$$

or

$$\frac{\partial t'}{\partial t} = \frac{1}{(1 + \dot{r}/s)}. \tag{610}$$

Taking the derivatives indicated in Eq. (609) we obtain

$$-\frac{\partial \mathbf{A}}{\partial t} = -\left[\frac{q}{4\pi p(1 + \dot{r}/s)^2}\left\{\frac{\mathbf{a}/s^2}{r} + \frac{\mathbf{V}/s}{1 + \dot{r}/s}\left(\frac{V_r/s}{r^2} - \frac{V^2/s^2}{r^2} + \frac{a_r/s^2}{r}\right)\right\}\right]_{(t - r/s)} \tag{611}$$

359b. To obtain $-$ grad E or $-\left\{\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right\}E$. Equation (605$a$) may be written

$$E = \frac{q}{4\pi p}\left[\left(1 + \frac{\dot{r}}{s}\right)^{-1}\left\{(x - X)^2 + (y - Y)^2 + (z - Z)^2\right\}^{-\frac{1}{2}}\right]_{(t - r/s)}. \tag{605b}$$

The operation of finding the value of $\frac{\partial E}{\partial x}$ for the point P at a given instant t is the operation of computing the values E_1 and

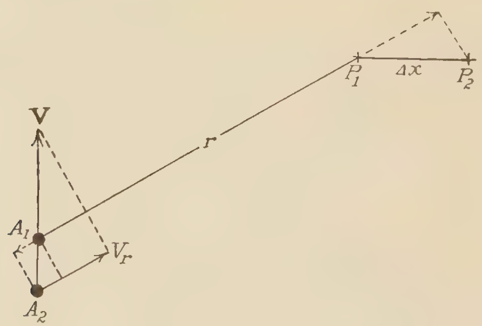

FIG. 302.—Shift in position of charge required by shift in P.

E_2 for two points P_1 and P_2 separated by a short distance Δx, and finding the value of the quotient $\frac{(E_2 - E_1)}{\Delta x}$. But in computing the values of E for P_1 and P_2, the coordinates X, Y, and Z of the charge must in each case be those corresponding to such instants $t_1' = (t - r_1/s)$ and $t_2' = (t - r_2/s)$ that the effects reach P_1 and P_2 at the same instant, t.

The relations between P_1, P_2, the two positions of the charge, A_1, and A_2, and the direction of the velocity are represented in Fig. 302. Suppose that at the instant t at which we want the value of the intensity at P_1, the influence is that from the charge when its center was located at A_1. Then since P_2 is $\left\{\begin{matrix}\text{farther from}\\ \text{nearer to}\end{matrix}\right\}$ A_1 than is P_1, the charge will have to be in some $\left\{\begin{matrix}\text{earlier}\\ \text{later}\end{matrix}\right\}$ position A_2, in order that its effect may reach P_2 at the same instant at which the effect from A_1 reaches P_1.

The distance A_2P_2 is algebraically greater than the distance A_1P_1 by the amount,

$$\frac{\Delta x(x - X)}{r} - V_r\Delta t'.$$

(NOTE: $\Delta t'$ is a negative quantity for Fig. 302.)

Now t_2' must come earlier than (or be less than) t_1' by the extra time required for the effect to travel this extra distance. Whence

$$t_1' - t_2' \{\text{or } t_1' - (t_1' + \Delta t')\} \text{ or } - \Delta t' = \frac{\dfrac{\Delta x(x - X)}{r} - V_r\Delta t'}{s}. \tag{612}$$

Solving

$$\frac{\Delta t'}{\Delta x} = -\frac{x - X}{sr}\frac{1}{1 - \dfrac{V_r}{s}} = -\frac{x - X}{sr}\frac{1}{1 + \dot{r}/s}$$

or

$$\frac{\partial t'}{\partial x} = -\frac{x - X}{sr}\frac{1}{1 + \dot{r}/s}. \tag{613a}$$

Likewise

$$\frac{\partial t'}{\partial y} = -\frac{y - Y}{sr}\frac{1}{1 + \dot{r}/s} \tag{613b}$$

and

$$\frac{\partial t'}{\partial z} = -\frac{z - Z}{sr}\frac{1}{1 + \dot{r}/s}. \tag{613c}$$

From the argument above, it follows that

$$-\frac{dE}{dx} = -\left[\frac{\partial E}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x}\right] \tag{614a}$$

and

$$-\frac{dE}{dy} = -\left[\frac{\partial E}{\partial y} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial y}\right] \tag{614b}$$

and

$$-\frac{dE}{dz} = -\left[\frac{\partial E}{\partial z} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial z}\right]. \tag{614c}$$

Taking the derivatives indicated above and combining, the following expression is obtained

$$-\mathbf{grad}\, E = \left[\frac{q}{4\pi p(1 + \dot{r}/s)^3}\left\{\mathbf{r}_1\left(\frac{1}{r^2} - \frac{V^2/s^2}{r^2} + \frac{a_r/s^2}{r}\right) \cdot \frac{(1 + \dot{r}/s)\mathbf{V}/s}{r^2}\right\}\right]_{(t-r/s)} \quad (615)$$

Combining Eqs. (611) and (615), the following expression is obtained for the electric intensity at the point P

$$\mathbf{F} = \left[\frac{q}{4\pi p}\left\{\frac{-\mathbf{a}_1 a/s^2}{r(1 + \dot{r}/s)^2} + \frac{\mathbf{r}_1 - \mathbf{V}_1 V/s}{(1 + \dot{r}/s)^3}\left(\frac{1}{r^2} - \frac{V^2/s^2}{r^2} + \frac{a_r/s^2}{r}\right)\right\}\right]_{(t-r/s)} \quad (616)$$

in which $\mathbf{a}_1$ represents a unit vector in the direction of the acceleration.
$\mathbf{r}_1$ represents a unit vector in the direction from the charge to the point P.
$\mathbf{V}_1$ represents a unit vector in the direction of the velocity.
a represents the acceleration.
a_r represents the component of the acceleration in the direction $\mathbf{r}_1$.
$\dot{r}$ represents the negative of the component of the velocity in the direction $\mathbf{r}_1$.

360. The Calculation of B.—An analysis similar to the above yields the following expression for the magnetic flux densities in the field of a small moving charge.

$$\mathbf{B} = \left[\frac{\mu q}{4\pi(1 + \dot{r}/s)^2}\left\{\frac{a/s}{r}(\mathbf{a}_1 \times \mathbf{r}_1) + \frac{V}{1 + \dot{r}/s}\left(\frac{1}{r^2} - \frac{V^2/s^2}{r^2} + \frac{a_r/s^2}{r}\right)(\mathbf{V}_1 \times \mathbf{r}_1)\right\}\right]_{(t-r/s)}, \quad (617)$$

in which $(\mathbf{a}_1 \times \mathbf{r}_1)$ represents a unit vector at right angles to the plane determined by $\mathbf{a}_1$ and $\mathbf{r}_1$ in that direction along the normal in which a right-hand screw would advance if turned as $\mathbf{a}_1$ must be turned to make it coincide with $\mathbf{r}_1$.

361. Application of Equations to Electric Circuits.—In electric circuits, the velocity of flow of electrons at economic current densities is estimated to be of the order of 30 centimeters per second or less. Consequently, in Eqs. (616) and (617) the value of $\dot{r}/s$ is 10^{-9} or less and the following simplifications apply with great precision to electric circuits.

$$\mathbf{F} = \frac{q}{4\pi p}\left[\frac{\mathrm{a}_r\mathbf{r}_1 - \mathrm{a}\mathbf{a}_1}{rs^2} + \frac{1}{r^2}\mathbf{r}_1\right]_{(t-r/s)} \tag{618}$$

$$\mathbf{F} = \frac{q}{4\pi p}\left[\frac{\mathrm{a}_n}{rs^2}\mathbf{n}_{ar} + \frac{1}{r^2}\mathbf{r}_1\right]_{(t-r/s)} \tag{619}$$

in which $\mathbf{n}_{ar}$ represents a unit vector normal to $\mathbf{r}_1$ and lying in the plane determined by $\mathbf{a}_1$ and $\mathbf{r}_1$, the angle between $\mathbf{n}_{a,r}$ and $\mathbf{a}_1$ being equal to or greater than 90°.

a_n represents the acceleration normal to $\mathbf{r}_1$.

The second term in each of these equations is the electrostatic intensity given by the inverse-square law.

$$\mathbf{B} = \left[\frac{\mu q \mathrm{a}}{4\pi sr}(\mathbf{a}_1 \times \mathbf{r}_1) + \frac{\mu q V}{4\pi r^2}(\mathbf{V}_1 \times \mathbf{r}_1)\right]_{(t-r/s)}. \tag{620}$$

To throw these equations into a form suitable for computing the field at distant points due to current in a short length dl of wire.

Let q_1 represent the quantity of moving electrons per unit length of wire, and let V represent the velocity of drift through the wire. Then the current in the wire is $i = q_1V$.

The total charge moving is: $q = q_1dl = \dfrac{idl}{V}$

Whence, in the above equations,

$$qV = idl.$$

$$q\mathrm{a} = dl\frac{di}{dt}.$$

Substituting these values in Eqs. (618) and (620), we obtain

$$d\mathbf{F} = \left[\frac{\mu dl}{4\pi r}\frac{di}{dt}(\mathbf{r}_1 \cos(l,r) - \mathbf{a}_1)\right]_{(t-r/s)} \tag{621}$$

or
$$d\mathbf{F} = \left[\frac{\mu dl}{4\pi r}\frac{di}{dt}\sin(l,r)\mathbf{n}_{ar}\right]_{(t-r/s)} \tag{621a}$$

and
$$d\mathbf{B} = \frac{\mu}{4\pi}\left[\frac{dl}{sr}\frac{di}{dt}(\mathbf{a}_1 \times \mathbf{r}_1) + \frac{idl}{r^2}(\mathbf{V}_1 \times \mathbf{r}_1)\right]_{(t-r/s)} \tag{622}$$

or
$$d\mathbf{B} = \frac{\mu}{4\pi}\left[\frac{dl\sin(l,r)}{sr}\frac{di}{dt} + \frac{idl\sin(l,r)}{r^2}\right]_{(t-r/s)}. \tag{622a}$$

The second term in Eq. (622) is the term from Ampere's formula for steady flow. Equation (622) may therefore be regarded as an extended form of Ampere's formula.

Let us consider the application of these equations to the radiating system shown in Fig. 303. It consists of two extended circular parallel plates or networks of wires connected by a straight wire AC of length l along the axis of the circles.

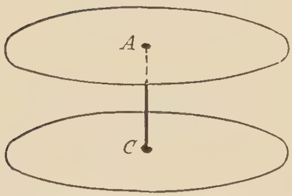

FIG. 303.—Radiating circuit.

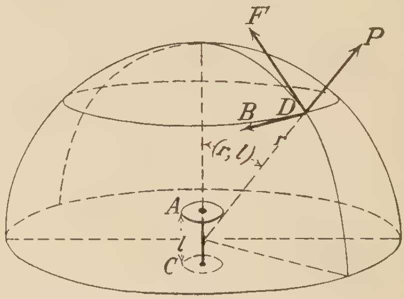

FIG. 304.—Directions of the vectors F, B and P in the field of the radiator.

An alternating-current generator (not shown) connected in the wire AC causes the current,

$$i = I_m \cos 2\pi ft,$$

to flow in the vertical wire AC.

Sufficient study will show that if the frequency of alternation is low enough so that all points of each plate are at substantially the same potential, then the only current contributing appreciably to the field at distant points is that in the vertical wire AC.

Consequently, at the distant point D in Fig. 304, the electric intensity is tangential to a meridian of longitude through D and has the value

$$F \text{ (volts per cm.)} = \frac{-\mu l I_m}{4\pi} \left[\frac{2\pi f}{r} \sin (r, l) \sin 2\pi ft \right]_{(t-r/s)} \qquad (623)$$

and the magnetic flux density is tangential to a circle of latitude through D and has the value

$$B \text{ (webers per sq. cm.)} = \frac{\mu I_m l}{4\pi} \sin (r, l) \left[\frac{-2\pi f}{sr} \sin 2\pi ft + \frac{1}{r^2} \cos 2\pi ft \right]_{(t-r/s)}. \qquad (624)$$

In Eq. (624) the amplitudes of the two terms become equal when

$$\frac{2\pi f}{sr} = \frac{1}{r^2}$$

or when

$$r = \frac{1}{2\pi}\frac{s}{f} = \frac{\lambda}{2\pi}, \tag{625}$$

in which λ is the wave length of the disturbance. At all points beyond $\lambda/(2\pi)$, the second term is the smaller since it decreases as the square of the distance.

Now by Poynting's theorem (Sec. 363) the direction of flow of energy at a point in a field is normal to the plane determined by the F and the B vectors at the point and the rate of flow per unit area is

$$P \text{ (watts per sq. cm.)} = \frac{FB}{\mu} \sin (F,B). \tag{631a}$$

Since F and B are at right angles at the point D, the expression for the power is

$$P \text{ (watts per sq. cm.)} = \frac{\mu I_m^2 l^2}{16\pi^2}\left[\frac{4\pi^2 f^2}{sr^2}\sin^2(r, l)\sin^2(2\pi ft) - \frac{2\pi f}{r^3}\sin^2(r,l)\sin(2\pi ft)\cos(2\pi ft)\right]_{(t-r/s)} \tag{626}$$

The second term is alternately + and − in value and represents a pulsation of power back and forth past the point.

The first term is always positive in value and represents the loss of power by radiation. By integrating for the power, P_s, radiated over the surface of the entire sphere, the following expression is obtained.

$$P_s \text{ (watts over sphere)} = (80\pi^2 l^2 f^2 s^{-2})I_m^2 \sin^2(2\pi ft). \tag{627}$$

This is seen to be independent of the radius of the sphere, and consequently represents the rate at which energy is lost to the system by radiation into space.

Now if this same current were flowing in a conductor having a resistance R_r of the value

$$R_r(\text{ohms}) = 80\pi^2 l^2 f^2 s^{-2}, \tag{628}$$

the loss in the resistance would be

$$P(\text{watts}) = R_r I_m^2 \sin^2(2\pi ft) = (80\pi^2 l^2 f^2 s^{-2})I_m^2 \sin^2(2\pi ft).$$

Consequently, we may call R_r the effective resistance of radiation of the system, or the **radiation resistance** of the system.

362. Motional Magnetic Intensity [DEDUCTION].—When **a** body moves relatively to an electrostatic field, **a magnetic intensity is induced** in the moving body. The induced magnetic intensity at any point is normal to the plane determined by the two vectors representing respectively the velocity of the body at the point and the electric displacement. The induced intensity is in that direction along the normal in which a right-hand screw would advance if rotated in the direction in which the displacement vector must be turned to bring it into parallelism with the velocity vector. The magnitude of the induced intensity is equal to the product of the displacement **D** times the component of the velocity normal to the electric displacement. In vector notation

$$\mathbf{H} = \mathbf{D} \times \mathbf{V} \text{ (vector product).} \tag{629}$$

363. Flow of Energy.—By postulating that the energy stored in any portion of space is given by the expression

$$W\text{(joules)} = \frac{1}{2\mu}\int^{\text{vol.}} B^2 dv + \frac{p}{2}\int^{\text{vol.}} F^2 dv$$

and by taking the derivative of W with respect to time and converting the volume integration into an integration over the surface bounding the volume, the following result is obtained

$$\frac{dW}{dt}\text{(watts)} = -\int^{\text{vol.}} \mathbf{F}\cdot(\rho\mathbf{V})dv - \frac{1}{\mu}\int^{\text{Os}} (\mathbf{F}\times\mathbf{B}) \cos\{(\mathbf{F}\times\mathbf{B}), \mathbf{n}\} da \tag{630}$$

The first term represents the rate at which energy is being dissipated in the volume by conduction currents. The second term represents the rate at which energy is leaving the volume by way of the surface. This result was obtained by Poynting and is known as Poynting's theorem. The theorem may be stated as follows:

363a. Poynting's Theorem (DEDUCTION).—The energy transmitted by an electric circuit flows or streams through the dielectric surrounding the conductors. The direction of flow at any point in the dielectric is perpendicular to the plane determined by the vectors representing the electric intensity and the magnetic flux density at the point. The flow is in that direction along the perpendicular in which a right-hand screw would advance if

rotated in the direction in which the **F** vector must be turned to bring it into parallelism with the **B** vector.

The rate P at which energy streams across unit area at the point is given by the expression (Poynting's theorem)

$$P = \frac{FB \sin (F,B)}{\mu} \tag{631a}$$

in which, (F,B) represents the angle between the **F** and **B** vectors at the point.

Or, in vector notation,

$$\mathbf{P} = \frac{1}{\mu}(\mathbf{F} \times \mathbf{B}) \text{ (vector product)} \tag{631}$$

364. Electromotive Intensities Due to Elements of Flux.—It has been shown that the equations

$$\text{curl } \mathbf{B} = \mu \mathbf{J}$$

and

$$dB = \frac{\mu I dl \sin (l, r)}{4\pi r^2}$$

are alternative ways of expressing the same relation.

By analogy we may infer that, since

$$\text{curl } \mathbf{F} = -\frac{d\mathbf{B}}{dt},$$

then the electric intensity dF due to any elementary portion of a tube of magnetic flux will be given by an expression analogous to Ampere's formula, namely,

$$dF = \frac{-\frac{d\Phi}{dt} dl \sin (l, r)}{4\pi r^2}. \tag{632}$$

365. Exercises.

1. Two very extended circular copper plates having the radii r_2 are mounted a short distance apart in parallel planes as in Fig. 305. A straight copper wire AB of radius r_1 connects the plates at the center and a cylindrical tube connects them at the edges.

(*a*) Suppose that a current flows as indicated. Derive the expression for the magnetic flux density at any point between the plates at a distance x from the axis.

(*b*) Derive an expression for the inductance of the circuit, neglecting the flux within the metal.

2. Suppose that the tubular return path which connected the edges of the circular plates in exercise 1 is removed. Any current which then flows in the wire AB is the result of transfer of charge from plate to plate and

therefore is accompanied by a displacement current in the region between the plates. Assume that the current in AB is not at the natural frequency of the system but is a forced current of such low frequency that at any given instant all points of the upper plate are substantially at the same potential.

(*a*) Derive the expression for the flux densities at any point between the plates.

(*b*) Neglecting the flux within the metal, derive an expression for the inductance of the system. If $r_2 > 100r_1$, what simple form does the expres-

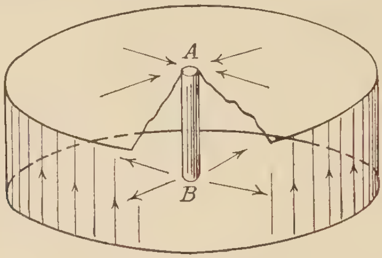

FIG. 305.

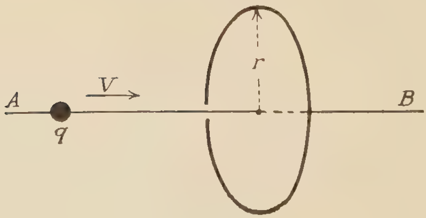

FIG. 306.

sion for the inductance assume? From this expression, what is the radius of the equivalent return path of the displacement current?

3. A positive charge q passes with a uniform velocity V from A to B along the axis of the circle of radius r shown in Fig. 306. Plot two curves, the first showing the flux from left to right over the area bounded by the circle for different positions of the charge on the axis, and the second showing the magnetic intensity around the circle for the different positions of the charge.

4. Let a wire of infinite length extend along the axis AB of the circle of Fig. 306. Suppose that this wire contains q coulombs of free electrons per centimeter of length and that this atmosphere is moving through the wire with the velocity V.

From the notion of the displacement current compute the magnetic intensity at the circle which would be due to the moving charge in a length dx of the wire. Does this agree with the value which would be given by Ampere's formula?

APPENDIX A

SYMBOLS

The symbols used at all systematically in this text are listed below in alphabetical order. Opposite each symbol is the name of the quantity which is symbolized, followed by the name, *in italics*, of the Practical Unit in which the value of the quantity is expressed. This is followed by the number of the section in which the symbol is defined or is first used.

1. *English letters*	*Numbers refer to sections*
A, A	magnetic vector potential, *weber-cm.*, 267.
A, A	retarded vector potential, *weber-cm.*, 353.
a, a	acceleration, *cm. per sec. per sec.*, 12.
a	area, *sq. cm.*, 54.
B, B	magnetic flux density, *weber per sq. cm.*, 229, 234.
C	capacitance of condenser, *farad*, 72.
$C_{1,2}$	mutual capacitance of condensers, *farad*, 111.
D, D	electrostatic flux density, *coulomb per sq. cm.*, 101.
E, e	potential at a point, *volt*, 45; potential increase, 49; potential difference, magnitude only, 49*c*.
E, e	value of electromotive force in specified direction along a path, *volt*, 147*b*.
F, F	electric intensity at a point, *volt per cm.*, 39, 41.
F_d	electric intensity at a point within a dielectric, 98.
F_m	electromotive intensity, *volt per cm.*, 148*a*.
$\mathfrak{F}$	value of magnetomotive force around a closed loop, *ampere-turn*, 246.
f, f	force, *dyne-seven*, 12.
f	frequency of alternation.
G	conductance of a conductor, *mho*, 170.
H, H	magnetic intensity, *ampere-turn per cm.*, 246.
I, i	value of electric current in a direction usually specified by an arrow, *ampere*, 118*a*.
I_o, I_u	initial and ultimate values of current after switching, 299.
J, J	current density, *ampere per sq. cm.*, 120*a*.
J	intensity of magnetization, *weber per sq. cm.*, 318.
K	magnetic susceptibility of a material, 320.
K, k	miscellaneous constant coefficients, 39 (footnote).
k	dielectric constant of a medium, 29; defined, 37.
k_r	relative dielectric constant of a medium, 37.
l	distance, *cm.*, 46.

L	self-inductance of a circuit, *henry*, 293
m	mass, *gram-seven*, 12.
M	electric moment of doublet, 112; magnetic moment of coil, 232; moment of inertia, 136.
M	mutual inductance between two circuits, *henry*, 292.
m	value or strength of magnetic pole, *weber*, 265.
N	number of turns.
$\mathbf{n}_1$	unit vector normal to a small surface.
P	hydrostatic pressure in dielectric, *dyne-seven per sq. cm.*, 106*g*.
P, p	electric power, *watt*, 150.
$\mathbf{P}$, P	Poynting's power vector, *watt per sq. cm.*
$\mathcal{P}$	permeance of magnetic circuit, *weber per ampere-turn*, 312.
p	permittivity of a medium, 38.
p_o	permittivity of free space, 38.
p_r	relative permittivity of a material, 38*b*.
Q, q	quantity of electric charge, *coulomb*, 29.
Q_t	quantity of charge used as a test charge, 39.
R	resistance of a conductor, *ohm*, 170.
R, r	distance, *cm.*, 29.
$\mathcal{R}$	reluctance of a magnetic circuit, *ampere-turn per weber*, 313.
s	velocity of propagation, *cm. per sec.*
s	elastivity of a medium, *daraf-cm.*, 104.
S	elastance of condenser, *daraf*, 72.
S_m, $S_{1,2}$	mutual elastances of condensers, *daraf*, 109.
T, t	temperature above a chosen zero, *degree Centigrade*, 151.
t	values of time measured from some chosen zero, *second*, 128.
U	magnetic potential at a point, *ampere-turn*, 262.
$\mathbf{V}$, V	velocity, *cm. per sec.*, 234.
$\mathbf{V}$	any vector quantity, 47.
v	volume, *cu. cm.*, 54*a*.
W, w	work or energy, *joule*, 12, 46.

2. *Greek letters*

α	(Alpha)	resistance-temperature coefficient of conductors, 175.
γ	(Gamma)	conductivity of conducting materials, *mho,cm.*, 177.
η	(Eta)	hysteresis coefficient of a material, 323.
θ	(Theta)	angular deflection of instrument, 69.
Λ	(Lambda)	magnetic flux-linkage, *weber-turn* 277.
μ	(Mu)	permeability of medium, (*weber per sq. cm.*) *per* (*ampere-turn per cm.*), 240, 246.
μ_o		permeability of free space.
μ_r		relative permeability of a medium, a ratio, 319.
ν	(Nu)	reluctivity of medium, (*ampere-turn per cm.*) *per* (*weber per sq. cm.*), 313.
ν_r		relative reluctivity of a material, 321.
ρ	(Rho)	volume density of charge, *coulomb per cu. cm.*, 54*a*.
ρ		resistivity of conducting material *ohm-cm.*, 177.

σ	(Sigma)	surface density of charge, *coulomb per sq. cm.*, 54*a*.
σ_c		quantity of concealed charge per unit area, 98.
σ_o		quantity of obvious charge per unit area, 98.
σ_n		net quantity of charge per sq. cm. on boundary surface of dielectrics, 98.
τ	(Tau)	torque, *dyne-seven-cm.*, 70.
Φ	(Phi)	magnetic flux across a surface, *weber*, 252.
ϕ		thermoelectric power of a metal, 151*g*.
ϕ		e.m.f. of electron affinity, 157.
Ψ	(Psi)	electrostatic flux, *coulomb*, 102.
ω	(Omega)	solid angle, 85.
ω		angular frequency $= 2\pi f$, radians per sec.

3. *Vector notation*

div **F**	divergence of a vector-point function, 87.
grad E	gradient of a scaler-point function, 52*a*.
pot q	potential of a scaler-point function, 54*a*.
curl **H**	curl of a vector-point function, 250.
V	bold-face letters designate vector quantities, the same letters in italics the absolute value of the quantity without reference to direction.

4. *Mathematical symbols*

(F,l)	the angle between the vector F and the chosen direction along a path l, 46.
(F,n)	the angle between the vector F and the normal n erected to a given surface in a given sense, 84.
$\int^{\bigcirc l} f\, dl$	line-integral around the *closed* loop designated by the letter following the circle, 308.
$\int^{\bigcirc s} f\, da$	surface-integral over the closed surface designated by the letter following the circle.
log	natural logarithm = 2.301 $\log_{10}$.
∂	partial differential sign.
$\dot{r}$	superdot indicates the time derivative $\dot{r} = dr/dt$.
i, **j**, **k**	unit vectors, parallel to the x, y, and z axes.
$\mathbf{V}_1$	a vector of unit length parallel to **V**.
$_{1\ 2\ 3}$	signifying the x, y, and z components of a vector.
∇^2	read "nabla square," the operator $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\}$.
$[\ \ ]_{(t-r/s)}$	signifying that the value of the quantity in the brackets is to be taken at the time $(t - r/s)$
$\mathbf{a} \cdot \mathbf{r}$	scalar product of two vector quantities; $= ar \cos (\mathrm{a},r)$.
$\mathbf{a} \times \mathbf{r}$	vector product of two vector quantities; a vector at right angles to plane determined by **a** and **r** and having a value of ar sin (a,r).

APPENDIX B

CONVERSION FACTORS

Length

1 inch	2.54 cm.
1 foot	30.48 cm.
1 mile	1.60935 km.

Area

1 sq. inch	6.452 sq. cm.

Volume

1 cu. inch	16.387 cu. cm.
1 liter	1000 cu. cm.
1 gal. (U. S. liquid)	3.785 liters

Mass

1 pound	453.6 grams
1 pound	4.536×10^{-5} gram-sevens

Force

1 gram	980.66 dynes
1 gram	9.8066×10^{-5} dyne-sevens
1 pound	0.04448 dyne-sevens

Pressure

1 bar	1 dyne per sq. cm.
1 standard atmosphere (76 cm. at 0°C., $g = 980.66$)	1.0132 megabars

Work or Energy

1 joule	1 dyne-seven, cm.
1 foot-pound	1.356 joules
1 (minor) calorie (gram, degree C.)	4.184 joules
1 B.t.u. (pound, degree F.)	1054.9 joules
1 kw.-hr	3.6×10^6 joules
1 watt-sec	1 joule

Power

1 watt	1 joule per sec.
1 h.p	746 watts
1 boiler h.p	9804 watts

Miscellaneous

1 degree of arc	0.01745 radian

APPENDIX C

THE DEFINING FORMULA VERSUS THE DERIVED FORMULAS FOR A PHYSICAL QUANTITY

The formulas for computing the value of a physical quantity may be classified as follows:

1. The defining formula
2. Derived formulas:
   a. Of general application.
   b. For specific cases.

The defining formula for a quantity is the formula which is obtained by:

a. Representing the quantity and the **antecedent** quantities in terms of which it is defined by means of symbols.

b. Expressing, in the form of an equation, the mathematical operations which must be performed to determine the numerical value of the newly defined quantity from the measured or computed values of the antecedent quantities.

The **defining formula** for a quantity should be clearly distinguished from the **derived formulas** for computing its value by means of quantities other than those appearing directly in the definition (and in the corresponding defining formula), or for computing its value in specific physical cases.

Because the quantities to be defined present many aspects, it is possible to frame many definitions for their measurements. For example, an electric current exhibits a heating effect, an electrochemical effect, a mechanical force effect, and a redistribution of electric charge effect. It is therefore possible to define the unit by which current is to be measured in terms of any one of these effects. But it should be recognized that if current is defined by one effect in one field of investigation and by another effect in another field, there is always the possibility that these two independent definitions will lead to irreconcileable conclusions in some part of the superstructure.

If it is desired to insure that our account of phenomena shall be a connected account, free of inconsistencies, each measurable quantity must be defined in one, and only one, way. That is, one and only one formula must be recognized as the **defining formula.** All other formulas, for computing the value of the quantity or for expressing its relations to other quantities, must be recognized as **derived formulas.**

By way of illustration, the defining formula and a few of the scores of derived formulas for the **electric intensity at a point** are given on p. 632.

1. The defining formula:

$$F \text{ (volts per cm.)} = \frac{f}{Q_t} \frac{\text{(dyne-sevens)}}{\text{(coulombs)}}.$$

2*a*. A derived formula of general application:

$$F_x \text{ (volts per cm.)} = -\frac{dE}{dx} \frac{\text{(volts)}}{\text{(cm.)}}.$$

2*b*. Derived formulas for specific cases:

(1) In the field of a point charge:

$$F \text{ (volts per cm.)} = \frac{Q}{4\pi p x^2} \frac{\text{(coulombs)}}{\text{(cm.)}}.$$

(2) In the field between oppositely charged parallel plates:

$$F \text{ (volts per cm.)} = \frac{E}{l} \frac{\text{(volts)}}{\text{(cm.)}}.$$

$$F \text{ (volts per cm.)} = \frac{\sigma}{p} \text{ (coulombs per sq. cm.)}.$$

(3) In the field of a charge Q which is uniformly distributed over a spherical surface:

Inside the sphere, $F = 0$.

Outside the sphere, $F = \dfrac{Q}{4\pi p x^2}$.

APPENDIX D

THE METHOD OF SPECIFYING THE UNITS IN FORMULAS

All formulas should contain a specification on each side of the equation of the names of the units by which the quantities which appear on that side are measured. The plural forms of the names should be used for all units except for centimeter, gram, and second, which are to be written: cm., gm., and sec. Commas (and not hyphens) are to be used to separate the names of the units.

The number of times the unit appears as a factor in a formula need not be designated. If the right member of the formula is in fractional form, the names of the units of the quantities appearing in the numerator should be written after the numerator and in line with it; likewise, the names of the units of the quantities in the denominator should be written after the denominator, except that any unit which appears in both the numerator and the denominator should be written only after that member of the fraction which contains it to the greater power. The (fraction) bar should not be inserted between the units of the numerator and of the denominator.

Examples

$$w \quad \text{(joules)} \quad = fl \cos (f,l) \quad \text{(dyne-sevens, cm.)}$$

$$f \quad \text{(dyne-sevens)} \quad = \frac{Q_1 Q_2}{4\pi p l^2} \quad \begin{matrix}\text{(coulombs)}\\ \text{(cm.)}\end{matrix}$$

$$R \quad \text{(ohms)} \quad = \frac{E}{I} \quad \begin{matrix}\text{(volts)}\\ \text{(amperes)}\end{matrix}$$

$$P \quad \text{(watts)} \quad = I^2R \quad \text{(amperes, ohms)}$$

$$dH \quad \text{(amp.-turns per cm.)} \quad = \frac{I\,(dl) \sin (r, l)}{4\pi r^2} \quad \begin{matrix}\text{(amperes)}\\ \text{(cm.)}\end{matrix}$$

APPENDIX E

THE STUDY OF DEFINITIONS

To insure a thorough grasp of the fundamental relations between physical quantities, the student should start from the definition of the unit of each newly defined quantity and should trace back the measuring operations necessary to set up or produce this unit by the most direct group of intermediate measurements to the fundamental experiences from which all physical science has grown: namely, to experiments involving measurements of relative position, time, and mass. He should trace the connections back to the fundamental experiments not merely once for each newly defined quantity, but should make it a practice to repeat this from time to time as his knowledge of the field broadens. To this end he should compile his own tabular summary of the important experimentally determined relations, definitions, etc. This compilation if worked out along the lines indicated in the table on page 636 will be found to be very helpful.

COMPILATION OF THE EXPERIMENTALLY DETERMINED RELATIONS, DEFINING EQUATIONS, AND DEDUCED RELATIONS

In the sequence in which the relations have been presented, the quantities defined, and the system of units has been built up. The units are those of the practical system

Name of the physical quantity, or of the relation	Statement of the exp. det. relation	Definition of the physical quantity	Smybol	Defining equation	Names of the unit (The name in general use is asterisked thus *): The physically descriptive name	Coined and awarded by decree	Derived from secondary (mathematical) relations	Deduced relations	A few of the derived formulas by which the value of the quantity may be computed in specific cases
Length	......	Fundamental independent units	l	....	....	cm.			
Mass	......		m	....	....	gram-seven			
Time	......		t	....	....	second			
Velocity	......	....	V	$=\frac{l}{t}$	cm. per sec.				
Acceleration	......	....	a	$=\frac{dV}{dt}$	cm. per sec. per sec.				
Force	......	....	f	$= ma$	....	dyne-seven			
Work	......	....	W	$=\int f \cos (f, l) dl$	....	joule*			
Power	......	....	P	$=\frac{W}{t}$	joule per sec.	watt*			
Electricity	......	Its attribute is the display of extraordinary forces							
	Two kinds only	+, as on glass −, as on resin							
Two charges are equal.	......	if, when placed in a hollow conductor, they cause identical outside forces							

Principle of the equal development of the two kinds..........	$\Sigma q = 0$								
Law of force between charges at rest............	$f \propto \frac{Q_1Q_2}{l^2}$								
Unit of electricity is defined by assigning a value to the permittivity p_0 of free space.	......	f(dyne-sevens) $= \frac{Q_1Q_2}{4\pi p_0 l^2}$ $p_0 = 8.85 \times 10^{-14}$	Q	$= \sqrt{4\pi p_0 l^2}$	....	coulomb*			
Electric intensity at a point P.	......	is a *vector* quantity, direction and magnitude is that of force which would be everted on unit positive charge at P by etc.	F	$= \frac{f}{Q_t}$	dyne-seven per coulomb	....	volt per cm.*	....	In field of: Pt. charge $F = \frac{Q}{4\pi pl^2}$; Par'l plates $F = \frac{\sigma}{p}$; Charge on a sphere: inside $F = 0$, outside $F = \frac{Q}{4\pi pl^2}$

APPENDIX F

UNITS, STANDARDS, AND ABSOLUTE MEASUREMENTS

The operation of measuring a physical quantity consists in having for reference a definite or known quantity of the same kind, called the **unit,** and in determining how many of these units will constitute the quantity being measured.

A **unit** of any physical quantity is a definite amount of that physical quantity, specified in some particular way. It is a reference quantity employed in measuring quantities of the same kind. Units are of two kinds, **independent** and **derived** (see below).

A **standard** is either the concrete representation or the experimental realization of a unit.

An **independent unit** of a physical quantity is the amount of that physical quantity in an **arbitrarily chosen standard.** The units of length, mass, and time are independent units.

A **derived unit** of a physical quantity is an amount of that quantity **specified in terms of related quantities.** The units of area and of velocity are examples of derived units. The former is defined to be the area of a square described upon the unit of length, and the latter is defined to be that velocity in which unit distance is passed over in unit time.

Classification of Standards.—In any system of units, each independent unit is necessarily represented by a **standard,** since it has been defined in terms of a standard. While derived units are not defined in terms of a standard, it is expedient (for the reasons stated below) to have standards representing many of them. "In fact, many of the units in practical use are each represented by a multitude of standards. Certain of the standards representing the unit of a given physical quantity are copies of certain other standards, so that they may be graded into **primary, secondary,** and **working** standards. The **primary** standard is the particular standard which is taken to represent the unit; it is maintained in general either at the International Bureau of Weights and Measures, at Sevres, France, or at one or more of the national standardizing laboratories. In the case of an **independent unit** the primary standard is that standard in terms of which the unit is defined; in the case of a **derived unit** it is simply a standard closely representing the unit and accepted for practical and legal purposes, its value having been fixed by certain measuring processes. Secondary or **reference** standards are copies (not necessarily duplicates) of a primary standard, used in the work of a standardizing laboratory. They should be occasionally tested and evaluated in terms of the primary standard. The third class, **working** standards, are standardized in terms of the second-

ary standards and are used in everyday measurements. In short, any unit may be represented by a certain definite standard, which is known as the primary standard of that unit; it fixes the values of secondary standards, which in turn are used for the standardization of working standards."

Fundamental Units and Unified Systems of Units.—Any **unified system of units** is based upon the smallest possible number of independent units. These independent units are called the **fundamental units** of that system. The units of all other physical quantities are defined in the simplest possible manner, either directly in terms of these fundamental units, or in terms of previously defined derived units, which, in turn, go back to the fundamental units of the system.

Absolute Measurements.—The value of a quantity which is to be expressed in terms of a derived unit can be determined by **absolute measurements.** By absolute measurements we mean measurements of a physical quantity made by a method which requires the measurement of no quantities other than the fundamental quantities of the system—mass, length, and time. Therefore it is not necessary to have standards to represent the derived units. In practice, however, it is found to be extremely advantageous to have standards or concrete representatives of the derived electrical units for the following reasons:

1. Absolute measurements are costly to make, because they require, in most cases, elaborate apparatus and the expenditure of much time.

2. The accuracy of absolute methods of measurement is limited in comparison with the accuracy obtainable in directly comparing two quantities of the same kind.

International Electrical Units.—All electrical units which have been defined in this text have been derived through the unit quantity of electricity. The unit quantity of electricity has been defined by the equation,

$$f(\text{dyne-sevens}) = \frac{Q_1Q_2}{4\pi pr^2}.$$

The value assigned to the permittivity p in this defining equation is

$$p = \frac{10^9}{4\pi s^2} = 8.8527 \times 10^{-14},$$

in which s, the velocity of light, equals $(2.9982 \pm 0.0003)\ 10^{10}$. This system of units may be called the Electrostatically Derived Practical System.

Because of the difficulty in precisely standardizing or calibrating instruments in terms of the fundamental units by **absolute** methods, the International Electrical Congress at Paris in 1881 recommended that an international technical commission be charged with the responsibility of formulating, from the results of absolute measurements, specifications for standards to represent certain units of the practical system. This commission drew up specifications for a standard of electric current, a standard of electrical resistance, and a standard of electromotive force. These specifications received the consideration and approval of subsequent Inter-

national Electrical Congresses. These electrical standards are known as the **International Standards,** and the units derived from them are known as the **International Units.** By the legislative actions of the various governments, these International Standards have been made the legal standards of all the civilized governments of the world.

While the International Standards were intended to represent certain of the derived units of the Electrostatically Derived Practical System,* they fail to do this precisely. Subsequent absolute measurements carried on by the National Standardizing Bureaus indicate that the greatest discrepancy is about 0.05 of 1 per cent. For all industrial purposes, this discrepancy is negligibly small. For scientific purposes, however, it is necessary to bear in mind the distinction, as sketched above, between the International Units and the units of the Electrostatically Derived Practical System. All instruments are calibrated in terms of International Units, and the electrical constants of materials are expressed in terms of these units.

The International Ampere and the International Ohm are defined in Secs. 130*b* and 173*a*, respectively.

The most recent determinations indicate that the International Ampere is 0.009 of 1 per cent smaller than the ampere of the Electrostatically Derived Practical System, and that 1 international ohm = 1.00052 ± 0.00004 ohm of the Practical System.

* As a matter of fact, the international standards were intended to represent certain specified multiples of the Electromagnetically Derived Units. But since these units are either multiples or submultiples of the Electrostatically Derived Units by the first or the second power of the velocity of light, we may be permitted to say that the international standards were also intended to represent the Electrostatically Derived Practical Units. For the history of the Electrical units see *Electrical Units and Standards,* Bulletin 60, U. S. Bureau of Standards, 1916.

APPENDIX G

CIRCUIT ANALOGIES

Analogies between conducting, magnetic, and dielectric circuits, and a metal bar in tension:

For the conducting circuit, read line 1.
For the magnetic circuit, read line 2.
For the dielectric circuit, read line 3.
For the metal bar in tension, read line 4.

1. An electromotive force E (in volts)
2. A magnetomotive force $\mathfrak{F}$ (in ampere-turns)
3. An electromotive force E (in volts)
4. A tension T (in dynes)

is accompanied by—

1. a conduction current I (in amperes)
2. a magnetic flux Φ (in webers)
3. an electrostatic flux Ψ (in coulombs)
4. an elongation E (in centimeters)

and the energy—

1. dissipated is at the rate of EI joules per sec.
2. stored in the magnetic circuit is $\frac{1}{2}\, \mathfrak{F}\Phi$ joules.
3. stored in the dielectric circuit is $\frac{1}{2}\, E\Psi$ joules.
4. stored in the metal bar is $\frac{1}{2}\, TE$ ergs.

The proportionality constants are called—

1. resistance $R = E/I$ ohms
2. reluctance $\mathcal{R} = \mathfrak{F}/\Phi$ ampere-turns per weber*
3. elastance $S = E/\Psi$ darafs
4. no name $= T/E$ dynes per cm.

or their reciprocals are called—

1. conductance $G = I/E$ mhos.
2. permeance $\mathcal{P} = \Phi/\mathfrak{F}$ webers per amp.-turn.*

* In the case of a ferromagnetic material, the ratio of the magnetic flux density to the magnetic intensity is not a constant. Hence in such materials, only for ranges in flux density between narrow limits may the reluctance, reluctivity, permeance, and permeability be treated as approximately constant.

3. permittance $C = \Psi/E$ farads.
4. no name $= E/T$ centimeters per dyne.

When considering a centimeter cube of the material,

1. an electric intensity F (in volts per cm.)
2. a magnetic intensity H (in amp.-turns per cm.)
3. an electric intensity F (in volts per cm.)
4. a stress e (in dynes per sq. cm.)

is accompanied by

1. a current density J (in amperes per sq. cm.)
2. a magnetic flux density B (in webers per sq. cm.)
3. an electrostatic flux density D (in coulombs per sq. cm.)
4. a strain t (in centimeters per cm.)

and the energy—

1. dissipated per cu. cm. is at the rate of FJ joules per sec.
2. stored per cu. cm. is $\frac{1}{2}\, HB$ joules.
3. stored per cu. cm. is $\frac{1}{2}\, FD$ joules.
4. stored per cu. cm. is $\frac{1}{2}\, et$ ergs.

The proportionality constants are called—

1. resistivity $\rho = F/J$ ohm,cm.
2. reluctivity* $\nu = H/B$ amp.-turns per cm. per weber per sq. cm.
3. elastivity $s = F/D$ volts per cm. per coulomb per sq. cm.
4. modulus of elasticity $M = e/t$ dynes per sq. cm. per cm. per cm.

or their reciprocals are called—

1. conductivity $\gamma = J/F$ mho,cm..
2. permeability* $\mu = B/H$ webers per sq. cm. per amp.-turn per cm.
3. permittivity $p = D/F$ coulombs per sq. cm. per volt per cm.
4. no name $= t/e$ cm. per cm. per dyne per sq. cm.

Breakdown of the material—

1 and 2. There are no phenomena in the conducting and magnetic circuits analogous to the failure of the dielectric circuit and of the metal bar.

3. The electric intensity which causes failure of the material is called the Dielectric Strength, and the corresponding flux density is called the Disruptive Flux Density.

4. The stress which causes failure of the material is called the Ultimate Tensile Strength, and the corresponding strain is called the Disruptive Strain.

* In the case of a ferromagnetic material, the ratio of the magnetic flux density to the magnetic intensity is not a constant. Hence in such materials, only for ranges in flux density between narrow limits may the reluctance, reluctivity, permeance, and permeability be treated as approximately constant.

CONSTANTS OF CYLINDERS OF MATERIAL OF LENGTH l AND CROSS-SECTIONAL AREA a

1. Resistance = resistivity $\times \frac{l}{a}$ $\quad R = \rho \frac{l}{a}$
2. Reluctance = reluctivity $\times \frac{l}{a}$ $\quad \mathcal{R} = \nu \frac{l}{a}$
3. Elastance = elastivity $\times \frac{l}{a}$ $\quad S = s \frac{l}{a}$

1. Conductance = conductivity $\times \frac{a}{l}$ $\quad G = \gamma \frac{a}{l}$
2. Permeance = permeability $\times \frac{a}{l}$ $\quad \mathcal{P} = \mu \frac{a}{l}$
3. Permittance = permittivity $\times \frac{a}{l}$ $\quad C = p \frac{a}{l}$

APPENDIX H

CHRONOLOGY OF EARLY ELECTRICAL DISCOVERIES AND A FEW CONTEMPORARY EVENTS

B.C. 600	Thales of Greece records the attraction of light bodies to amber which has been rubbed.
425 and later	Euripides and other Greek and Roman writers record the attraction of iron to lodestone.
A.D. 1000	Compass in use by Europeans in navigation, possibly introduced from China. Pointing of compass attributed to the influence of the pole star.
1268	Treatises by Roger Bacon of Oxford, with emphasis on experimental methods.
1269	Letter of Peter Peregrinus of Picardy giving an acute study of magnetic properties of lodestone.
1454	Invention of printing by Gutenberg in Germany.
1530	Copernicus, of Poland, conceives that the earth revolves around the sun.
1570	Gilbert of England observes that many bodies other than amber will attract light bodies when rubbed.
1581	Galileo of Pisa observes equal periods of vibration of pendulum, derives laws of falling bodies in 1590, and constructs telescopes in 1609.
1587–08	Shakespeare produces his plays.
1600	Gilbert's *De Magnete* published. Gilbert conceives that the earth itself possesses the properties of a magnet.
1620	Francis Bacon's *Novum Organum* published.
1640	von Guericke of Magdeberg constructs a sulphur sphere frictional machine, and is the first to record electrical repulsion.
1676	Independent invention of the method of the differential calculus by Newton in England and Leibnitz of Leipzig.
1686–87	Newton's *Principia* appears.
1729	Stephen Gray of England evolves the conception of conductors and non-conductors of electricity, which supplants Gilbert's fortuitous classification of "electrics and non-electrics."
1733	Du Fay of Paris discovers that there are two and only two kinds of electricity, which he calls *vitreous* and resinous. Announces law "like charges repel and unlike attract."
1745	von Kleist and Musschenbroeck independently discover the principle of the Leyden jar.

1746	Franklin advances the single-fluid theory of electricity and proposes the designations + and −.
1750	Michell's *Treatise on Artificial Magnets* contains inverse-square law of force between poles.
1752	Experiments by Franklin and many others on discharge from pointed rods during thunder storms.
1752	Franklin, by kite experiment, identifies atmospheric with frictional electricity.
1753	Canton directs attention to, and elucidates phenomena of, electrostatic induction.
1759	Symmer advances the two-fluid theory of electricity.
1766	Priestley infers the inverse-square law for the force between charges.
1767	Lane devises his discharging-jar electrometer.
1767	*Priestley's History of Electricity.*
1772	Henley devises his electrometer.
1774	Priestley discovers (dephlogisticated air) oxygen.
1775	Volta of Padua invents the electrophorus.
1778	Volta's *Dissertation on the Capacity of a Conductor.*
1782	Volta devises his condensing electroscope.
1785	Coulomb devises his balance and experimentally verifies the inverse-square law for charges and for magnetic poles.
1786	Bennett devises the gold-leaf electroscope.
1786	Galvani makes observations on muscular contractions produced by electric discharges in decapitated frogs and advances a theory of animal electricity.
1788	Nicholson devises the rotating doubler.
1794	Volta demonstrates contact electrification by means of his condensing electroscope.
1799	Volta constructs voltaic pile and voltaic battery.
1800	Decomposition of a liquid by electrolysis discovered by Nicholson and Carlisle.
1805	Grotthuss theory of electrochemical decomposition.
1807	Davy produces sodium and potassium by electrolysis.
1811	Poisson's treatment of electric and magnetic potential.
1813	Gauss's theorem.
1820	Oersted discovers deflection of a compass by a current and deflection of a circuit by a magnet.
1820–3	Ampere demonstrates forces between currents, and demonstrates the magnetic equivalence of an electric circuit and a magnetic shell.
1821	Faraday produces rotation of a wire carrying a current around a pole.
1822	Fourier's *Analytical Theory of Heat.*
1823	Thermal e.m.f. discovered by Seebeck.
1824	Arago causes a compass to rotate by rotating a copper disk near it.

1824	Carnot's pronouncement about futility of thermodynamic arguments in which the gas is not left in the initial condition.
1825	Sturgeon constructs electromagnet.
1827	Ohm's law announced.
1828	Green's essay on the potential function.
1831	Faraday discovers motional e.m.fs. and e.m.fs. of mutual inductance, devises disk dynamo.
1831	Joseph Henry discovers e.m.f. of self-inductance.
1833	Faraday discovers laws of electrochemical decomposition.
1834	Faraday's study of self-induction.
1834	Heating and cooling effect of current at a junction discovered by Peltier.
1834	Lenz law announced.
1837	Faraday discovers that the intervening medium affects the force between charges.
1841	Joule's law.
1842	Oscillatory character of Leyden discharge recognized by Henry.
1842	R. Mayer's attempt to obtain the mechanical equivalent of heat.
1843–5	Joule's determinations of the mechanical equivalent of heat.
1846	Weber's molecular current system of electrodynamics.
1847	Helmholtz's *Memoir on the Conservation of Force (Energy)*.
1848–52	Establishment of thermodynamics by Clausius and Kelvin.
1849	Fizeau's determination of the velocity of light.
1853	Kelvin's complete mathematical treatment of condenser discharge.
1856	Measurements by Weber and Kohlrausch give the value for s, the ratio of E.S. and E.M., units, as 3.1×10^{10}.
1857	Kirchhoff in an investigation of propagation of disturbance along aerial wires finds that velocity is s as given above and equals that of light.
1856	Maxwell's first paper.
1861	Maxwell treats a varying electrostatic flux as a displacement current and postulates the magnetic effect of displacement currents.
1864	Maxwell asserts identity of light waves and electromagnetic waves.
1873	Maxwell's *Treatise* on electricity and magnetism.
1876	Rowland demonstrates magnetic effect of a moving charge.
1888	Hertz furnishes experimental justification of Maxwell's postulates by demonstrating electromagnetic wave phenomena.

APPENDIX I

THE PRACTICAL SYSTEM OF ELECTRICAL UNITS

Symbols, defining equations, names, dimensions, and relations to other systems

Quantity	Symbol and defining equation	Name of the practical unit	Dimensional formulas $[U] = [l^a m^b t^c p^d]$				Relative magnitudes of the units	
			a	b	c	d	$\frac{\text{E.S.}}{\text{P.}}$	$\frac{\text{E.M.}}{\text{P.}}$
Length	l, fundamental	centimeter	1	0	0	0	1	1
Time	t, fundamental	second	0	0	1	0	1	1
Mass	m, fundamental	gram-seven	0	1	0	0	10^{-7}	10^{-7}
Area	$a = l_1 l_2$	sq. cm.	2	0	0	0	1	1
Volume	$v = l_1 l_2 l_3$	cu. cm.	3	0	0	0	1	1
Velocity	$V = dl/dt$	cm. per second	1	0	-1	0	1	1
Acceleration	$\mathfrak{a} = dV/dt$	cm. per sec. per sec.	1	0	-2	0	1	1
Force	$f = m \times \mathfrak{a}$	dyne-seven	1	1	-2	0	10^{-7}	10^{-7}
Energy, work	$W = f \times l$	joule	2	1	-2	0	10^{-7}	10^{-7}
Power	$P = W/t$	watt	2	1	-3	0	10^{-7}	10^{-7}
Dielectric constant	$k = 4\pi p$	....	0	0	0	1	$1/(9.10^{11})$	10^9
Permittivity	p, fundamental	Coulomb per sq. cm. per volt per cm.	0	0	0	1	$1/(9.10^{11})$	10^9
Quantity of electricity	$Q = \sqrt{4\pi f p l^2}$	coulomb	$\frac{3}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$1/(3.10^9)$	10
Electric intensity	$\mathbf{F} = \mathbf{f}/Q_t$	volt per cm.	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	300	10^{-8}
Electric potential	$E = W/Q$	volt	$\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	300	10^{-8}
Capacitance	$C = Q/E$	farad	1	0	0	1	$1/(9.10^{11})$	10^9
Elastance	$S = E/Q$	daraf	-1	0	0	-1	9.10^{11}	10^{-9}
Mutual elastance	$S_m = E_1/Q_2$	daraf	-1	0	0	-1	9.10^{11}	10^{-9}
Surface density of charge	$\sigma_c = dQ/da$	coulomb per sq. cm.	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$1/(3.10^9)$	10
Volume density of charge	$\rho = dQ/dv$	coulomb per cu. cm.	$-\frac{3}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$1/(3.10^9)$	10
Electrostatic flux density	$\mathbf{D} = p\mathbf{F}$	coulomb per sq. cm.	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$1/(3.10^9)$	10
Electrostatic flux	$\Psi = Da$	coulomb	$\frac{3}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$1/(3.10^9)$	10
Elastivity	$s = 1/p$	daraf-cm.	0	0	0	-1	9.10^{11}	10^{-9}
Electric current	$I = \frac{dq}{dt} + \frac{d\Psi}{dt}$	ampere	$\frac{3}{2}$	$\frac{1}{2}$	-2	$\frac{1}{2}$	$1/(3.10^9)$	10

Current density	$\mathbf{J} = dI/da$	ampere per sq. cm.	$-\frac{1}{2}$	$\frac{1}{2}$	-2	$\frac{1}{2}$	$1/(3.10^9)$	10
Electromotive force	$E = W/Q$	volt	$\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	300	10^{-8}
Resistance	$R = E/I$	ohm	-1	0	1	-1	9.10^{11}	10^{-9}
Conductance	$G = I/E$	mho	1	0	-1	1	$1/(9.10^{11})$	10^9
Resistivity	$\rho = F/J$	ohm-cm.	0	0	1	-1	9.10^{11}	10^{-9}
Conductivity	$\gamma = J/F$	mho,cm.	0	0	-1	1	$1/(9.10^{11})$	10^9
Magnetic flux density	$\mathbf{B} = \dfrac{f}{Il}$	weber per sq. cm.	$-\frac{3}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	300	10^{-8}
Permeability	$\mu = \dfrac{\int B \cos (B,l) dl}{NI}$	webers per sq. cm. per amp.-turn per cm.	-2	0	2	-1	9×10^{11}	10^{-9}
Magnetic intensity	$\mathbf{H} = \dfrac{\mathbf{B}}{\mu}$	amp.-turn per cm.	$\frac{1}{2}$	$\frac{1}{2}$	-2	$\frac{1}{2}$	$1/(3.10^9)$	10
Magnetomotive force	$\mathfrak{F} = \int H \cos (H,l)\, dl$	amp.-turn	$\frac{3}{2}$	$\frac{1}{2}$	-2	$\frac{1}{2}$	$1/(3.10^9)$	10
Magnetic flux	$\Phi = \int B \cos (B, n) da$	weber	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	300	10^{-8}
Magnetic linkages	$\Lambda = \Phi_1 N$	weber-turn	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	300	10^{-8}
Magnetic potential	$U = -\int H \cos (H,l) dl$	amp.-turn	$\frac{3}{2}$	$\frac{1}{2}$	-2	$\frac{1}{2}$	$1/(3.10^9)$	10
Vector potential	$\mathbf{A} = \dfrac{\mu}{4\pi} \int \dfrac{\mathbf{J} dv}{r}$	weber-cm.	$\frac{3}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$1/(3.10^9)$	10
Mutual inductance	$M = e_2 \Big/ \dfrac{di_1}{dt}$	henry	-1	0	2	-1	9.10^{11}	10^{-9}
Inductance (self)	$L = e \Big/ \dfrac{di}{dt}$	henry	-1	0	2	-1	9.10^{11}	10^{-9}
Permeance	$\mathcal{P} = \Phi/\mathfrak{F}$	webers per amp.-turn	-1	0	2	-1	9.10^{11}	10^{-9}
Reluctance	$\mathcal{R} = \mathfrak{F}/\Phi$	amp.-turn per weber	1	0	-2	1	$1/(9.10^{11})$	10^9
Reluctivity	$\nu = H/B$	amp.-turn per cm. per weber per sq. cm.	2	0	-2	1	$1/(9.10^{11})$	10^9
Magnetic pole strength	$m = \Phi$	weber	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	300	10^{-8}
Intensity of magnetization	$J = (B - \mu_0 H)/4\pi$	weber per sq. cm.	$-\frac{3}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	300	10^{-8}
Susceptibility	$K = J/H$	....	-2	0	2	-1	9×10^{11}	10^{-9}

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